

Oscillation and comparison theorems for second-order half-linear differential equations

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1 Introduction

Over the past four decades a great deal of articles have been devoted to the study of oscillation of solutions of half-linear differential equations. For example, those results can be found in [1–6, 9–12]. Especially, it is well-known that all nontrivial solutions of a half-linear differential equation of the form

$$(|x'|^{\alpha-1}x')' + \frac{\lambda}{t^{\alpha+1}}|x|^{\alpha-1}x = 0, \quad t > t_0 \quad (1.1)$$

with $\alpha > 0$, $\lambda > 0$ and $t_0 \geq 0$, are oscillatory if

$$\lambda > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1};$$

otherwise, they are nonoscillatory. This fact means that $(\alpha/(\alpha+1))^{\alpha+1}$ is the lower bound for all nontrivial solutions of (1.1) to be oscillatory. Such a number is generally called the *oscillation constant* (for example, see [7, 8, 13–15]).

Let us add a perturbation to equation (1.1) when λ is the oscillation constant and consider the perturbed half-linear differential equation

$$(|x'|^{\alpha-1}x')' + \frac{1}{t^{\alpha+1}} \left\{ \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} + \left(\frac{\alpha}{\alpha+1} \right)^{\alpha} \delta(t) \right\} |x|^{\alpha-1}x = 0, \quad (E_{\alpha})$$

where $\delta(t)$ is positive and continuous on some half-line (t_0, ∞) . Elbert and Schneider [6] have investigated the asymptotic behaviour of solutions of (E_{α}) . Using their results, we can present the following statements.

Theorem A. *Let $\alpha > 1$. If equation (E_{α}) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_1) are oscillatory.*

Theorem B. *Let $0 < \alpha < 1$. If equation (E_1) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_{α}) are oscillatory.*

It follows from the fact mentioned in the first paragraph and Sturm's comparison theorem for half-linear differential equations that if

$$\liminf_{t \rightarrow \infty} \delta(t) > 0, \quad (1.2)$$

then all nontrivial solutions of (E_α) are oscillatory. As to Sturm's separation and comparison theorems, for example, see [5, 11, 12]. On the other hand, if condition (1.2) fails to hold, then there is some possibility that equation (E_α) has a nonoscillatory solution. One of the most interesting case is that $\delta(t) = \lambda/(\log t)^2$ with $\lambda > 0$. In this case, if $\lambda > 1/2$, then all nontrivial solutions of (E_α) are oscillatory; otherwise, they are nonoscillatory (for details, see [6]).

We may regard Theorems A and B as comparison theorems between the linear differential equation

$$x'' + \frac{1}{t^2} \left\{ \frac{1}{4} + \frac{1}{2} \delta(t) \right\} x = 0 \quad (E_1)$$

and half-linear differential equations of the form (E_α) . Let α and β be positive numbers satisfying $\alpha < 1 < \beta$. Then, combining Theorems A and B, we get the following conclusion: if equation (E_β) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory. A natural question now arises as to whether or not the converse proposition is also true.

The first purpose of this paper is to extend Theorems A and B to a comparison theorem between any two half-linear differential equations. The second purpose is to give an answer to the above question. Our main results are stated as follows:

Theorem 1.1. *Let $0 < \alpha < \beta$. If equation (E_β) has a nontrivial oscillatory solution, then all nontrivial solutions of (E_α) are oscillatory.*

Remark 1.1. Theorem 1.1 is a generalization of Theorems A and B. To put it precisely, Theorem 1.1 coincides with Theorem A (respectively, Theorem B) when $\alpha = 1$ (respectively, $\beta = 1$).

Theorem 1.2. *Let $0 < \alpha < \beta$. If equation (E_α) has a nontrivial oscillatory solution, then all nontrivial solutions of*

$$(|x'|^{\beta-1} x')' + \frac{1}{t^{\beta+1}} \left\{ \left(\frac{\beta}{\beta+1} \right)^{\beta+1} + \nu \delta(t) \right\} |x|^{\beta-1} x = 0 \quad (1.3)$$

are oscillatory, where $\nu > (\beta/(\beta+1))^\beta$.

