

Recent advances of analytic method
in the theory of numbers

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In this lecture, I shall state two analytic methods induced by Gallagher [2] and Fogels [1] and extend these so as to be able to apply in case of algebraic number fields. By virtue of these tools we shall have a prospect of natural extension of density theorem concerning the number $N(\sigma, T)$ of zeros $\rho = \beta + i\gamma$ in the rectangle

$$\sigma \leq \beta \leq 1, \quad -T \leq \gamma \leq T$$

of all Hecke's zeta-functions $\zeta(s, \chi)$ with same moduli $\tilde{m}$. Hirano is now working out this problem applying the extended Ingham's method [5, Chapt. 9].

Let k be an algebraic number field of degree n over the rational number field. Let k have r real conjugates $k^{(i)}$ ($1 \leq i \leq r_1$) and r_2 pairs of complex conjugates $k^{(m)}, k^{(m+r_2)}$ ($r_1+1 \leq m \leq r_1+r_2$), namely $n=r_1+2r_2$. A formal product

$$\tilde{m} = m \cdot p_{\infty}^{(1)} \cdots p_{\infty}^{(r)} \quad (0 \leq i \leq r_1)$$

is called a divisor of k , where m is an integral ideal and $p_{\infty}^{(i)}$ is an infinite prime spot. Let $A(\tilde{m})$ be the group of all ideals prime to m in k and $S(\tilde{m})$ be the group of all principal ideals (v) generated by v satisfying the multiplicative congruence

$$v \equiv 1 \pmod{\tilde{m}},$$

namely

$$v \in k, \quad v \equiv 1 \pmod{m} \quad \text{and} \quad v^{(1)}, \dots, v^{(r)} > 0.$$

A class of the factor group $A(\tilde{m})/S(\tilde{m})$ is called a ray class mod $\tilde{m}$. We denote by $h(\tilde{m})$ the number of classes of this factor group. Let $\mathcal{F}$ be the different and d be the discriminant of k , so that $N\mathcal{F} = |d|$. Let R be the regulator of k , h be the number of absolute ideal classes of k and w be the number of roots of unity in k [4].

Throughout the paper, c or c with suffix will be used to denote a positive constant depending only on k , and $c(*)$ will be used when c depends further on several parameters $*$. $A \ll B$

or $A = O(B)$, where B is positive, means that there exists c satisfying

$$|A| \leq cB$$

in the region under consideration.

In the first place, we shall estimate the number $T(t)$ of integral ideals in a ray class $C \bmod \tilde{m}$ whose norms don't exceed t . The details of this subject will be stated another paper. We shall therefore only point out the main results.

Theorem 1. Let C be a ray class $\bmod \tilde{m}$. Take c sufficiently large, and suppose that

$$t \geq cN\tilde{m}.$$

Then

$$T(t) = \sum_{N\alpha \leq t, \alpha \in C} 1 = \frac{1}{h(\tilde{m})} \prod_{p|\tilde{m}} \left(1 - \frac{1}{Np}\right) \frac{2^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} |R| h}{\sqrt{|d|} w} t + O \left\{ \frac{1}{h(\tilde{m})} \prod_{p|\tilde{m}} \left(1 - \frac{1}{Np}\right) N\tilde{m}^{\frac{1}{2}} t^{1-\frac{1}{2}} \right\}.$$

Theorem 2. If $0 \leq y \ll x$, then

$$\sum_{x \leq N\alpha \leq x+y, \alpha \in C} 1 \ll \frac{1}{h(\tilde{m})} \left(y + N\tilde{m}^{\frac{1}{2}} x^{1-\frac{1}{2}} + N\tilde{m} \right).$$

Secondly, we shall introduce a new elegant real analytic method due to Gallagher.

Theorem 3. If $\sum_{n=1}^{\infty} |a_n| < \infty$, then

$$\int_{-T}^T \left| \sum_{n=1}^{\infty} a_n n^{it} \right|^2 dt \leq \frac{\pi}{2} T^2 \int_{\tau^{-1}}^{\infty} \left| \sum_{z \leq n \leq z\tau} a_n \right|^2 \frac{dz}{z}$$

where $\tau = e^{\frac{\pi}{T}}$.

