

On the unitarizability of irreducible  
representation of  $GL(n, k)$

by

京大理 松本 茂樹  
(Shigeki Matsumoto)

Introduction.

Let  $k$  be a non-archimedean local field with the standard norm  $|\cdot|$ . Zelevinskii [2] parametrized all the irreducible smooth representations of  $GL(n, k)$  using the multisets of segments of cuspidal representations. In the present paper we determine when the irreducible representations of  $GL(n, k)$  have non-degenerate Whittaker models in Zelevinskii's parametrization. We also study for degenerate Whittaker models.

Bernstein [1] gave a criterion of unitarizability of irreducible representations of  $GL(n, k)$  along Zelevinskii's parameter. Applying his criterion, we find the unitarizability

condition of irreducible representations of  $GL(2,k)$ ,  $GL(3,k)$ , and of multiplicity free support.

In the final section we compute values of Zelevinskii's duality  $t$  and ascertain Bernstein's unitarizability conjecture for some special cases.

### 1. Zelevinskii's parametrization and Whittaker models.

If  $(n_1, n_2, \dots, n_r)$  is a partition of the number  $n$  and  $\rho_i$  is a irreducible representation of  $GL(n_i, k)$ , then we have the tensor product representation  $\tau = \rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_r$  of  $\prod_{i=1}^r GL(n_i, k)$ , which is isomorphic to the block diagonal subgroup  $D$  of  $GL(n, k)$ .

The representation  $\tau$  can be extended to the representation  $\tilde{\tau}$  of the standard parabolic subgroup  $P$  of  $GL(n, k)$  by the canonical epimorphism  $P \longrightarrow D$ . We call the induced representation

$\text{Ind}_P^{GL(n, k)} \tilde{\tau}$  the product representation of  $\rho_i$  and denote it by  $\rho_1 \times \rho_2 \times \dots \times \rho_r$ .

Let  $\rho$  be a cuspidal representation of  $GL(n, k)$  and  $\alpha$  be a real number. We denote by  $\gamma^\alpha \rho$  the cuspidal representation defined by  $g \longmapsto |\det g|^\alpha \rho(g)$  ( $g \in GL(n, k)$ ). A finite

set  $\Delta$  is called a segment of length  $m$  if it is of the form  $\Delta = \{ \rho, \nu^1 \rho, \nu^2 \rho, \dots, \nu^{m-1} \rho \}$ , where  $\rho$  is a cuspidal representation of  $GL(n, k)$ .

Let  $\Delta_1 = \{ \rho_1, \nu^1 \rho_1, \dots, \nu^{m-1} \rho_1 \}$ ,  $\Delta_2 = \{ \rho_2, \nu^1 \rho_2, \dots, \nu^{m'-1} \rho_2 \}$  be segments. We say that  $\Delta_1$  and  $\Delta_2$  are linked if the union  $\Delta_1 \cup \Delta_2$  is a segment different from  $\Delta_1, \Delta_2$ . If  $\Delta_1$  and  $\Delta_2$  are linked and  $\rho_2 = \nu^k \rho_1$  for some  $k > 0$  then we say that  $\Delta_1$  precedes  $\Delta_2$ .

Let  $\Delta = \{ \rho, \nu^1 \rho, \dots, \nu^{m-1} \rho \}$  be a segment. Then the product representation  $\rho \times \nu^1 \rho \times \dots \times \nu^{m-1} \rho$  is reducible if  $m > 1$  and has a unique irreducible subrepresentation, which we denote by  $\langle \Delta \rangle$ .

Let  $a = \{ \Delta_1, \Delta_2, \dots, \Delta_r \}$  be a multiset of segments. (Each element of a multiset may have multiplicity. See [2].) Suppose for each pair of indices  $i, j$  such that  $i < j$ ,  $\Delta_i$  does not precede  $\Delta_j$ . Then the product representation  $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \dots \times \langle \Delta_r \rangle$  has a unique subrepresentation. We denote it by  $\langle a \rangle$ .