Remark 1.2. It is essential that ν is greater than $(\beta/(\beta+1))^\beta$ in Theorem 1.2. Unfortunately, even if equation (E_α) has a nontrivial oscillatory solution, we cannot judge whether all nontrivial solutions of (E_β) are oscillatory or not.

Remark 1.3. From Theorems 1.1 and 1.2, we see that the oscillation constant for equation (E_α) with $\delta(t) = \lambda/(\log t)^2$ is $1/2$ for any $\alpha > 0$.

2 Riccati technique

Consider the half-linear differential equation

$$(|x'|^{p-1}x')' + \frac{1}{t^{p+1}} \left\{ \left(\frac{p}{p+1} \right)^{p+1} + h(t) \right\} |x|^{p-1}x = 0 \quad (2.1)$$

with $p > 0$ a fixed real number, where $h(t)$ is positive and continuous on $(0, \infty)$. Using Riccati's transformation, we prepare some lemmas below. To this end, we denote

$$H_p(\xi) = p \left\{ \xi^{(p+1)/p} - \xi + \frac{p^p}{(p+1)^{p+1}} \right\}$$

for $\xi > 0$ and

$$\gamma_p = \left(\frac{p}{p+1} \right)^p.$$

Lemma 2.1. *Let $\xi(s)$ be a positive function on $[s_0, \infty)$ with $s_0 > 0$ satisfying*

$$\dot{\xi}(s) + H_p(\xi(s)) \leq 0. \quad (2.2)$$

Then it is nonincreasing and tends to γ_p as $s \rightarrow \infty$.

Proof. From

$$H_p(\gamma_p) = p \left\{ \left(\frac{p}{p+1} \right)^{p+1} - \left(\frac{p}{p+1} \right)^p + \frac{p^p}{(p+1)^{p+1}} \right\} = 0$$

and

$$\frac{d}{d\xi} H_p(\xi) = (p+1)\xi^{1/p} - p,$$

we see that $H_p(\xi) \geq 0$ for $\xi > 0$ and $H_p(\xi) = 0$ if and only if $\xi = \gamma_p$.

Since $\xi(s)$ is positive for $s \geq s_0$, we have

$$\dot{\xi}(s) \leq -H_p(\xi(s)) \leq 0$$

by (2.2), namely, $\xi(s)$ is nonincreasing. Hence, there exists a $\mu \geq 0$ such that $\xi(s) \searrow \mu$ as $s \rightarrow \infty$. Suppose that $\mu \neq \gamma_p$. If $\mu > \gamma_p$, then $\xi(s) > \mu > (\mu + \gamma_p)/2 > \gamma_p$ for $s \geq s_0$. If $\mu < \gamma_p$, then $\mu < \xi(s) < (\mu + \gamma_p)/2 < \gamma_p$ for s sufficiently large. In either case,

$$\dot{\xi}(s) \leq -H_p(\xi(s)) \leq -H_p((\mu + \gamma_p)/2) < 0$$

for s sufficiently large, which yields that $\xi(s)$ tends to $-\infty$ as $s \rightarrow \infty$. This contradicts the assumption that $\xi(s)$ is positive for $s \geq s_0$. Thus, $\xi(s)$ tends to γ_p as $s \rightarrow \infty$. The proof of Lemma 2.1 is complete. $\square$

We next give a sufficient condition for all nontrivial solutions of (2.1) to be nonoscillatory.

Lemma 2.2. Let $\xi(s)$ be a positive function on $[s_0, \infty)$ with $s_0 > 0$ satisfying

$$\dot{\xi}(s) + H_p(\xi(s)) + h(e^s) \leq 0, \quad (2.3)$$

where h is the function defined in equation (2.1). Then all nontrivial solutions of (2.1) to be nonoscillatory.

