

MODULI SPACE OF PARABOLIC BUNDLES AND PARABOLIC CONNECTIONS

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Introduction

Rank two case

- $(C, \mathbf{t})$: an irreducible smooth projective curve over $\mathbb{C}$ with n distinct points $\mathbf{t} = (t_1, \dots, t_n)$
- (L, ∇_L) : a $\mathrm{tr}(\boldsymbol{\nu})$ -connection over $(C, \mathbf{t})$ of rank one and degree $2g - 1$
- $M_{(C, \mathbf{t})}^\alpha(\boldsymbol{\nu}, (L, \nabla_L)) := \{(E, \nabla, l_*) \mid \alpha\text{-stable}, (\det E, \mathrm{tr} \nabla) \cong (L, \nabla_L)\} / \sim$
- $P_{(C, \mathbf{t})}^\alpha(L) := \{(E, l_*) \mid \alpha\text{-stable}, \det E \cong L\} / \sim$
- $N := \frac{1}{2} \dim M_{(C, \mathbf{t})}^\alpha(\boldsymbol{\nu}, (L, \nabla_L)) = \dim P_{(C, \mathbf{t})}^\alpha(L)$

Theorem (Loray-Saito, Fassarella-Loray, Fassarella-Loray-Muniz, M)

For a suitable α , the rational map

$$\mathrm{App} \times \mathrm{Bun}: M_{(C, \mathbf{t})}^\alpha(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \rightarrow \mathbb{P}^N \times P_{(C, \mathbf{t})}^\alpha(L)$$

is birational.

Problem

Is $\mathrm{App} \times \mathrm{Bun}$ also birational for higher rank case ?

In this poster, we investigate the moduli space of rank 3 parabolic bundles and parabolic logarithmic connections over $\mathbb{P}^1$ with 3 points, which provides a counterexample of the problem.

Moduli space of parabolic connections

$$\circ \boldsymbol{\nu} = (\nu_{i,j})_{0 \leq j \leq 2}^{1 \leq i \leq 3} \in \mathbb{C}^9: \sum_{i=1}^3 \sum_{j=0}^2 \nu_{i,j} = -d \in \mathbb{Z}$$

$\boldsymbol{\nu}$ -parabolic connection

- $(E, \nabla, l_* = \{l_{i,*}\}_{1 \leq i \leq 3})$
- E : a vector bundle of rank 3 and degree d ,
- ∇ : $E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(t_1 + t_2 + t_3)$: a logarithmic connection
- ▷ ∇ is locally written by

$$\nabla = d + \sum_{i=1}^3 \frac{A_i}{z - t_i} dz + \text{holomorphic}, \quad A_i \in M_3(\mathbb{C}).$$

- $l_{i,*}$: a filtration $E|_{t_i} = E \otimes k(t_i) = l_{i,0} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq l_{i,3} = 0$ such that $(\mathrm{res}_{t_i}(\nabla) - \nu_{i,j} \mathrm{id})(l_{i,j}) \subset l_{i,j+1}$ for $1 \leq i \leq 3, 0 \leq j \leq 2$.

- $\alpha = \{\alpha_{i,j}\}_{1 \leq i \leq 3, 0 \leq j \leq 2} \in (0, 1)^9$: a set of rational numbers satisfying $0 < \alpha_{i,1} < \alpha_{i,2} < \alpha_{i,3} < 1$ for each $i = 1, 2, 3$, and $\alpha_{i,j} \neq \alpha_{i',j'}$ for $(i, j) \neq (i', j')$

α -stability

(E, ∇, l_*) is α -stable

$\stackrel{\text{def}}{\iff}$ for any subbundle $0 \neq F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega_{\mathbb{P}^1}^1(t_1 + t_2 + t_3)$, the inequality