We denote by  $\mathcal{Q}$  the set of all multisets of segments.

Let  $a \in \mathcal{Q}$ . Call an elementary operation on the multiset  $a$  the replacement in it of linked segments  $\Delta_1, \Delta_2$  by  $\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2$ . Further we can define an order  $\leq$  in  $\mathcal{Q}$ :  $b < a$  if  $b$  may be obtained from  $a$  by a chain of elementary operations.

We denote by  $R_n$  the Grothendieck group of the abelian category of smooth  $GL(n, k)$ -modules of finite length. We regard  $GL(0, k)$  as the trivial group.

We introduce the product  $\times$  on  $R = \bigoplus_{n=0}^{\infty} R_n$  by the induction functors (product representations)

$$R_n \times R_m \ni (\sigma, \tau) \longmapsto \sigma \times \tau \in R_{n+m}.$$

Then the algebra  $R$  is associative and commutative.

For a multiset  $a = \{\Delta_1, \Delta_2, \dots, \Delta_r\}$ ,  $\pi(a)$  denotes the element  $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \dots \times \langle \Delta_r \rangle$  of  $R$ .

We put  $\text{Irr} = \bigcup_{n=0}^{\infty} \{\text{irreducible smooth representations of } GL(n, k)\}.$

Then we have the following

Theorem 1 (Zelevinskii [2]).

- (1)  $\mathcal{Q} \ni a \longmapsto \langle a \rangle \in \text{Irr}$  is bijective.
- (2)  $(\pi(a))_{a \in \mathcal{Q}}$  is a  $\mathbb{Z}$ -basis of  $R$ . In other words,  $R$  is the polynomial ring over  $\mathbb{Z}$  in variables  $\langle \Delta \rangle$  ( $\Delta \in \mathcal{S}$ ), where  $\mathcal{S}$  is the set of segments.

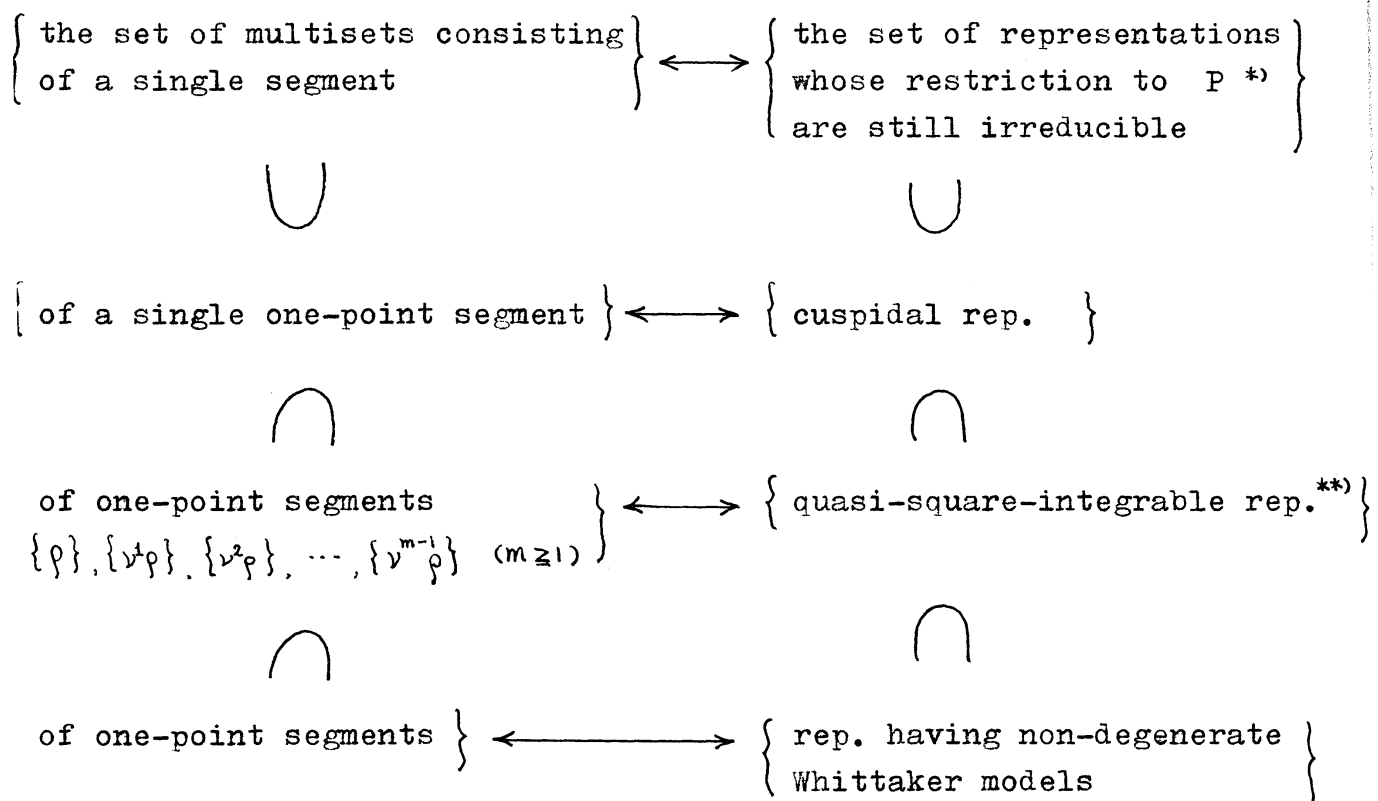
For the above correspondence (1), we have furthermore a proposition.