Proof. Define

$$c(s) = -\dot{\xi}(s) - H_p(\xi(s))$$

for $s \geq s_0$. Then we have

$$c(s) \geq h(e^s) \quad \text{for } s \geq s_0. \quad (2.4)$$

Let $u(s)$ be the positive function defined by

$$u(s) = \exp \left(\int_{s_0}^s \xi(\sigma)^{1/p} d\sigma \right)$$

for $s \geq s_0$. Then we get

$$\dot{u}(s) = u(s)\xi(s)^{1/p} > 0$$

for $s \geq s_0$, namely,

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)} \right)^p \quad \text{for } s \geq s_0.$$

Differentiate $\xi(s)$ to obtain

$$\dot{\xi}(s) = \frac{(\dot{u}(s)^p) \cdot u(s)^p - pu(s)^{p-1} \dot{u}(s)^{p+1}}{u(s)^{2p}} = \frac{(\dot{u}(s)^p) \cdot}{u(s)^p} - p \left(\frac{\dot{u}(s)}{u(s)} \right)^{p+1}$$

for $s \geq s_0$. Hence, we have

$$\begin{aligned} c(s) &= -\frac{(\dot{u}(s)^p) \cdot}{u(s)^p} + p \left(\frac{\dot{u}(s)}{u(s)} \right)^{p+1} - p \left\{ \left(\frac{\dot{u}(s)}{u(s)} \right)^{p+1} - \left(\frac{\dot{u}(s)}{u(s)} \right)^p + \frac{p^p}{(p+1)^{p+1}} \right\} \\ &= -\frac{(\dot{u}(s)^p) \cdot}{u(s)^p} + p \left(\frac{\dot{u}(s)}{u(s)} \right)^p - \left(\frac{p}{p+1} \right)^{p+1}, \end{aligned}$$

and therefore, we see that the positive function $u(s)$ is a nonoscillatory solution of the equation

$$(|\dot{u}|^{p-1} \dot{u})' - p|\dot{u}|^{p-1} \dot{u} + \left\{ \left(\frac{p}{p+1} \right)^{p+1} + c(s) \right\} |u|^{p-1} u = 0. \quad (2.5)$$

Changing variable $t = e^s$, we can transform equation (2.5) into the equation

$$(|x'|^{p-1} x')' + \frac{1}{t^{p+1}} \left\{ \left(\frac{p}{p+1} \right)^{p+1} + c(\log t) \right\} |x|^{p-1} x = 0. \quad (2.6)$$

Let $x(t)$ be the solution of (2.6) corresponding to $u(s)$. Then $x(t)$ is positive for $t \geq e^{s_0}$. From (2.4) it follows that

$$c(\log t) \geq h(t) \quad \text{for } t \geq e^{s_0}.$$

Hence, by Sturm's comparison theorem for half-linear differential equations, all nontrivial solutions of (2.1) are nonoscillatory. This completes the proof of Lemma 2.2. $\square$

3 Proof of the main theorems

By means of Lemmas 2.1 and 2.2, we can prove our comparison theorems for half-linear differential equations of the form (E_α) .

Proof of Theorem 1.1. By way of contradiction, we suppose that equation (E_β) has an oscillatory solution and equation (E_α) has a nonoscillatory solution $x(t)$. We may assume that $x(t)$ is eventually positive, because the proof of the case that $x(t)$ is eventually negative is carried out in the same way. Hence, there exists a $T > t_0$ such that $x(t) > 0$ for $t \geq T$, and therefore,

$$(|x'(t)|^{\alpha-1}x'(t))' = -\frac{1}{t^{\alpha+1}}\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \gamma_\alpha\delta(t)\right\}|x(t)|^{\alpha-1}x(t) < 0 \quad (3.1)$$

for $t \geq T$. From this we see that $x'(t)$ is also positive for $t \geq T$. In fact, if there exists a $t_1 \geq T$ such that $x'(t_1) \leq 0$, then by (3.1) we have

$$|x'(t)|^{\alpha-1}x'(t) < |x'(t_1)|^{\alpha-1}x'(t_1) \leq 0$$

for $t > t_1$. Hence, we can find a $t_2 > t_1$ such that $x'(t_2) < 0$. By (3.1) again, we obtain

$$|x'(t)|^{\alpha-1}x'(t) \leq |x'(t_2)|^{\alpha-1}x'(t_2) < 0$$

for $t \geq t_2$. We therefore conclude that $x'(t) \leq x'(t_2) < 0$ for $t \geq t_2$, which implies that

$$x(t) \leq x'(t_2)(t - t_2) + x(t_2) \rightarrow -\infty$$

as $t \rightarrow \infty$. This is a contradiction to the assumption that $x(t)$ is eventually positive.

