

On Several Properties of Spiral-like Functions

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1 Introduction

We consider an analytic function f on the unit disk $\mathbb{D}$ normalized so that $f(0) = f'(0) - 1 = 0$. For a constant $\beta \in (-\pi/2, \pi/2)$, such a function f is called β -spiral-like if f is univalent on $\mathbb{D}$ and for any $z \in \mathbb{D}$, the β -logarithmic spiral $\{f(z) \exp(-e^{i\beta}t); t \geq 0\}$ is contained in $f(\mathbb{D})$. It is equivalent to the analytic condition that $\Re(e^{-i\beta}zf'(z)/f(z)) > 0$ in $\mathbb{D}$. We denote by $SP(\beta)$ the set of β -spiral-like functions. We call $f_\beta(z) := z(1-z)^{-2e^{i\beta}\cos\beta} \in SP(\beta)$ the β -spiral Koebe function. Note that $SP(0)$ is the set of starlike functions and that $f_0(z) = z(1-z)^{-2}$ is the Koebe function. The β -spiral Koebe function conformally maps the unit disk onto the complement of the β -logarithmic spiral $\{f_\beta(-e^{-2i\beta}) \exp(-e^{i\beta}t); t \leq 0\}$ in $\mathbb{C}$. For the known results about these classes of the functions, see, for example, [1].

2 Norm estimates

For a locally univalent holomorphic function f , we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,$$

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which are said to be the *pre-Schwarzian derivative* (or nonlinearity) and the *Schwarzian derivative* of f , respectively. For a locally univalent function f in $\mathbb{D}$, we define the norms of T_f and S_f by

$$\|T_f\|_1 = \sup_{z \in \mathbb{D}} (1 - |z|^2) |T_f(z)| \quad \text{and} \quad \|S_f\|_2 = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f(z)|,$$

respectively.

As well as $\|S_f\|_2$, the norm $\|T_f\|_1$ has a significant meaning in the theory of Teichmüller spaces. For example, see [8], [2] and [13].

We shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class $SP(\beta)$.

Main Theorem 1 ([9]). *For any $f \in SP(\beta)$, where $\beta \in (-\pi/2, \pi/2)$, we have the following.*

I) *In the case $|\beta| \leq \pi/3$, we have*

$$\|T_f\|_1 \leq \|T_{f_\beta}\|_1 = 2|2 + e^{2i\beta}|. \quad (1)$$

II) *In the case $|\beta| > \pi/3$, we have $\|T_f\|_1 \leq \|T_{f_\beta}\|_1$, where*

$$\|T_{f_\beta}\|_1 = \max_{0 \leq m \leq \frac{4}{3} \sin |\beta|} 2m \cos \beta \left(1 + \sqrt{\frac{m^2 + 4 - 4m \sin |\beta|}{m^2 + 1 - 2m \sin |\beta|}} \right) \quad \text{and} \quad (2)$$

$$2|2 + e^{2i\beta}| < \|T_{f_\beta}\|_1 < 2 \left(1 + \frac{4}{3} \sin 2|\beta| \right). \quad (3)$$

In particular, $\|T_{f_\beta}\|_1 \rightarrow 2$ as $|\beta| \rightarrow \pi/2$.

In both cases, the equality $\|T_f\|_1 = \|T_{f_\beta}\|_1$ holds if and only if f is a rotation of the β -spiral Koebe function, i.e., $f(z) = (1/\varepsilon)f_\beta(\varepsilon z)$ for some $|\varepsilon| = 1$.

The proof of Main Theorem 1 is in [9]. From the proof, if $|\beta| \leq \pi/3$, the function $(1 - |z|^2)|T_{f_\beta}(z)|$ does not attain its supremum in $\mathbb{D}$. However if $|\beta| > \pi/3$, it does since

$$\max_{\partial \mathbb{D} \ni z_0} \limsup_{\mathbb{D} \ni z \rightarrow z_0} (1 - |z|^2) |T_{f_\beta}(z)| = 2|2 + e^{2i\beta}| < \|T_{f_\beta}\|_1.$$

This phenomenon of *phase transition* seems to be quite interesting.