Proposition 2.

Let  $a$  be a multiset and  $\pi$  be the corresponding irreducible representation. Then the representation  $\pi$  has a non-degenerate Whittaker model if and only if the multiset  $a$  consists of one-point segments.

The proof depends on the arguments of derivatives of representations (see [2]).

We give a list of the correspondence between some classes of multisets and irreducible representations (see [1,2] ).



\*) Here we denote by  $P$  the subgroup of  $GL(n, k)$  consisting of matrices whose final rows are  $0, 0, 0, \dots, 0, 0, 1$ .

\*\*) A representation of  $GL(n, k)$  is called quasi-square-integrable if its matrix coefficients become square-integrable modulo the center of  $GL(n, k)$  after multiplying by a suitable character of  $GL(n, k)$ .

For degenerate Whittaker models we have the following proposition. Let  $a = \{ \Delta_1, \Delta_2, \dots, \Delta_r \} \in \mathcal{O}$ , where  $\Delta_i$  is a segment consisting of cuspidal representations of  $GL(n_i, k)$ . The level (of non-degeneracy) of the representation  $\langle a \rangle$  is an integer  $\sum_{i=1}^r n_i$ .

Proposition 3. Let  $U$  be the subgroup of upper triangular matrices in  $GL(n, k)$  and  $\psi$  be a non-trivial additive character of  $k$ . For a finite set  $S$  satisfying

$$\{ n-r+1, n-r+2, \dots, n-1 \} \subset S \subset \{ 1, 2, 3, \dots, n-1 \}$$

we denote by  $\chi_S$  the character of  $U$  defined by

$$U \ni (u_{ij}) \longmapsto \psi \left( \sum_{i \in S} u_{ii+1} \right) \in \mathbb{C}.$$

Then the level of any irreducible subrepresentation of the induced representation  $\text{Ind}_U^{GL(n, k)} \chi_S$  is greater than or equal to  $r$ .

Remark. Proposition 3 for  $r = n$  coincides with "only if" part of Proposition 2.

## 2. Bernstein's unitarizability criterion.

Let  $\pi$  be an irreducible representation of  $GL(n, k)$ . We say the representation  $\pi$  is hermitian if there exists a  $GL(n, k)$ -invariant, non-degenerate sesquilinear form on the representation space of  $\pi$ . And we say  $\pi$  is unitarizable if there exists a  $GL(n, k)$ -invariant, positive-definite sesquilinear form. Any unitarizable representation is hermitian, but a hermitian representation is not necessarily unitarizable even if it is irreducible and its central character is unitary.