Making the change of variable $s = \log t$, we can rewrite equation (E_α) in the form

$$(|\dot{u}|^{\alpha-1}\dot{u})' - \alpha|\dot{u}|^{\alpha-1}\dot{u} + \left\{\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \gamma_\alpha\delta(e^s)\right\}|u|^{\alpha-1}u = 0. \quad (3.2)$$

Let $u(s)$ be the solution of (3.2) which corresponds to $x(t)$. Then $u(s) = x(t) > 0$ and $\dot{u}(s) = tx'(t) > 0$ for $s \geq \log T$. Define

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)}\right)^\alpha$$

and differentiate $\xi(s)$ to obtain

$$\dot{\xi}(s) = \frac{(\dot{u}(s)^\alpha)' }{u(s)^\alpha} - \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^{\alpha+1}.$$

Using (3.2), we have

$$\begin{aligned} \dot{\xi}(s) &= \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} - \gamma_\alpha\delta(e^s) - \alpha\left(\frac{\dot{u}(s)}{u(s)}\right)^{\alpha+1} \\ &= -\alpha\left\{\xi(s)^{(\alpha+1)/\alpha} - \xi(s) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right\} - \gamma_\alpha\delta(e^s) \\ &= -H_\alpha(\xi(s)) - \gamma_\alpha\delta(e^s) \end{aligned} \quad (3.3)$$

for $s \geq \log T$.

We here show that there exists an $\varepsilon_0 > 0$ such that

$$\frac{\gamma_\alpha}{\gamma_\beta} H_\beta \left(\frac{\gamma_\beta}{\gamma_\alpha} \xi \right) \leq H_\alpha(\xi) \quad (3.4)$$

for $\gamma_\alpha \leq \xi \leq \gamma_\alpha + \varepsilon_0$. For this purpose, we define

$$F_1(\xi) = H_\alpha(\xi) - \frac{\gamma_\alpha}{\gamma_\beta} H_\beta \left(\frac{\gamma_\beta}{\gamma_\alpha} \xi \right).$$

Then, differentiating $F_1(\xi)$ three times, we obtain

$$\begin{aligned} \frac{d}{d\xi} F_1(\xi) &= (\alpha + 1)\xi^{1/\alpha} - \alpha - (\beta + 1) \left(\frac{\gamma_\beta}{\gamma_\alpha} \right)^{1/\beta} \xi^{1/\beta} + \beta, \\ \frac{d^2}{d\xi^2} F_1(\xi) &= \frac{\alpha + 1}{\alpha} \xi^{(1-\alpha)/\alpha} - \frac{\beta + 1}{\beta} \left(\frac{\gamma_\beta}{\gamma_\alpha} \right)^{1/\beta} \xi^{(1-\beta)/\beta}, \\ \frac{d^3}{d\xi^3} F_1(\xi) &= \frac{1 - \alpha^2}{\alpha^2} \xi^{(1-2\alpha)/\alpha} - \frac{1 - \beta^2}{\beta^2} \left(\frac{\gamma_\beta}{\gamma_\alpha} \right)^{1/\beta} \xi^{(1-2\beta)/\beta}, \end{aligned}$$

so that

$$F_1(\gamma_\alpha) = \frac{d}{d\xi} F_1(\xi) \Big|_{\xi=\gamma_\alpha} = \frac{d^2}{d\xi^2} F_1(\xi) \Big|_{\xi=\gamma_\alpha} = 0 \quad (3.5)$$

and

$$\frac{d^3}{d\xi^3} F_1(\xi) \Big|_{\xi=\gamma_\alpha} = \frac{\beta - \alpha}{\alpha\beta} \left(\frac{\alpha + 1}{\alpha} \right)^{2\alpha} > 0. \quad (3.6)$$

From (3.6) we can choose an $\varepsilon_0 > 0$ such that

$$\frac{d^3}{d\xi^3} F_1(\xi) > 0 \quad \text{for } \gamma_\alpha \leq \xi \leq \gamma_\alpha + \varepsilon_0.$$

Hence, taking account of this estimation and (3.5), we see that $F_1(\xi) \geq 0$ for $\gamma_\alpha \leq \xi \leq \gamma_\alpha + \varepsilon_0$, as required.

