

On a subclass of n -starlike functions

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ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and w is a fixed point in U . In [1] the authors introduced the classes of functions close to convex and α -convex, which are normalized in the same way. All these definitions are somewhat similar to the ones for the uniformly type functions and it is easy to see that for $w = 0$ are obtained the well-known classes of starlike, convex, close to convex and α -convex functions. In this paper we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in U .

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

We recall here the definitions of the well-known classes of starlike and convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U \right\},$$

$$S^c = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in U \right\}$$

Let w be a fixed point in U and $A(w) = \{f \in \mathcal{H}(U) : f(w) = f'(w) - 1 = 0\}$.

In [3] S. Kanas and F. Ronning introduced the following classes:

$$S(w) = \{f \in A(w) : f \text{ is univalent in } U\}$$

$$ST(w) = S^*(w) = \left\{ f \in S(w) : \operatorname{Re} \frac{(z-w)f'(z)}{f(z)} > 0, \quad z \in U \right\}$$

$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re} \frac{(z-w)f''(z)}{f'(z)} > 0, \quad z \in U \right\}.$$

It is obvious that exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \text{ if and only if } f(z) = (z-w)g'(z) \in S^*(w).$$

Let denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$ that are regular in U and satisfy $p(w) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in U$.

2 Preliminary results

It is easy to see that a function $f(z) \in A(w)$ have the series expansions:

$$f(z) = (z - w) + a_2(z - w)^2 + \dots$$

In [8] J. K. Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$:

Teorema 2.1 *If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$, then*

$$(1) \quad |B_n| \leq \frac{2}{(1+d)(1-d)^n}, \quad \text{where } d = |w| \text{ and } n \geq 1.$$

Using the above result, S. Kanas and F. Ronning obtain in [3]:

Teorema 2.2 *Let $f \in S^*(w)$ and $f(z) = (z - w) + b_2(z - w)^2 + \dots$. Then*

$$(2) \quad \begin{aligned} |b_2| &\leq \frac{2}{1-d^2}, \quad |b_3| \leq \frac{3+d}{(1-d^2)^2}, \\ |b_4| &\leq \frac{2}{3} \cdot \frac{(2+d)(3+d)}{(1-d^2)^3}, \quad |b_5| \leq \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4} \end{aligned}$$

where $d = |w|$.

Remark 2.1 *It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.*

In [1] are also defined the following sets:

$$D(w) = \left\{ z \in U : \operatorname{Re} \left[\frac{w}{z} \right] < 1 \text{ and } \operatorname{Re} \left[\frac{z(1+z)}{(z-w)(1-z)} \right] > 0 \right\} \text{ for } w \neq 0 \text{ and } D(0) = U;$$

$$s(w) = \{f : D(w) \rightarrow \mathbb{C}\} \cap S(w); \quad s^*(w) = S^*(w) \cap s(w)$$

where w is a fixed point in U .

The authors consider the integral operator $L_a : A(w) \rightarrow A(w)$ defined by

$$(3) \quad f(z) = L_a F(z) = \frac{1+a}{(z-w)^a} \cdot \int_w^z F(t) \cdot (t-w)^{a-1} dt, \quad a \in \mathbb{R}, \quad a \geq 0.$$

The next theorem is results of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [3], [4], [5]).

Teorema 2.3 *Let h convex in U and $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and p satisfied the Briot - Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } p(z) \prec h(z).$$

3 Main results

Definition 3.1 *Let w be a fixed point in U , $n \in \mathbb{N}$. We denote by D_w^n the differential operator:*

$$D_w^n : A(w) \rightarrow A(w) \text{ with :}$$

$$D_w^0 f(z) = f(z)$$

$$D_w^1 f(z) = \bar{D}_w f(z) = (z-w) \cdot f'(z)$$

$$D_w^n f(z) = D_w(D_w^{n-1} f(z)).$$

Remark 3.1 *For $f \in A(w)$, $f(w) = (z-w) + \sum_{j=2}^{\infty} a_j (z-w)^j$, we have*

$$D_w^n f(z) = (z-w) + \sum_{j=2}^{\infty} j^n \cdot a_j \cdot (z-w)^j.$$

It easy to see that if we take $w = 0$ we obtain the Şălăgean differential operator (see [7]).