For a segment  $\Delta = \{ \rho, v^1 \rho, \dots, v^{m-1} \rho \}$ ,  
 we put  $v^{\frac{1}{2}} \Delta = \{ v^{\frac{1}{2}} \rho, v^{\frac{3}{2}} \rho, \dots, v^{m-\frac{1}{2}} \rho \}$ ,  
 $\Delta' = \{ v^{\frac{1}{2}} \rho, v^{\frac{3}{2}} \rho, \dots, v^{m-\frac{3}{2}} \rho \}$ .

By virtue of the latter half (2) of Theorem 1, the correspondence

$$\langle \Delta \rangle \longmapsto \langle v^{\frac{1}{2}} \Delta \rangle + \langle \Delta' \rangle$$

for segments can be extended to a ring endomorphism  $\mathcal{D}$ . For

a multiset  $a = \{\Delta_1, \Delta_2, \dots, \Delta_r\}$ , we put  $a' = \{\Delta'_1, \Delta'_2, \dots, \Delta'_r\}$

For a representation  $\pi$  of  $GL(n, k)$ , we denote by  $\deg(\pi)$

the integer  $n$  and denote by  $e(\pi)$  the real number which

satisfies the following equality  $|\chi(\lambda)| = |\lambda|^{e(\pi)}$

( $\lambda \in k^\times$ ), where  $\chi$  is the central character of  $\pi$  and

the center of  $GL(n, k)$  is canonically identified with the

multiplicative group  $k^\times$ .

Criterion 4 (Bernstein [1]). Let  $a \in \mathcal{O}$ . The representation

$\langle a \rangle$  is unitarizable if and only if the following three

conditions are satisfied:

(i)  $\langle a \rangle$  is hermitian,

(ii)  $\langle a' \rangle$  is unitarizable,

(iii) In the expression  $\mathcal{D}(\langle a \rangle) = \sum_{b \in \mathcal{O}} c_b \cdot \pi(b)$ ,

coefficients  $c_b$  is zero for such a  $b \in \mathcal{O}$

that  $\deg(\langle b \rangle) > \deg(\langle a' \rangle)$  and  $e(\langle b \rangle) \leq$

### 3. Unitarizability condition for some representations.

Using Criterion 4, we can write down the following lists of unitarizability condition for irreducible representations of  $GL(2,k)$  and  $GL(3,k)$ .

#### Case of $GL(2,k)$ .

| Multiset consisting of                                                   | Unitarizability condition                                                                                                                                            |
|--------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\{\rho\} \quad (\rho \in C_2)$                                          | $e(\rho) = 0$                                                                                                                                                        |
| $\{\mu, \nu\mu\} \quad (\mu \in C_1)$                                    | $e(\mu) = -1/2$                                                                                                                                                      |
| linked segments $\{\mu\}, \{\nu\mu\}$<br>( $\mu \in C_1$ )               | $e(\mu) = -1/2$                                                                                                                                                      |
| non-linked segments $\{\mu_1\}, \{\mu_2\}$<br>( $\mu_1, \mu_2 \in C_1$ ) | (i) $e(\mu_1) = e(\mu_2) = 0$ , or<br>(ii) $\mu_1$ and $\mu_2$ are hermitian contragredient each other and<br>$-1/2 < e(\mu_1) < 1/2$ ,<br>$-1/2 < e(\mu_2) < 1/2$ . |

Here  $C_n$  is the set of cuspidal representations of  $GL(n,k)$ .