Remark. Clearly, the β -spiral Koebe function f_β converges to $id_{\mathbb{D}}$ (which is bounded) locally uniformly on $\mathbb{D}$ as $|\beta| \rightarrow \pi/2$ but does not converge to it with respect to the norm $\|\cdot\|_1$ since $\lim_{|\beta| \rightarrow \pi/2} \|T_{f_\beta}\|_1 = 2$. On the other hand, it is known that a normalized analytic function f is bounded if $\|T_f\|_1 < 2$. In fact, the value 2 is the least one of the norms of unbounded normalized analytic functions.

We would also like to mention the related works about norm estimates of pre-Schwarzian derivatives in other classes by Shinji Yamashita [11] and Toshiyuki Sugawa [10].

Theorem 2.1. *Let $0 \leq \alpha < 1$ and f be a normalized analytic function.*

If f is starlike of order α , i.e., $\Re(zf'(z)/f(z)) > \alpha$, then $\|T_f\|_1 \leq 6 - 4\alpha$.

If f is convex of order α , i.e., $\Re(1 + zf''(z)/f'(z)) > \alpha$, then $\|T_f\|_1 \leq 4(1 - \alpha)$.

If f is strongly starlike of order α , i.e., $\arg(zf'(z)/f(z)) < \pi\alpha/2$, then $\|T_f\|_1 \leq M(\alpha) + 2\alpha$, where $M(\alpha)$ is a specified constant depending only on α satisfying $2\alpha < M(\alpha) < 2\alpha(1 + \alpha)$.

All of the bounds are sharp.

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of β -spiral-like functions.

Main Theorem 2 ([9]). *Assume $|\beta| < \pi/2$. For any $f \in SP(\beta)$, $\|S_f\|_2 \leq \|S_{f_\beta}\|_2 = 6$.*

Proof. From direct calculation, it follows that

$$\begin{aligned} S_{f_\beta} &= (T_{f_\beta})' - \frac{1}{2}(T_{f_\beta})^2 \\ &= -c \frac{e^{2i\beta} \{e^{2i\beta} (e^{2i\beta} - 1)z^2 + 4(e^{2i\beta} - 1)z + 6\}}{2(1 - z)^2(1 + ze^{2i\beta})^2} \end{aligned}$$

and that

$$(1 - |z|^2)^2 |S_{f_\beta}(z)| = |c| \frac{(1 - |z|^2)^2 |e^{2i\beta} (e^{2i\beta} - 1)z^2 + 4(e^{2i\beta} - 1)z + 6|}{2|1 - z|^2 |1 + ze^{2i\beta}|^2}.$$

We can easily see that $(1 - |z|^2)^2 |S_{f_\beta}(z)| \rightarrow 6$ as $z \rightarrow -e^{-2i\beta}$ radially. By the Kraus-Nehari theorem, we obtain $\|S_{f_\beta}\|_2 = 6$ and the extremality of f_β in $SP(\beta)$ for any $|\beta| < \pi/2$. $\square$

3 Order estimates of the coefficients

Knowing the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of f . For example, the following holds.

Theorem 3.1 (cf. [7]). *Let $(3/2) < \lambda \leq 3$. For a normalized analytic function $f(z) = z + a_2z^2 + a_3z^3 + \dots$ such that $\|T_f\|_1 \leq 2\lambda$, it holds that $a_n = O(n^{\lambda-2})$ as $n \rightarrow +\infty$. This order estimate is best possible.*

However the sharp estimate of coefficients of $f \in SP(\beta)$ has been already obtained by Zamorski [12] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f \in SP(\beta)$ from this.

Theorem 3.2 (Zamorski). *If $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is in $SP(\beta)$ and $|\beta| < \pi/2$, then*

$$|a_n| \leq \prod_{k=1}^{n-1} \left| 1 + \frac{e^{2i\beta}}{k} \right| \quad (4)$$

for any $n \geq 2$. The equality in (4) holds for some $n \geq 2$ if and only if f is a rotation of the β -spiral Koebe function f_β .