Proof. Write

$$a_n = c \left(\frac{\log n}{2\pi} \right).$$

Therefore,

$$S(t) = \sum_{n=1}^{\infty} a_n n^{it} = \sum_{\nu} c(\nu) e^{2\pi i \nu t}$$

where ν runs over

$$0, \frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}, \dots$$

Put

$$F_{\delta}(x) = \delta^{-1} \text{ or } 0,$$

according as $|x| \leq \frac{1}{2}\delta$ or not, and write

$$f(x) = \sum_{\nu} c(\nu) F_{\delta}(x - \nu) = \frac{1}{\delta} \sum_{N-x \leq \frac{1}{2}\delta} c(\nu) \quad (1)$$

Accordingly,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(t)|^p dt &= \frac{1}{\delta^p} \int_{-\infty}^{\infty} \left| \sum_{\left| \frac{\log n}{2\pi} - t \right| \leq \frac{1}{2}\delta} a_n \right|^p dt \\
&= \frac{1}{2\pi\delta^p} \int_{\tau^{-1}}^{\infty} \left| \sum_{z \leq n \leq z\tau} a_n \right|^p \frac{dz}{z} \quad (2)
\end{aligned}$$

for $p=1$ and 2 , where

$$\delta = \frac{1}{2\pi} \quad \text{and} \quad \tau = e^{\frac{\pi}{T}}.$$

Assume that the right side of Theorem 3 is finite, since the result is trivially true otherwise. Hence, $f(x)$ belongs to L^2 because of (2). We have also

$$\int_{-\infty}^{\infty} |f(t)| dt \leq \frac{1}{2\pi\delta} \int_{\tau^{-1}}^{\infty} \sum_{z \leq n \leq z\tau} |a_n| \frac{dz}{z}$$

from (1), where the coefficient of $|a_n|$ of the left side turns out to be

$$\frac{1}{2\pi\delta} \int_{\tau^{-1}}^{\tau} \frac{dz}{z} \ll \frac{1}{2\pi\delta} \log \tau = 1$$

It follows therefore from the assumption of the theorem that $f(x)$ belongs to L^1 .

By these properties,

$$\begin{aligned}
\hat{f}(x) &= \int_{-\infty}^{\infty} e^{2\pi i x t} f(t) dt = \sum_{\nu} c(\nu) \int_{-\infty}^{\infty} e^{2\pi i x t} F_{\delta}(t - \nu) dt \\
&= \sum_{\nu} c(\nu) e^{2\pi i x \nu} \frac{\sin \pi \delta x}{\pi \delta x} \quad (3)
\end{aligned}$$

belongs to L^2 [3] and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx.$$

This implies from (1) and (3) that

$$\begin{aligned}
\frac{1}{\delta^2} \int_{-\infty}^{\infty} \left| \sum_{|v-x| \leq \frac{\delta}{2}} c(\nu) \right|^2 dx &= \int_{-\infty}^{\infty} \left| \sum_{\nu} c(\nu) e^{2\pi i x \nu} \frac{\sin \pi \delta x}{\pi \delta x} \right|^2 dx \\
&\geq \left(\frac{2}{\pi} \right)^2 \int_{-T}^T \left| \sum_{\nu} c(\nu) e^{2\pi i x \nu} \right|^2 dx.
\end{aligned}$$

In view of (1) and (2), it follows therefore

$$\int_{-T}^T \left| \sum_{n=1}^{\infty} a_n n^{it} \right|^2 dt \leq \frac{\pi^2}{4} \int_{-\infty}^{\infty} \left| \frac{1}{\delta} \sum_{|v-x| \leq \frac{\delta}{2}} c(\nu) \right|^2 dx = \frac{\pi}{8\delta^2} \int_{\tau^{-1}}^{\infty} \left| \sum_{z \leq n \leq z\tau} a_n \right|^2 \frac{dz}{z}.$$

Now we shall apply Theorem 3 to the sum of the form

$$S(\chi, t) = \sum A(\alpha) \chi(\alpha) N(\alpha)^{it}$$

where α runs over all integral ideals of k , χ is a character mod $\tilde{m}$ and $A(\alpha)$ is a function defined over all integral ideals in k . Put

$$a_n = \sum_{N\alpha=n} A(\alpha) \chi(\alpha),$$

and make use of Theorem 1, we have

$$\begin{aligned} \sum_{\chi \bmod \tilde{m}} \int_{-T}^T |S(\chi, t)|^2 dt &= \sum_{\chi \bmod \tilde{m}} \int_{-T}^T \left| \sum_{n=1}^{\infty} a_n n^{it} \right|^2 dt \\ &\leq \frac{\pi}{2} T^2 \int_{\tau^{-1}}^{\infty} \sum_{\chi \bmod \tilde{m}} \left| \sum_{z \leq N\alpha \leq z\tau} A(\alpha) \chi(\alpha) \right|^2 \frac{dz}{z}. \quad (4) \end{aligned}$$