Because of (3.3), Lemma 2.1 is available for $p = \alpha$ and $s_0 = \log T$, and therefore, there exists an $s_1 > s_0$ such that

$$\gamma_\alpha \leq \xi(s) \leq \gamma_\alpha + \varepsilon_0$$

for $s \geq s_1$. Hence, together with (3.3) and (3.4), we get

$$\dot{\xi}(s) + \frac{\gamma_\alpha}{\gamma_\beta} H_\beta \left(\frac{\gamma_\beta}{\gamma_\alpha} \xi(s) \right) + \gamma_\alpha \delta(e^s) \leq 0$$

for $s \geq s_1$. Let $\eta(s) = \gamma_\beta \xi(s) / \gamma_\alpha$. Then we see that $\eta(s)$ satisfies

$$\dot{\eta}(s) + H_\beta(\eta(s)) + \gamma_\beta \delta(e^s) \leq 0$$

for $s \geq s_1$. Hence, from Lemma 2.2 with $p = \beta$ and $h(e^s) = \gamma_\beta \delta(e^s)$ we conclude that all nontrivial solutions of (E_β) are nonoscillatory. This contradicts the assumption that equation (E_β) has an oscillatory solution. Thus, we have completed the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Suppose to the contrary that equation (E_α) has an oscillatory solution and equation (1.3) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. Let $T > t_0$ be a number satisfying $x(t) > 0$ for $t \geq T$. From the same manner as in the proof of Theorem 1.1, we see that $x'(t)$ is also positive for $t \geq T$.

By putting $t = e^s$, equation (1.3) becomes

$$(|\dot{u}|^{\beta-1}\dot{u})' - \beta|\dot{u}|^{\beta-1}\dot{u} + \left\{ \left(\frac{\beta}{\beta+1} \right)^{\beta+1} + (\gamma_\beta + \varepsilon)\delta(e^s) \right\} |u|^{\beta-1}u = 0$$

for some $\varepsilon > 0$, where $u(s) = x(e^s) = x(t)$. Define

$$\xi(s) = \left(\frac{\dot{u}(s)}{u(s)} \right)^\beta,$$

which is positive for $s \geq \log T$. A simple calculation shows that

$$\dot{\xi}(s) = -H_\beta(\xi(s)) - (\gamma_\beta + \varepsilon)\delta(e^s) \quad (3.7)$$

for $s \geq \log T$. Hence, it follows from Lemma 2.1 with $p = \beta$ and $s_0 = \log T$ that

$$\xi(s) \searrow \gamma_\beta \quad \text{as } s \rightarrow \infty. \quad (3.8)$$

Let

$$c = \frac{\gamma_\beta + \varepsilon}{\gamma_\alpha} \quad \text{and} \quad \eta(s) = \frac{\xi(s) + \varepsilon}{c}.$$

Then, from (3.7) and (3.8) it turns out that

$$\dot{\eta}(s) + \frac{1}{c}H_\beta(c\eta(s) - \varepsilon) + \gamma_\alpha\delta(e^s) = 0 \quad (3.9)$$

for $s \geq s_0$ and

$$\eta(s) \searrow \gamma_\alpha \quad \text{as } s \rightarrow \infty, \quad (3.10)$$

respectively.

