

Related Topics to Transferable Utility Games — An Infinite Market Game —

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1 Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space throughout this paper. A *game* v is a nonnegative real valued function, defined on the σ -field $\mathcal{F}$, which maps the empty set to zero. An *outcome* of a game v is a finitely additive real valued function α on $\mathcal{F}$ such that $\alpha(\Omega) = v(\Omega)$. For an outcome α of v , an integrable function f satisfying $\int_S f d\mu = \alpha(S)$ for all $S \in \mathcal{F}$ is said to be an *outcome density* of α with respect to μ . An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The *core* of v is the set of outcomes α satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathcal{F}$.

To every game v we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^n \lambda_i v(S_i) : \sum_{i=1}^n \lambda_i \chi_{S_i} \leq \chi_{\Omega} \right\},$$

where $n = 1, 2, \dots$, $S_i \in \mathcal{F}$, λ_i is a real number. The notation χ_A denotes the characteristic function of a subset A of Ω . For a game v with $|v| < \infty$, we define two games $\bar{v}$ and $\hat{v}$ by

$$\begin{aligned} \bar{v}(S) &= \sup \left\{ \sum_{i=1}^n \lambda_i v(S_i) : \sum_{i=1}^n \lambda_i \chi_{S_i} \leq \chi_S \right\}, \quad S \in \mathcal{F}, \\ \hat{v}(S) &= \min \{ \alpha(S) : \alpha \text{ is additive, } \alpha \geq v, \alpha(\Omega) = |v| \}, \quad S \in \mathcal{F}, \end{aligned}$$

following [3]. A game v is said to be *balanced* if $v(\Omega) = |v|$, *totally balanced* if $v = \bar{v}$ and *exact* if $v = \hat{v}$, respectively. It is proved in [3] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game v is said to be *monotone* if $S \subset T$ implies $v(S) \leq v(T)$ for any S and T in $\mathcal{F}$. A game v is said to be *continuous* at Ω if it follows that $\lim_{n \rightarrow \infty} v(S_n) = v(\Omega)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = \Omega$.

2 Market Games

We denote utilities of players by a Carathéodory type function u defined on $\Omega \times R_+^l$ to R_+ , where R_+^l denotes the nonnegative orthant of the l -dimensional Euclidean space R^l , and R_+ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player ω getting goods x . We always use the ordinary coordinatewise order when having concern with an order in R_+^l . We suppose that the function $u : \Omega \times R_+^l \rightarrow R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R_+^l$;
2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all ω in Ω ;
3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_+\} < \infty$, where $B_+ = \{x \in R_+^l : \|x\| \leq 1\}$, and $\|x\|$ denotes the Euclidean norm of $x \in R_+^l$.

For any set S in $\mathcal{F}$, the set of integrable functions on S to R_+^l is denoted by $L_1(S, R_+^l)$. We take an element e of $L_1(\Omega, R_+^l)$ as the density of initial endowments for the players. For any S in $\mathcal{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega)) d\mu(\omega) : x \in L_1(S, R_+^l), \int_S x d\mu = \int_S e d\mu \right\}.$$

The set function v defined above is called a *market game* derived from the market $(\Omega, \mathcal{F}, \mu, u, e)$.

It is well known that the function $\omega \mapsto u(\omega, x(\omega))$ is measurable for any $x \in L_1(S, R_+^l)$. Moreover we need to show that the function $\omega \mapsto u(\omega, x(\omega))$ is integrable in order to define $v(S)$ as a real number.

Lemma 1 If $x \in L_1(S, R_+^l)$, then $u(\cdot, x(\cdot)) \in L_1(S, R_+)$ for any $S \in \mathcal{F}$ and the map $x \mapsto u(\cdot, x(\cdot))$ is continuous with respect to the norm topologies of $L_1(S, R_+^l)$ and $L_1(S, R_+)$.