Case of  $GL(3, k)$ .

| Multiset consisting of                                                                       | Unitarizability condition                                                            |
|----------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| $\{p\} \quad (p \in C_3)$                                                                    | $e(p) = 0$                                                                           |
| $\{p_1\}, \{p_2\} \quad (p_1 \in C_1, p_2 \in C_2)$                                          | $e(p_1) = e(p_2) = 0$                                                                |
| $\{p, \nu^1 p, \nu^2 p\} \quad (p \in C_1)$                                                  | $e(p) = -1$                                                                          |
| $\{p_1, \nu^1 p_1\}, \{p_2\} \quad (p_1, p_2 \in C_1)$                                       | $e(p_1) = -1/2, e(p_2) = 0$                                                          |
| $\{p\}, \{\nu^1 p\}, \{\nu^2 p\} \quad (p \in C_1)$                                          | $e(p) = -1$                                                                          |
| $\{p_1\}, \{\nu^1 p_1\}, \{p_2\} \quad (p_1, p_2 \in C_1, \\ p_2 \neq \nu^1 p_1, \nu^2 p_1)$ | $e(p_1) = -1/2, e(p_2) = 0$                                                          |
| $\{p_1\}, \{p_2\}, \{p_3\} \quad (p_1, p_2, p_3 \in C_1, \\ \text{no pairs are linked})$     | $-1/2 < e(p_i) < 1/2, \\ \nu^{-2e(p_i)} p_i \in \{p_1, p_2, p_3\} \quad (i=1, 2, 3)$ |

We also examine unitarizability of composition factors of the product representation.

Proposition 5. Let  $\rho$  be a cuspidal representation.

Then the product representation  $\rho \times \nu^1 \rho \times \dots \times \nu^{m-1} \rho$  is of length  $2^{m-1}$ .

If  $e(\nu^{\frac{m-1}{2}} \rho) \neq 0$ , then no composition factors are unitarizable.

If  $e(\nu^{\frac{m-1}{2}} \rho) = 0$ , then exactly two factors, corresponding to the multisets  $\{\{\rho, \nu^1 \rho, \nu^2 \rho, \dots, \nu^{m-1} \rho\}\}$  and  $\{\{\rho\}, \{\nu^1 \rho\}, \dots, \{\nu^{m-1} \rho\}\}$ , are unitarizable, and others are not unitarizable.

In order to apply the criterion, we explicitly calculate the value  $e(\langle b \rangle)$  for some  $b \in \mathcal{O}$ .

#### 4. Zelevinskii's duality and Bernstein's conjecture.

Let us consider another ring endomorphism  $t$  of  $R$  extending the correspondence

$$\langle \{\rho, \nu^1 \rho, \dots, \nu^{m-1} \rho\} \rangle \longmapsto \langle \{\{\rho\}, \{\nu^1 \rho\}, \dots, \{\nu^{m-1} \rho\}\} \rangle.$$

The endomorphism  $t$  is involutive and maps  $\text{Irr}$  into  $\text{Irr}$ ,

which we call duality after Zelevinskii. Bernstein states in [1]

the following

Conjecture 6. Duality  $t$  maps irreducible unitarizable representations into unitarizable ones.

A partial answer to this conjecture is given by Bernstein himself in [1]. He proves that  $t(\pi)$  is unitarizable if  $\pi$  is a unitarizable representation of the form  $\pi = \pi(a) = \langle a \rangle$  ( $a \in \mathcal{O}$ ).

Combining the results of the previous section and the following propositions, we can ascertain the conjecture for representations of  $GL(2, k)$ ,  $GL(3, k)$ , and for composition factors of the product representation  $\rho \times \nu^1 \rho \times \dots \times \nu^{m-1} \rho$ .

Proposition 7. Let  $\rho_i$  be a cuspidal representation of  $GL(n_i, k)$  ( $i = 1, 2$ ). We assume that the segments  $\Delta_1 = \{\rho_1, \nu^1 \rho_1\}$  and  $\Delta_2 = \{\rho_2\}$  are not linked. Then the product module  $\rho_1 \times \nu^1 \rho_1 \times \rho_2$  is of length 2. Its composition factors are  $\langle \{\rho_1, \nu^1 \rho_1\}, \{\rho_2\} \rangle$  and  $\langle \{\rho_1\}, \{\nu^1 \rho_1\}, \{\rho_2\} \rangle$  which are dual (under  $t$ ) of each other.