To show that there exists an $\varepsilon_0 > 0$ such that

$$H_\alpha(\eta) \leq \frac{1}{c}H_\beta(c\eta - \varepsilon) \quad (3.11)$$

for $\gamma_\alpha \leq \eta \leq \gamma_\alpha + \varepsilon_0$, we define

$$F_2(\eta) = \frac{1}{c}H_\beta(c\eta - \varepsilon) - H_\alpha(\eta).$$

Differentiating $F_2(\eta)$ twice, we have

$$\begin{aligned}\frac{d}{d\eta}F_2(\eta) &= (\beta+1)(c\eta - \varepsilon)^{1/\beta} - \beta - (\alpha+1)\eta^{1/\alpha} + \alpha, \\ \frac{d^2}{d\eta^2}F_2(\eta) &= \frac{c(\beta+1)}{\beta}(c\eta - \varepsilon)^{(1-\beta)/\beta} - \frac{\alpha+1}{\alpha}\eta^{(1-\alpha)/\alpha},\end{aligned}$$

so that

$$F_2(\gamma_\alpha) = \frac{d}{d\xi}F_2(\eta)\Big|_{\eta=\gamma_\alpha} = 0$$

and

$$\frac{d^2}{d\xi^2}F_2(\eta)\Big|_{\eta=\gamma_\alpha} = \frac{\varepsilon}{\gamma_\alpha\gamma_\beta} > 0.$$

Hence, we can select an $\varepsilon_0 > 0$ such that

$$\frac{d^2}{d\xi^2}F_2(\eta) > 0 \quad \text{for } \gamma_\alpha \leq \eta \leq \gamma_\alpha + \varepsilon_0,$$

and therefore, $F_2(\eta) \geq 0$ for $\gamma_\alpha \leq \eta \leq \gamma_\alpha + \varepsilon_0$. Thus, the inequality (3.11) is shown.

By (3.10), there exists an $s_1 > s_0$ such that

$$\gamma_\alpha \leq \eta(s) \leq \gamma_\alpha + \varepsilon_0$$

for $s \geq s_1$. Hence, together with (3.9) and (3.11), we have

$$\dot{\eta}(s) + H_\alpha(\eta(s)) + \gamma_\alpha\delta(e^s) \leq 0$$

for $s \geq s_1$. Using Lemma 2.2 with $p = \alpha$ and $h(e^s) = \gamma_\alpha\delta(e^s)$, we see that all nontrivial solutions of (E_α) are nonoscillatory. This is a contradiction to the assumption that equation (E_α) has an oscillatory solution. We have thus proved Theorem 1.2. $\square$

4 Discussion and another comparison theorem

Let us now look at Theorem 1.2 from a different angle. To this end, we consider the more general half-linear differential equation

$$\left(|x'|^{\alpha-1}x'\right)' + a(t)|x|^{\alpha-1}x = 0, \quad (4.1)$$

where $\alpha > 0$ and $a(t)$ is positive and continuous on (t_0, ∞) for some $t_0 \geq 0$. Then, we can guarantee that all solutions of (4.1) are continuable in the future. Hence, it is worth while to discuss whether solutions of (4.1) are oscillatory or not.

The Hille-Wintner comparison theorem has been widely studied by many authors. For example, Kusano and Yoshida [9] presented the following comparison theorem of Hille-Wintner type for half-linear differential equations (see also [10]).

Theorem C. *Consider*

$$(|x'|^{\alpha-1}x')' + b(t)|x|^{\alpha-1}x = 0, \quad (4.2)$$

where $b(t)$ is positive and continuous on (t_0, ∞) . Suppose that

$$\int_t^\infty a(s)ds \leq \int_t^\infty b(s)ds$$

for all sufficiently large t . If all nontrivial solutions of (4.1) are oscillatory, then those of (4.2) are also oscillatory.

We can regard the number α in equations (4.1) and (4.2) as a positive parameter. In Theorem C, needless to say, the parameter α is fixed and the integral of the coefficient $a(t)$ is compared with that of the coefficient $b(t)$. Let us fix the coefficient $a(t)$ and move the parameter α to the contrary. Then we have another comparison theorem for half-linear differential equations.

Theorem 4.1. *Consider*

$$(|x'|^{\beta-1}x')' + a(t)|x|^{\beta-1}x = 0, \quad (4.3)$$

where $a(t)$ is the same as in equation (4.1). Suppose that $0 < \alpha < \beta$. If all nontrivial solutions of (4.1) are oscillatory, then those of (4.3) are also oscillatory.