Proof Let $x \in L_1(S, R_+^l)$. Since $u(\omega, \cdot)$ is concave, for any $x \in R_+^l$ with $\|x\| > 1$, we have the inequality

$$\frac{u(\omega, x) - u(\omega, x/\|x\|)}{\|x - x/\|x\|\|} \leq \frac{u(\omega, x/\|x\|) - u(\omega, 0)}{\|x/\|x\| - 0\|},$$

and hence we have $u(\omega, x) \leq \|x\|\sigma$ for any $\omega \in \Omega$ and $x \in R_+^l$ with $\|x\| > 1$. It is obvious from the definition of σ that $u(\omega, x) \leq \sigma$ for any $\omega \in \Omega$ and $x \in R_+^l$ with $\|x\| \leq 1$. Thus we have $u(\omega, x) \leq \sigma(1 + \|x\|)$ for any $(\omega, x) \in \Omega \times R_+^l$ and this leads to the inequalities

$$\begin{aligned} \int_S u(\omega, x(\omega)) d\mu(\omega) &\leq \int_S \sigma(1 + \|x(\omega)\|) d\mu(\omega) \\ &= \sigma \left(\mu(S) + \int_S \|x(\omega)\| d\mu(\omega) \right) \\ &< \infty. \end{aligned}$$

Thus it follows that $u(\cdot, x(\cdot)) \in L_1(S, R_+)$. The second part of the assertion is verified in Theorem 2.1 of [2]. Although Theorem 2.1 of [2] is proved under the hypotheses that S is a measurable set in R^l and the second argument x of the function u runs over R , the proof of Theorem 2.1 of [2] is valid even in our setting. Thus the map $x \mapsto u(\cdot, x(\cdot))$ is norm continuous. Q.E.D.

Lemma 2 A market game v is actually a game and is monotone.

Proof It is obvious $v(\emptyset) = 0$. The finiteness of $v(S)$ follows since the inequalities

$$\begin{aligned} \int_S u(\omega, x(\omega)) d\mu(\omega) &\leq \sigma \int_S (1 + \|x\|) d\mu \\ &\leq \sigma \left(\mu(S) + \sum_{i=1}^l \int_S x^i d\mu \right) = \sigma \left(\mu(S) + \sum_{i=1}^l \int_S e^i d\mu \right) \end{aligned}$$

hold if

$$\int_S x d\mu = \int_S e d\mu,$$

where x^i and e^i are the i -th coordinate functions of x and e , respectively. Moreover v is monotone because u has nonnegative values. Q.E.D.

3 Cores of Market Games

We start with a lemma on concave functions.

Lemma 3 If $f : R_+^l \rightarrow R$ is concave and $f(0) = 0$, then for any $x_1, \dots, x_n \in R_+^l$ and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i \leq 1$, it follows that

$$\sum_{i=1}^n \lambda_i f(x_i) \leq f\left(\sum_{i=1}^n \lambda_i x_i\right).$$

Proof We can assume that $\lambda = \sum_{i=1}^n \lambda_i > 0$ without loss of generality. It follows that

$$\begin{aligned} \sum_{i=1}^n \lambda_i f(x_i) &= \lambda \sum_{i=1}^n \frac{\lambda_i}{\lambda} f(x_i) \\ &\leq \lambda f\left(\sum_{i=1}^n \frac{\lambda_i}{\lambda} x_i\right) \\ &= (1 - \lambda) f(0) + \lambda f\left(\frac{1}{\lambda} \sum_{i=1}^n \lambda_i x_i\right) \\ &\leq f\left(\sum_{i=1}^n \lambda_i x_i\right). \end{aligned}$$

Q.E.D.

Let S be a measurable set. For any $x \in L_1(S, R_+^l)$, define $\bar{x} \in L_1(\Omega, R_+^l)$ by

$$\bar{x}(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in S; \\ 0, & \text{if } \omega \in S^c. \end{cases}$$

Proposition 1 A market game v is totally balanced.

Proof Take any $S \in \mathcal{F}$ and $S_i \in \mathcal{F}$ and $\lambda_i > 0$, $i = 1, \dots, n$ with $\sum_{i=1}^n \lambda_i \chi_{S_i} \leq \chi_S$. We can assume that $\mu(S) > 0$ without loss of generality.

Let ϵ be an arbitrary positive number. Take $x_i \in L_1(S_i, R_+^l)$ such that

$$\int_{S_i} x_i d\mu = \int_{S_i} e d\mu \quad \text{and} \quad v(S_i) - \frac{\epsilon}{n} < \int_{S_i} u(\omega, x_i(\omega)) d\mu(\omega),$$

and define $y \in L_1(S, R_+^l)$ by

$$y = \sum_{i=1}^n \lambda_i \bar{x}_i.$$

Then we have the following:

$$\begin{aligned} \int_S y \, d\mu &= \sum_{i=1}^n \lambda_i \int_S \bar{x}_i \, d\mu \\ &= \sum_{i=1}^n \lambda_i \int_{S_i} e \, d\mu \\ &= \int_S e \sum_{i=1}^n \lambda_i \chi_{S_i} \, d\mu \\ &\leq \int_S e \, d\mu. \end{aligned}$$

Define $y' \in L_1(S, R_+^l)$ by

$$y' = y + \frac{1}{\mu(S)} \left(\int_S e \, d\mu - \int_S y \, d\mu \right).$$

Then it is easily seen that $\int_S y' \, d\mu = \int_S e \, d\mu$.