Remark. This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [4](1980).

Corollary 3.1. *Let $|\beta| < \pi/2$ and $f(z) = z + a_2z^2 + a_3z^3 + \dots$ be a β -spiral-like function. Then it holds that*

$$a_n = O(n^{\cos 2\beta}) \quad (n \rightarrow +\infty). \quad (5)$$

This order estimate is sharp.

Proof. From the inequality (4), we have that for $|\beta| < \pi/2$,

$$\begin{aligned} \log |a_n| &\leq \frac{1}{2} \sum_{k=1}^{n-1} \log \left(1 + \frac{2 \cos 2\beta}{k} + \frac{1}{k^2} \right) \\ &= \frac{1}{2} \sum_{k=1}^{n-1} \left(\frac{2 \cos 2\beta}{k} \right) + O(1) \\ &= \cos 2\beta \log n + O(1) \end{aligned}$$

as $n \rightarrow +\infty$. Therefore we obtain the estimate (5). $\square$

Remark. In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [3](1970).

4 Strongly normalized univalent functions are not always holomorphic.

The following is known.

Theorem 4.1. *For a holomorphic function ϕ on a simply connected domain A , there exists a locally univalent meromorphic function f on A such that*

$$S_f = \phi.$$

The solution is unique up to postcomposition of an arbitrary Möbius transformation.

We assume $A = \mathbb{D}$. Nehari showed that if $\|\phi\|_2 = \sup_{z \in \mathbb{D}} |\phi(z)|(1 - |z|^2)^2 \leq 2$, then f is univalent (meromorphic) on $\mathbb{D}$. It is well-known that if $\|\phi\|_2 = \sup_{z \in \mathbb{D}} |\phi(z)|(1 - |z|^2)^2 \leq 2$ and f is the *strongly normalized* solution, i.e., $f(0) = f'(0) - 1 = f''(0) = 0$, then f is holomorphic on $\mathbb{D}$. Since for a normalized analytic function $f(z) = z + a_2 z^2 + \dots$, $g := f/(a_2 f + 1)$ is strongly normalized and $\|S_f\|_2 = \|S_g\|_2$, we have the following.

Proposition 4.1 ([6] and [5] Corollary 2.). *If a normalized analytic function $f(z) = z + a_2 z^2 + \dots$ satisfies $\|S_f\|_2 \leq 2$, then f is univalent and $a_2 f + 1 \neq 0$ on $\mathbb{D}$.*

In [5] Chuaqui and Osgood remark that a strongly normalized univalent function f is not always holomorphic if $\|S_f\|_2 > 2$. Spiral-like functions are examples for this fact.

Theorem 4.2. *If $|\beta|$ is sufficiently close to $\pi/2$, the β -spiral-Koebe function $f_\beta(z) = z + a_2 z^2 + \dots$ satisfies $a_2 f_\beta(z) + 1 = 0$ for some $z \in \mathbb{D}$.*

Proof. By direct calculation, we have $a_2 = f''_\beta(0)/2 = e^{2i\beta} + 1$. The β -logarithmic spiral $\{f_\beta(-e^{-2i\beta}) \exp(e^{i\beta}t); t \geq 0\}$ is the complement of $f_\beta(\mathbb{D})$ in $\mathbb{C}$. Thus $a_2 f_\beta(z) + 1 \neq 0$ on $\mathbb{D}$ if and only if this spiral contains $-1/a_2$. We can see that if $f_\beta(-e^{-2i\beta}) \exp(e^{i\beta}t) = -1/a_2$, then

$$t = e^{i\beta} \log(1 + e^{-2i\beta}). \quad (6)$$

and that the imaginary part of the right side of (6) tends to $-\infty$ (resp. $+\infty$) if β tends to $+\pi/2$ (resp. $-\pi/2$). $\square$

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