Proof. The proof is by contradiction. We suppose that all nontrivial solutions of (4.1) are oscillatory and equation (4.3) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. As in the proof of Theorem 1.1, we see that $x'(t)$ is also eventually positive.

Define the function $\xi(t)$ by

$$\xi(t) = \left(\frac{x'(t)}{x(t)} \right)^\beta.$$

Then there exists a $T > t_0$ such that $\xi(t) > 0$ and

$$\xi'(t) = -a(t) - \beta\xi(t)^{(\beta+1)/\beta} < 0 \quad (4.4)$$

for $t \geq T$, namely, $\xi(t)$ is decreasing and bounded from below. Hence, we can find a $\mu \geq 0$ such that

$$\xi(t) \searrow \mu \quad \text{as } t \rightarrow \infty,$$

and therefore, we have

$$\xi'(t) = -a(t) - \beta\xi(t)^{(\beta+1)/\beta} \leq -\beta\mu^{(\beta+1)/\beta}$$

for $t \geq T$. If $\mu > 0$, then $\xi(t)$ has to tend to $-\infty$ as $t \rightarrow \infty$. This contradicts the fact that $\xi(t)$ is eventually positive. Thus, $\xi(t)$ tends to zero as $t \rightarrow \infty$. From this property of $\xi(t)$ and the assumption that $0 < \alpha < \beta$, we see that there exists a $t_1 > T$ such that

$$\alpha\xi(t)^{(\alpha+1)/\alpha} \leq \beta\xi(t)^{(\beta+1)/\beta}$$

for $t \geq t_1$. Hence, together with (4.4), we have

$$\xi'(t) \leq -a(t) - \alpha \xi(t)^{(\alpha+1)/\alpha} \quad (4.5)$$

for $t \geq t_1$.

It is easy to check that the function

$$y(t) = \exp \left(\int_{t_1}^t \xi(\tau)^{1/\alpha} d\tau \right)$$

is a nonoscillatory solution of

$$(|x'|^{\alpha-1} x')' + b(t) |x|^{\alpha-1} x = 0,$$

where $b(t) = -\xi'(t) - \alpha \xi(t)^{(\alpha+1)/\alpha}$. From (4.5) it follows that $a(t) \leq b(t)$ for $t \geq t_1$. Hence, Sturm's comparison theorem implies that (4.1) also has a nonoscillatory solution. This is a contradiction, thereby completing the proof of Theorem 4.1. $\square$

In the case that

$$t^{\alpha+1} a(t) > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \quad (4.6)$$

for t sufficiently large, we can rewrite equation (4.1) in the form (E_α) with

$$\delta(t) = \left(\frac{\alpha+1}{\alpha} \right)^\alpha t^{\alpha+1} a(t) - \frac{\alpha}{\alpha+1} > 0.$$

Suppose that all nontrivial solutions of (4.1) are oscillatory. Then, from Theorem 1.2 we see that all nontrivial solutions of

$$(|x'|^{\beta-1} x')' + c(t) |x|^{\beta-1} x = 0$$

with

$$c(t) = \frac{1}{t^{\beta+1}} \left\{ \left(\frac{\beta}{\beta+1} \right)^{\beta+1} + \left(\left(\frac{\beta}{\beta+1} \right)^\beta + \varepsilon \right) \delta(t) \right\}$$

are oscillatory. Since $0 < \alpha < \beta$, we have

$$\begin{aligned} c(t) &= \frac{1}{t^{\beta+1}} \left\{ \left(\frac{\beta}{\beta+1} \right)^{\beta+1} + \left(\left(\frac{\beta}{\beta+1} \right)^\beta + \varepsilon \right) \delta(t) \right\} \\ &< \frac{1}{t^{\alpha+1}} \left\{ \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} + \left(\frac{\alpha}{\alpha+1} \right)^\alpha \delta(t) \right\} = a(t) \end{aligned}$$

for t sufficiently large. Hence, from Theorem C we conclude that all nontrivial solutions of (4.3) are also oscillatory. This means that Theorem 1.2 is sharper than Theorem 4.1 in the case (4.6).

References

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