Definition 3.2 *Let w be a fixed point in U , $n \in \mathbb{N}$ and $f \in S(w)$. We say that f is a n - w -starlike functions if*

$$\operatorname{Re} \frac{D_w^{n+1} f(z)}{D_w^n f(z)} > 0, \quad z \in U.$$

We denote the class of all this functions by $S_n^(w)$.*

Remark 3.2 1. $S_0^*(w) = S^*(w)$ and $S_0^*(0) = S_n^*$, where S_n^* is the class of n -starlike functions introduced by Sălăgean in [7].

2. If $f(z) \in S_n^*(w)$ and we denote $D_w^n f(z) = g(z)$, we obtain $g(z) \in S^*(w)$.

3. Using the class $s(w)$, we obtain $s_n^*(w) = S_n^*(w) \cap s(w)$.

Teorema 3.1 Let w be a fixed point in U and $n \in \mathbb{N}$. If $f(z) \in s_{n+1}^*(w)$ then $f(z) \in s_n^*(w)$. This means

$$s_{n+1}^*(w) \subset s_n^*(w).$$

Proof. From $f(z) \in s_{n+1}^*(w)$ we have $\operatorname{Re} \frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} > 0$, $z \in U$.

We denote $p(z) = \frac{D_w^{n+1} f(z)}{D_w^n f(z)}$, where $p(0) = 1$ and $p(z) \in \mathcal{H}(U)$.

We obtain:

$$\begin{aligned} \frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} &= \frac{D_w (D_w^{n+1} f(z))}{D_w (D_w^n f(z))} = \\ &= \frac{(z-w) (D_w^{n+1} f(z))'}{(z-w) (D_w^n f(z))'} = \frac{(D_w^{n+1} f(z))'}{(D_w^n f(z))'} \\ p'(z) &= \frac{(D_w^{n+1} f(z))' \cdot (D_w^n f(z)) - (D_w^{n+1} f(z)) \cdot (D_w^n f(z))'}{(D_w^n f(z))^2} = \\ &= \frac{(D_w^{n+1} f(z))'}{(D_w^n f(z))'} \cdot \frac{(D_w^n f(z))'}{D_w^n f(z)} - p(z) \cdot \frac{(D_w^n f(z))'}{D_w^n f(z)}. \end{aligned}$$

Thus we have:

$$\begin{aligned} (z-w) \cdot p'(z) &= \frac{(D_w^{n+1} f(z))'}{(D_w^n f(z))'} \cdot \frac{(z-w) \cdot (D_w^n f(z))'}{D_w^n f(z)} - p(z) \cdot \frac{(z-w) \cdot (D_w^n f(z))'}{D_w^n f(z)} = \\ (z-w) \cdot p'(z) &= \frac{(D_w^{n+1} f(z))'}{(D_w^n f(z))'} \cdot p(z) - [p(z)]^2 \end{aligned}$$

and

$$\frac{(D_w^{n+1} f(z))'}{(D_w^n f(z))'} = p(z) + \frac{1}{p(z)} \cdot (z-w) \cdot p'(z).$$

From $\operatorname{Re} \frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} > 0$ we obtain $p(z) + \frac{1}{p(z)} \cdot (z-w) \cdot p'(z) \prec \frac{1+z}{1-z}$
or

$$p(z) + \frac{zp'(z)}{1 - \frac{w}{z} \cdot p(z)} \prec \frac{1+z}{1-z} \equiv h(z), \text{ with } h(0) = 1.$$

From hypothesis we have $\operatorname{Re} \left[\frac{1}{1 - \frac{w}{z}} \cdot h(z) \right] > 0$, and thus from Theorem 2.3 we obtain $p(z) \prec h(z)$ or $\operatorname{Re} p(z) > 0$. This means $f \in s_n^*(w)$.

Remark 3.3 From Theorem 3.1 we obtain $s_n^*(w) \subseteq s_0^*(w) \subseteq S^*(w)$, $n \in \mathbb{N}$.

Teorema 3.2 If $F(z) \in s_n^*(w)$ then $f(z) = L_a F(z) \in S_n^*(w)$, where L_a is the integral operator defined by (3).