On the other hand, let $\mathcal{A}$ be the family of all nonempty subsets A of $\{1, \dots, n\}$ such that $T_A \equiv \bigcap_{i \in A} S_i \cap \bigcap_{j \in A^c} (S \setminus S_j) \neq \emptyset$. Then it is easily seen that $S_i = \bigcup_{A \ni i} T_A$ for $i = 1, \dots, n$ and $\{T_A : A \in \mathcal{A}\}$ is a partition of $\bigcup_{i=1}^n S_i$, and $\sum_{i \in A} \lambda_i \leq 1$ for all $A \in \mathcal{A}$. For any i and A with $i \in A \in \mathcal{A}$, define $x_i^A = x_i|_{T_A}$, the restriction of x_i to T_A . Then we have

$$\bar{x}_i = \sum_{A \ni i} \bar{x}_i^A \quad \text{and} \quad y = \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \bar{x}_i^A.$$

Thus we have

$$\begin{aligned}
\sum_{i=1}^n \lambda_i v(S_i) - \epsilon &< \sum_{i=1}^n \lambda_i \int_{S_i} u(\omega, x_i(\omega)) d\mu(\omega) \\
&= \sum_{i=1}^n \sum_{A \ni i} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega)) d\mu(\omega) \\
&= \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \int_{T_A} u(\omega, x_i^A(\omega)) d\mu(\omega) \\
&= \sum_{A \in \mathcal{A}} \int_{T_A} \sum_{i \in A} \lambda_i u(\omega, x_i^A(\omega)) d\mu(\omega) \\
&\leq \sum_{A \in \mathcal{A}} \int_{T_A} u(\omega, \sum_{i \in A} \lambda_i x_i^A(\omega)) d\mu(\omega) \quad \text{by Lemma 3} \\
&= \int_S u(\omega, \sum_{A \in \mathcal{A}} \sum_{i \in A} \lambda_i \bar{x}_i^A(\omega)) d\mu(\omega) \quad \text{by } u(\omega, 0) = 0 \\
&= \int_S u(\omega, y(\omega)) d\mu(\omega) \\
&\leq \int_S u(\omega, y'(\omega)) d\mu(\omega) \quad \text{by monotonicity of } u(\omega, \cdot) \\
&\leq v(S).
\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^n \lambda_i v(S_i) \leq v(S).$$

Thus $\bar{v}(S) \leq v(S)$ and the reverse inequality is obvious. Hence we have $\bar{v} = v$. Q.E.D.

Proposition 2 A market game v is continuous at Ω .

Proof Let $\{S_n\}$ be a nondecreasing sequence of measurable sets with $\Omega = \bigcup_{n=1}^{\infty} S_n$ and ϵ an arbitrary positive number. Then, there is $x \in L_1(S, R_+^l)$ such that

$$v(\Omega) - \epsilon < \int_{\Omega} u(\omega, x(\omega)) d\mu(\omega) \quad \text{and} \quad \int_{\Omega} x d\mu = \int_{\Omega} e d\mu.$$

Let x_n be the restriction $x|_{S_n}$ and define a sequence $\{y_n\}$ of functions in $L_1(S_n, R_+^l)$ by

$$y_n^i = \begin{cases} \frac{\int_{S_n} e^i d\mu}{\int_{S_n} x_n^i d\mu} x_n^i, & \text{if } \int_{S_n} x_n^i d\mu > \int_{S_n} e^i d\mu; \\ x_n^i + \frac{1}{\mu(S_n)} \left(\int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right), & \text{if } \int_{S_n} x_n^i d\mu \leq \int_{S_n} e^i d\mu, \end{cases}$$

for $i = 1, \dots, l$. It is obvious that

$$\int_{S_n} y_n d\mu = \int_{S_n} e d\mu.$$

On the other hand, since

$$\lim_{n \rightarrow \infty} \int_{S_n} |y_n^i - x_n^i| d\mu = \lim_{n \rightarrow \infty} \left| \int_{S_n} e^i d\mu - \int_{S_n} x_n^i d\mu \right| = 0,$$