Divide integral ideals α satisfying $z \leq N\alpha \leq z\tau$ into several classes in such a way that each pair of integral ideals in a class $\mathfrak{A}$ is never congruent mod $\tilde{m}$ with each other and that the number M of classes is less than

$$O \left\{ \frac{1}{h(\tilde{m})} (z\tau - z + z^{1-\frac{1}{n}} N m^{\frac{1}{n}} + N m) \right\},$$

in view of Theorem 2. Accordingly,

$$\begin{aligned} &\sum_{\chi \bmod \tilde{m}} \left| \sum_{z \leq N\alpha \leq z\tau} A(\alpha) \chi(\alpha) \right|^2 \\ &\leq \sum_{\chi \bmod \tilde{m}} M \sum_{\mathfrak{A}} \left| \sum_{\alpha \in \mathfrak{A}} A(\alpha) \chi(\alpha) \right|^2 \\ &\ll \frac{1}{h(\tilde{m})} (z\tau - z + z^{1-\frac{1}{n}} N m^{\frac{1}{n}} + N m) \sum_{\mathfrak{A}} \sum_{\chi \bmod \tilde{m}} \left| \sum_{\alpha \in \mathfrak{A}} A(\alpha) \chi(\alpha) \right|^2 \\ &\ll (z\tau - z + z^{1-\frac{1}{n}} N m^{\frac{1}{n}} + N m) \sum_{z \leq N\alpha \leq z\tau} |A(\alpha)|^2. \end{aligned}$$

Substituting (4) for this result, we get

$$\begin{aligned} &\sum_{\chi \bmod \tilde{m}} \int_{-T}^T |S(\chi, t)|^2 dt \\ &\ll T^2 \int_{\tau^{-1}}^{\infty} \frac{z\tau - z + z^{1-\frac{1}{n}} N m^{\frac{1}{n}} + N m}{z} \sum_{z \leq N\alpha \leq z\tau} |A(\alpha)|^2 dz \\ &\ll \sum |A(\alpha)|^2 (N\alpha + N\alpha^{1-\frac{1}{n}} N m^{\frac{1}{n}} T + N m T), \end{aligned}$$

which was the result obtained by Hirano in his master dissertation.

Theorem 4. If $\sum |A(n)| < \infty$, then

$$\sum_{\chi \bmod \tilde{m}} \int_{-T}^T \left| \sum A(n) \chi(n) N n^{it} \right|^2 dt \ll \sum (N n + N n^{1-\frac{1}{m}} N n^{\frac{1}{m}} T + N m T) |A(n)|^2.$$

Thirdly we shall introduce a new Lindelöf principle due to Fogels.

Theorem 5. Let $f_j(s)$ ($1 \leq j \leq m$) be regular in the region D

$$\sigma_1 < \sigma < \sigma_2, \quad -\infty < t < \infty,$$

satisfying the conditions

$$F(s) = |f_1(s)|^p + \dots + |f_m(s)|^p \leq c_0 e^{c_1 |t|}$$

and be continuous over the closure of D , p being some positive integer. If the conditions

$$F(s) \leq A(|t|+2)^a \log^c C(|t|+2) \quad \text{on } \sigma = \sigma_1$$

$$F(s) \leq B(|t|+2)^b \log^c C(|t|+2) \quad \text{on } \sigma = \sigma_2$$

are fulfilled, then there exists $c(\sigma_1, \sigma_2)$ such that

$$F(s) \leq c(\sigma_1, \sigma_2) A^{\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}} B^{\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}} (|t|+2)^{a \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} + b \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}} \{ \log C(|t|+2) \}^c,$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, where a, b, c are given positive numbers and $A, B, C > 1$ are parameters not depending on t .