Proof. From (3) we obtain

$$(1 + a) \cdot F(z) = a \cdot f(z) + (z - w) \cdot f'(z).$$

By means of the application of the operator D_w^{n+1} we obtain

$$(1 + a) \cdot D_w^{n+1} F(z) = a \cdot D_w^{n+1} f(z) + D_w^{n+1} [(z - w) \cdot f'(z)]$$

or

$$(1 + a) \cdot D_w^{n+1} F(z) = a \cdot D_w^{n+1} f(z) + D_w^{n+2} f(z).$$

Similarly, by means of the application of the operator D_w^n we obtain

$$(1 + a) \cdot D_w^n F(z) = a \cdot D_w^n f(z) + D_w^{n+1} f(z).$$

Thus

$$\frac{D_w^{n+1} F(z)}{D_w^n F(z)} = \frac{\frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} \cdot \frac{D_w^{n+1} f(z)}{D_w^n f(z)} + a \cdot \frac{D_w^{n+1} f(z)}{D_w^n f(z)}}{\frac{D_w^{n+1} f(z)}{D_w^n f(z)} + a}.$$

Using the notation $\frac{D_w^{n+1} f(z)}{D_w^n f(z)} = p(z)$, with $p(0) = 1$, we have

$$\frac{(z - w) \cdot p'(z)}{p(z)} = \frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} - p(z)$$

or

$$\frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} = p(z) + \frac{(z - w) \cdot p'(z)}{p(z)}.$$

Thus

$$\begin{aligned} \frac{D_w^{n+1} F(z)}{D_w^n F(z)} &= \frac{p(z) \left[p(z) + \frac{(z - w)p'(z)}{p(z)} + a \right]}{p(z) + a} = \\ &= p(z) + \frac{z p'(z)}{\frac{1}{1 - \frac{w}{z}} p(z) + \frac{a}{1 - \frac{w}{z}}}. \end{aligned}$$

From $F(z) \in s_n^*(w)$ we obtain $\frac{D_w^{n+1}F(z)}{D_w^n F(z)} \prec \frac{1+z}{1-z} \equiv h(z)$ or

$$p(z) + \frac{zp'(z)}{\frac{1}{1-\frac{w}{z}}p(z) + \frac{a}{1-\frac{w}{z}}} \prec h(z).$$

From hypothesis we have $\operatorname{Re} \left[\frac{1}{1-\frac{w}{z}} \cdot h(z) + \frac{a}{1-\frac{w}{z}} \right] > 0$ and from Theorem 2.3 we obtain $p(z) \prec h(z)$ or $\operatorname{Re} \left\{ \frac{D_w^{n+1}f(z)}{D_w^n f(z)} \right\} > 0, z \in U$. This means $f(z) = L_a F(z) \in S_n^*(w)$.

Remark 3.4 If we consider $w = 0$ in Theorem 3.2 we obtain that the integral operator defined by (3) preserve the class of n -starlike functions, and if we consider $w = 0$ and $n = 0$ in the above Theorem we obtain that the integral operator defined by (3) preserve the well-known class of starlike functions.

Teorema 3.3 Let w be a fixed point in U and $f \in S_n^*(w)$ with $f(z) = (z - w) + \sum_{j=2}^{\infty} a_j \cdot (z - w)^j$. Then we have:

$$\begin{aligned} |a_2| &\leq \frac{1}{2^{n-1} \cdot (1 - d^2)}; \\ |a_3| &\leq \frac{3 + d}{3^n \cdot (1 - d^2)^2}; \\ |a_4| &\leq \frac{(2 + d)(3 + d)}{2^{2n-1} \cdot 3 \cdot (1 - d^2)^3}; \\ |a_5| &\leq \frac{(2 + d)(3 + d)(3d + 5)}{5^n \cdot 6 \cdot (1 - d^2)^4}, \end{aligned}$$

where $d = |w|$.

Proof. From Remark 3.2 for $f \in S_n^*(w)$ we obtain

$$(4) \quad D_w^n f(z) = g(z) \in S^*(w).$$

If we consider $g(z) = (z - w) + \sum_{j=2}^{\infty} b_j \cdot (z - w)^j$, using Remark 3.1, from (4) we obtain $j^n \cdot a_j = b_j, j = 2, 3, \dots$. Thus we have $a_j = \frac{1}{j^n} \cdot b_j, j = 2, 3, \dots$, and from the estimates (2) we get the result.

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