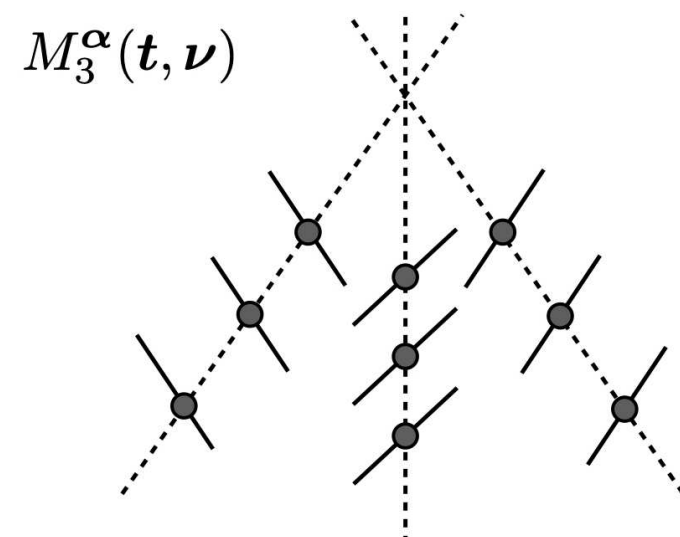
$$\frac{\deg F + \sum_{i=1}^3 \sum_{j=1}^2 \alpha_{i,j} d_{i,j}(F)}{\mathrm{rank} F} < \frac{\deg E + \sum_{i=1}^3 \sum_{j=1}^2 \alpha_{i,j}}{\mathrm{rank} E}$$

holds, where $d_{i,j}(F) := \dim(F|_{t_i} \cap l_{i,j-1}) / (F|_{t_i} \cap l_{i,j})$.

$$\circ M_3^\alpha(\mathbf{t}, \boldsymbol{\nu}) := \{(E, \nabla, l_*) \mid \alpha\text{-stable}, \deg E = -2\} / \sim$$

Description of $M_3^\alpha(\mathbf{t}, \boldsymbol{\nu})$

Take $\alpha = (\alpha_{i,j})_{1 \leq i,j \leq 3}$ such that $0 < \alpha_{i,j} \ll 1$ for any $1 \leq i, j \leq 3$. Then $M_3^\alpha(\mathbf{t}, \boldsymbol{\nu})$ is the surface obtained by removing the anti-canonical divisor from a blow-up of $\mathbb{P}^2$ at three points on each three lines meeting in a single point.



Moduli space of parabolic bundles

Parabolic bundle

- $(E, l_* = \{l_{i,*}\}_{1 \leq i \leq 3})$
- E : a vector bundle of rank 3 and degree d , and
- $l_{i,*}$: a filtration $E|_{t_i} = E \otimes k(t_i) = l_{i,0} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq l_{i,3} = 0$.

Assumption

For any $1 \leq i \leq 3, j = 1, 2$,

$$\alpha_{i,j+1} - \alpha_{i,j} = \text{constant} =: w$$

- $P^w(-2) := \{(E, l_*) \mid w\text{-stable}, \deg E = -2\} / \sim$
- (E, l_*) is w -stable $\implies E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