for $i = 1, \dots, l$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|\bar{y}_n - x\| d\mu = \lim_{n \rightarrow \infty} \int_{S_n} \|y_n - x\| d\mu + \lim_{n \rightarrow \infty} \int_{S_n^c} \|x\| d\mu = 0,$$

and hence $\bar{y}_n$ converges to x with respect to the norm topology of $L_1(\Omega, R_+^l)$. Therefore, by Lemma 1, it follows that

$$\lim_{n \rightarrow \infty} \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) = \lim_{n \rightarrow \infty} \int_{\Omega} u(\omega, \bar{y}_n(\omega)) d\mu(\omega) = \int_{\Omega} u(\omega, x(\omega)) d\mu(\omega)$$

and hence, for sufficiently large n ,

$$v(\Omega) - \epsilon < \int_{S_n} u(\omega, y_n(\omega)) d\mu(\omega) \leq v(S_n).$$

Thus we have $\lim_{n \rightarrow \infty} v(S_n) = v(\Omega)$. Q.E.D.

Now we have reached our main theorem combining Proposition 1 and Proposition 2.

Theorem 1 A market game v has a nonempty core, and every element α of the core is countably additive and has a unique outcome density f in $L_1(\Omega, R_+)$ with respect to μ , and hence it follows that

$$\alpha(S) = \int_S f d\mu, \quad S \in \mathcal{F}.$$

Proof The core is nonempty by Proposition 1. Since v is continuous at Ω by Proposition 2, any element α of the core is continuous at Ω , which implies that α is countably additive. To prove existence of an outcome density for α , it is sufficient to show that α is absolutely continuous with respect to μ by virtue of the Radon-Nikodym theorem. If $\mu(S) = 0$, then $v(S^c) = v(\Omega)$ by the definition of the market game v , and hence we have $\alpha(S^c) \geq v(S^c) = v(\Omega) = \alpha(\Omega)$, that is, $\alpha(S) = 0$. Q.E.D.

Remark 1 Similar to the assertion of Theorem 1, an exact game which is continuous at Ω has a nonempty core and every element of the core is countably additive. Moreover, there is a measure λ on $\mathcal{F}$ such that every element of the core is absolutely continuous with respect to λ according to [3]. The following example shows that there is a market game which is not exact and Theorem 1 is independent of the results of [3].

Example 1 [[1], pp. 192] Let $l = 1$, $\Omega = [0, 1]$ and μ be the Lebesgue measure. Define $u : [0, 1] \times R_+ \rightarrow R_+$ by

$$u(\omega, x) = \sqrt{x + \omega} - \sqrt{\omega} \quad \text{and} \quad e(\omega) = \frac{1}{32} \quad \text{for all } \omega \in [0, 1].$$

According to [1], the supremum is attained for every measurable sets in the definition of the market game, and the core has only one element α and the outcome density f of α is given by

$$f(\omega) = \begin{cases} (\frac{1}{2} - \sqrt{\omega})^2 + \frac{1}{32}, & \text{if } \omega \in [0, \frac{1}{4}); \\ \frac{1}{32}, & \text{if } \omega \in [\frac{1}{4}, 1]. \end{cases}$$

Thus it follows $\alpha([\frac{1}{2}, 1]) = \frac{1}{64}$, and hence $\hat{v}([\frac{1}{2}, 1]) = \frac{1}{64}$. On the other hand, we have

$$\sqrt{x + \omega} - \sqrt{\omega} \leq \sqrt{x + \frac{1}{2}} - \sqrt{\frac{1}{2}} \leq \sqrt{\frac{1}{2}}x$$

for $1/2 \leq \omega \leq 1$ and $x \geq 0$. Thus, if $x \in L_1([0, 1], R_+)$ satisfies

$$\int_{\frac{1}{2}}^1 x d\mu = \int_{\frac{1}{2}}^1 e d\mu = \frac{1}{64},$$

then

$$\int_{\frac{1}{2}}^1 u(\omega, x(\omega)) d\mu(\omega) \leq \int_{\frac{1}{2}}^1 \sqrt{\frac{1}{2}}x d\mu = \frac{1}{64\sqrt{2}} < \frac{1}{64}.$$

Therefore we have $v([\frac{1}{2}, 1]) < \hat{v}([\frac{1}{2}, 1])$ and v is not exact.

References

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