Let  $\Delta = \{\rho, \nu^1 \rho, \dots, \nu^{m-1} \rho\}$  be a segment of length  $m$ . For an element  $e = (e_1, e_2, \dots, e_{m-1})$  in  $\{1, -1\}^{m-1}$ , we associate an equivalence relation  $\sim_e$  on  $\Delta$  in such a manner that  $\nu^i \rho \sim_e \nu^j \rho$  if and only if  $e_i = 1$ . We denote by  $\Delta(e)$  the set of equivalence classes with respect to the equivalence relation  $\sim_e$ . We regard naturally  $\Delta(e)$  as an element in  $\mathcal{O}$ .

Proposition 8. Using the above notation, we have a bijection  $e \longmapsto \langle \Delta(e) \rangle$  of  $\{1, -1\}^{m-1}$  onto the set of all composition factors of the product  $\pi = \rho \times \nu^1 \rho \times \nu^2 \rho \times \dots \times \nu^{m-1} \rho$ .

Duality  $t$  permutes the composition factors of  $\pi$  as the following manner:

$$t(\langle \Delta(e_1, e_2, e_3, \dots, e_{m-1}) \rangle) = \langle \Delta(-e_1, -e_2, \dots, -e_{m-1}) \rangle.$$

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- [4] I. N. Bernstein and A. V. Zelevinskii, Induced representations of reductive p-adic groups I, Ann. Scient. Éc. Norm. Sup., 4<sup>e</sup> série, t. 10, 1977, 441-472.  
Note: Three papers [2], [3] and [4] form a trilogy on the representation theory of  $GL(n, k)$ . The paper [3] is by itself an excellent introduction to the representation theory of reductive p-adic groups basing on the works of Harish-Chandra, Jacquet, Gel'fand and Kajdan.
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 Note: This paper neatly summarizes the works [2] and [4] of Bernstein and Zelevinskii. He emphasizes the importance of Lemma 4.7 [4] in their works.
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- [10] I. M. Gel'fand and D. A. Kajdan, Representations of  $G(n, K)$ , in Lie groups and their Representations, Summer School of group representations, Budapest, 1971.  
 Note: One of the main methods in [2] - [4] is based on studying the restriction of representations of  $GL(n, k)$  to the subgroup  $P \subset GL(n, k)$  consisting of matrices with the last row  $0, 0, \dots, 0, 1$ . This is a method of Gel'fand and Kajdan [9], [10]. Bernstein and Zelevinskii formulate this method in terms of functors.
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 Note: In this volume, Zelevinskii applies the technique developed in [2] - [4] for the investigation of representations of general linear groups over p-adic fields to the representation theory of the groups  $GL(n, F_q)$ .

- [12] A. V. Zelevinskii,  $p$ -adic analog of the Kazhdan-Lusztig hypothesis, *Funkt. Anal. i Prilozhen.*, Vol. 15, No. 2, 9-21.  
 Note: In the reduction process of Bernstein's unitarizability criterion, we require a knowledge of the multiplicity matrix  $m = (m_{ab})$ , which describes the decomposition in Grothendieck group of induced representations into irreducible ones. In [12] Zelevinskii defined some polynomials  $P_{ab}(q)$ , analogous to the Kazhdan-Lusztig polynomials, and conjectured that  $m_{ab} = P_{ab}(1)$ .
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 Note: In this proceeding Kato states some generalization of Zelevinskii's conjecture in [12].
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 Note: In this paper Tadić proved the conjecture of Bernstein (see Conjecture 6 in the present paper) by a dexterous reduction.

Department of Mathematics,  
 Faculty of Science,  
 Kyoto University.

Added. When I sent this note to Professor N. Kawanaka, he kindly sent back to me the following preprint.

M. Tadić : Solution of the unitarizability problem for general linear group (non-archimedean case), preprint.