Description of $P^w(-2)$

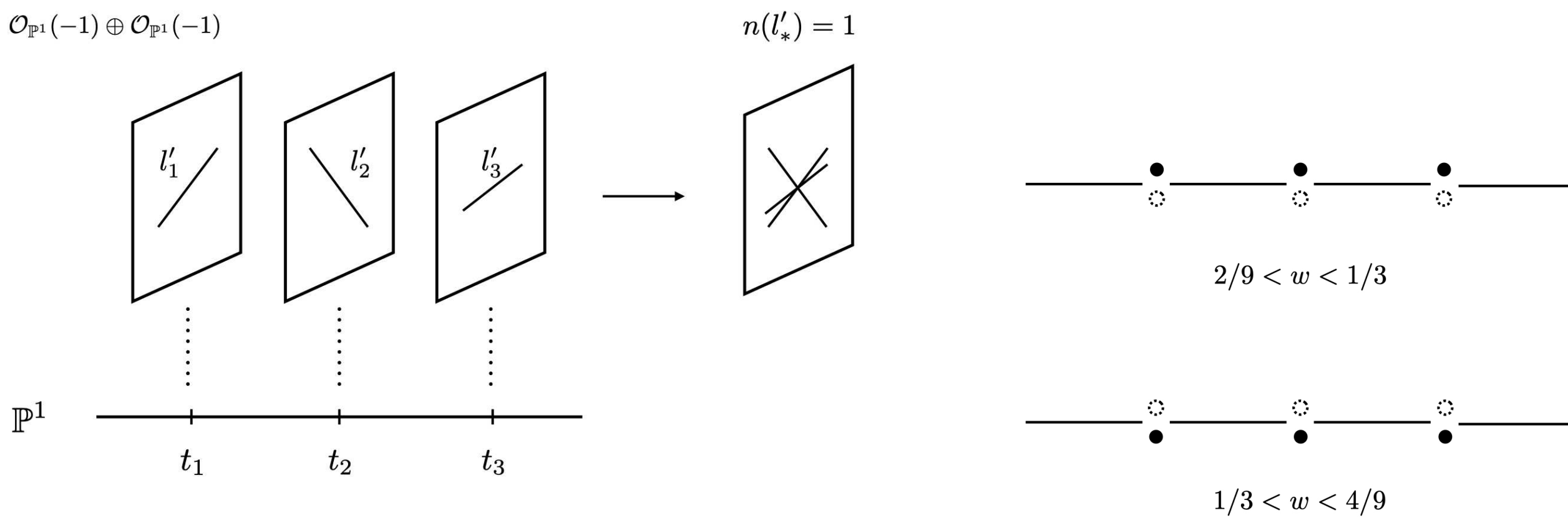
- (1) If $0 < w < 2/9, 4/9 < w < 1/2$, then $P^w(-2) = \emptyset$.
- (2) If $2/9 < w < 1/3$, then a w -stable parabolic bundle (E, l_*) fits into a nonsplit exact sequence

$$0 \longrightarrow (\mathcal{O}_{\mathbb{P}^1}, \emptyset) \longrightarrow (E, l_*) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*) \longrightarrow 0,$$

- where $n(l'_*) := \max_{\mathcal{O}_{\mathbb{P}^1}(-1) \cong F \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}} \#\{i \mid F|_{t_i} = l'_i\} = 1$. In particular, $P^w(-2) \cong \mathbb{P}^1$.
- (3) If $1/3 < w < 4/9$, then a w -stable parabolic bundle (E, l_*) is either type of the following:
 - (i) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 0$, $n(l'_*) = 1$, and the condition $(*)$ holds.
 - (ii) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 1$, $n(l'_*) = 1$, and the condition $(*)$ holds.

In particular, $P^w(-2) \cong \mathbb{P}^1$. Here the condition $(*)$ is the following.

- $(*)$ There is no subbundle $F \subset E$ such that $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l_{i,2} \subset F|_{t_i}$ and $F|_{t_j} = l_{j,1}$ for some i and any $j \neq i$.



A coordinate of $P^w(-2)$

Assume that $2/9 < w < 1/3$. Then we can take a homogeneous coordinate $(a : b)$ of $P^w(-2) \cong \mathbb{P}^1$ by

$$l_{1,2} = \mathbb{C} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad l_{1,1} = \mathbb{C} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad l_{2,2} = \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad l_{2,1} = \mathbb{C} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$l_{3,2} = \mathbb{C} \begin{pmatrix} a+b \\ 1 \\ 1 \end{pmatrix}, \quad l_{3,1} = \mathbb{C} \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} b \\ 0 \\ 1 \end{pmatrix}.$$

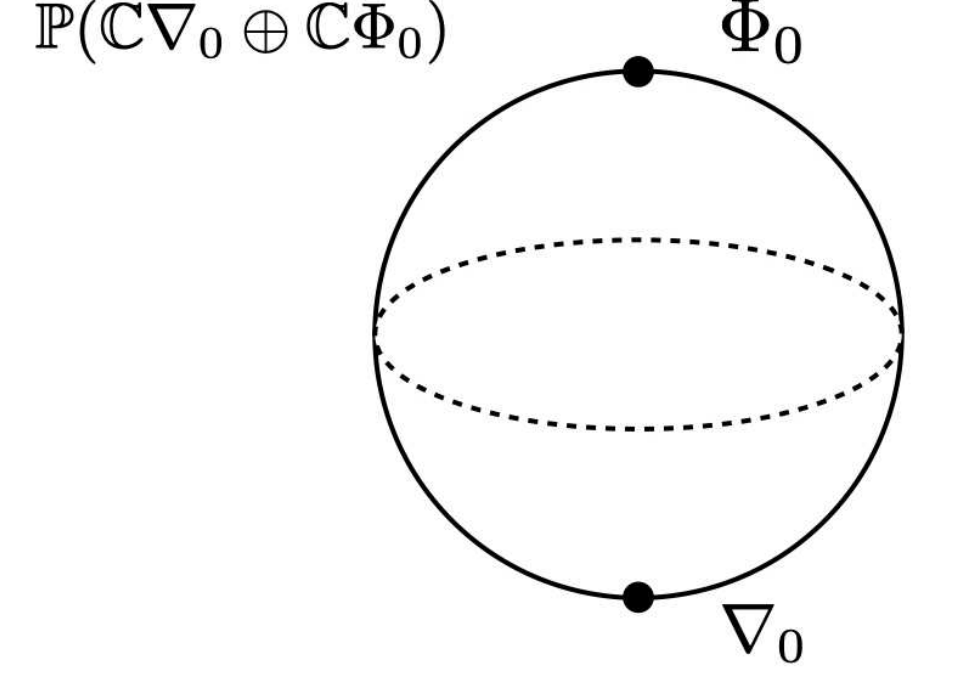
Compactification by parabolic Higgs fields

Parabolic Higgs field over (E, l_*)

- $\Phi: E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(t_1 + t_2 + t_3)$: a homomorphism satisfying $(\mathrm{res}_{t_i}(\Phi))(l_{i,j}) \subset l_{i,j+1}$ for $1 \leq i \leq 3, 0 \leq j \leq 2$.

- ∇_0 : a $\boldsymbol{\nu}$ -parabolic connection over (E, l_*)
- $\Phi_0 \neq 0$: a parabolic Higgs field over (E, l_*)

$$\nabla_0 + \mathbb{C}\Phi_0 \subset_{\text{open}} \mathbb{P}(\mathbb{C}\nabla_0 \oplus \mathbb{C}\Phi_0)$$



- $M_3^w(\mathbf{t}, \boldsymbol{\nu})^0 := \{(E, \nabla, l_*) \mid (E, l_*) \in P^w(-2)\} / \sim$
- $\overline{M_3^w(\mathbf{t}, \boldsymbol{\nu})}^0$: the compactification of $M_3^w(\mathbf{t}, \boldsymbol{\nu})^0$ by parabolic Higgs fields

Description of $\overline{M_3^w(\mathbf{t}, \boldsymbol{\nu})}^0$

Assume that $2/9 < w < 1/3$. Then we have

$$\overline{M_3^w(\mathbf{t}, \boldsymbol{\nu})}^0 \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0 \\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} = 0. \end{cases}$$

Proof

$$(\nabla_\infty(b), \Phi_\infty(b)) \cong (\nabla_0(a), \Phi_0(a)) \begin{pmatrix} 1 & 0 \\ -(\nu_{1,0} + \nu_{2,0} + \nu_{3,0})a^{-1} & a^{-2} \end{pmatrix}$$

App $\times$ Bun

Assume that $2/9 < w < 1/3$ and $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0$.

Apparent map

There exists a unique filtration $E =: F_0 \supset F_1 \supset F_2 \supset 0$ such that $F_2 \cong \mathcal{O}_{\mathbb{P}^1}$, $F_1 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $\nabla(F_2) \subset F_1 \otimes \Omega_{\mathbb{P}^1}^1(t_1 + t_2 + t_3)$. We define the apparent singularity $\mathrm{App}(E, \nabla, l_*)$ by the zero of the nonzero homomorphism

$$\mathcal{O}_{\mathbb{P}^1}(-1) \cong F_1/F_2 \xrightarrow{\nabla} (E/F_1) \otimes \Omega_{\mathbb{P}^1}^1(D(\mathbf{t})) \cong \mathcal{O}_{\mathbb{P}^1}.$$

- $V_0 := P^w(-2) \setminus \{(1 : 0), (1 : 1), (0 : 1)\}$
- $\mathrm{Bun}: \overline{M_3^w(\mathbf{t}, \boldsymbol{\nu})}^0 \rightarrow P^w(-2), (E, \nabla, l_*) \mapsto (E, l_*)$

App $\times$ Bun

Then the morphism

$$\mathrm{App} \times \mathrm{Bun}: \mathrm{Bun}^{-1}(V_0) \longrightarrow \mathbb{P}^1 \times V_0$$

is finite and its generic fiber consists of three points.

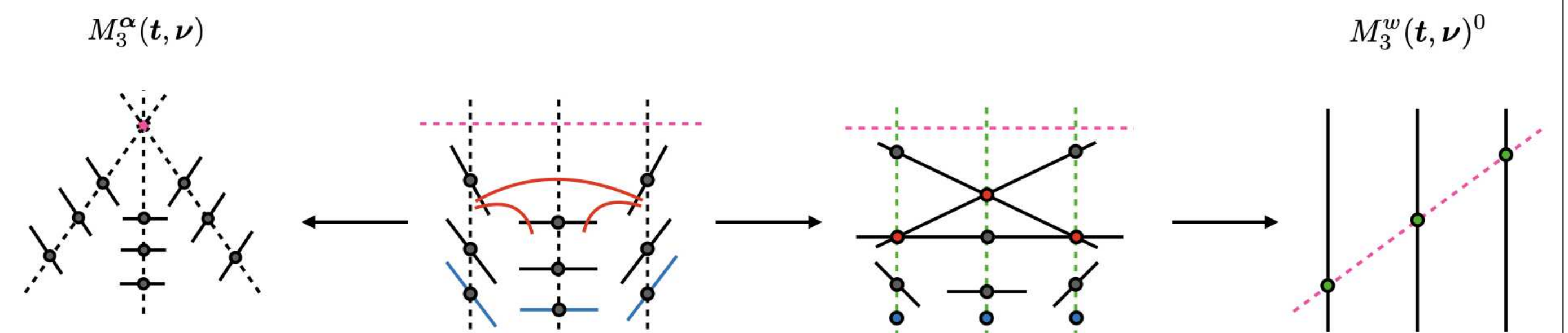
Proof

$$\mathrm{Bun}^{-1}((E, l_{a,*})) = \mathbb{P}(\mathbb{C}\nabla_0(a) \oplus \mathbb{C}\Phi_0(a))$$

$$\mathrm{App}(\lambda \nabla_0(a) + \mu \Phi_0(a)) = (f_1(a; \lambda, \mu) + f_2(a; \lambda, \mu) : t_1 f_1(a; \lambda, \mu) + t_2 f_2(a; \lambda, \mu)),$$

where $f_1(a; \lambda, \mu), f_2(a; \lambda, \mu)$ are homogeneous polynomials of degree 3 in variable λ, μ .

Relation between $M_3^\alpha(\mathbf{t}, \boldsymbol{\nu})$ and $M_3^w(\mathbf{t}, \boldsymbol{\nu})^0$



References

- [1] T. Matsumoto, Moduli space of rank three logarithmic connections on the projective line with three poles, in preparation.