Proof. We know that

$$|\Gamma(s)| = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left\{ 1 + O\left(\frac{1}{|t|}\right) \right\}$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \geq 1$. Hence

$$\left| \frac{\Gamma\left(2 - \frac{s - \sigma_1}{2(\sigma_2 - \sigma_1)}\right)}{\Gamma\left(1 + \frac{s - \sigma_1}{2(\sigma_2 - \sigma_1)}\right)} \right| = (|t|+2)^{1 - \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}} \left\{ 1 + O\left(\frac{1}{|t|+2}\right) \right\}, \quad (5)$$

and

$$\left| \frac{\Gamma\left(2 - \frac{\sigma_2 - s}{2(\sigma_2 - \sigma_1)}\right)}{\Gamma\left(1 + \frac{\sigma_2 - s}{2(\sigma_2 - \sigma_1)}\right)} \right| = (|t|+2)^{1 - \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}} \left\{ 1 + O\left(\frac{1}{|t|+2}\right) \right\}, \quad (6)$$

in the region $\sigma_1 \leq \sigma \leq \sigma_2$.

Take σ_0 such that $1 - \sigma_0 < \sigma_1$ and define

$$\log C(\delta + \sigma_0) = \int_{\Gamma} \frac{dz}{z},$$

the contour lying on the right half plane $\sigma \geq 0$. Then,

$$\log C(\delta + \sigma_0) = \log C(|t| + 2) \left\{ 1 + O\left(\frac{1}{\log C(|t| + 2)}\right) \right\}.$$

Now we define

$$G(s) = A^{\frac{\sigma_2 - \delta}{\sigma_2 - \sigma_1}} B^{\frac{\delta - \sigma_1}{\sigma_2 - \sigma_1}} \left\{ \frac{\Gamma(2 - \frac{\delta - \sigma_1}{2(\sigma_2 - \sigma_1)})}{\Gamma(1 + \frac{\delta - \sigma_1}{2(\sigma_2 - \sigma_1)})} \right\}^a \left\{ \frac{\Gamma(2 - \frac{\sigma_2 - \delta}{2(\sigma_2 - \sigma_1)})}{\Gamma(1 + \frac{\sigma_2 - \delta}{2(\sigma_2 - \sigma_1)})} \right\}^b \{ \log C(\delta + \sigma_0) \}^c.$$

Obviously $G(s)$ is regular in $\sigma_1 \leq \sigma \leq \sigma_2$ and there exists c_2 satisfying

$$|G(s)| \geq c_2$$

in the region stated above. By (5) and (6),

$$|G(s)| = A(|t| + 2)^a \log^c C(|t| + 2) \left\{ 1 + O\left(\frac{1}{\log C(|t| + 2)}\right) \right\}$$

on $\sigma = \sigma_1$ and

$$= B(|t| + 2)^b \log^c C(|t| + 2) \left\{ 1 + O\left(\frac{1}{\log C(|t| + 2)}\right) \right\}$$

on $\sigma = \sigma_2$. Further, from the assumption of the theorem,

$$\begin{aligned} \frac{F(s)}{|G(s)|} &\leq c_3 e^{c_1 |t|} && \text{for } \sigma_1 \leq \sigma \leq \sigma_2 \\ &\leq c_4 && \text{for } \sigma = \sigma_1 \text{ and } \sigma = \sigma_2 \end{aligned}$$

For any $\varepsilon > 0$,

$$|e^{\varepsilon s^2}| \frac{F(s)}{|G(s)|}$$

is subharmonic in $\sigma_1 < \sigma < \sigma_2$ and is continuous on $\sigma_1 \leq \sigma \leq \sigma_2$.

Applying the usual Lindelöf principle to this function and letting $\varepsilon \rightarrow 0$, we get the desired result.

From now on let χ be a primitive character mod $\tilde{m}$, $\tilde{m}$ being $m_2 p_\infty^{(1)} \dots p_\infty^{(q)}$ as before. We can find f_j ($1 \leq j \leq s = N(m)$) in $(m\tilde{f})^{-1}$ which satisfy the following conditions:

$$f_j \equiv 1 \pmod{p_\infty^{(i)}} \quad (1 \leq i \leq q),$$

each pair of ρ_j is never congruent mod $\mathfrak{f}^{-1}$, moreover, if $\rho \in (m\mathfrak{f})^{-1}$, then

$$\rho \equiv \rho_j \pmod{\mathfrak{f}^{-1}}$$

for some ρ_j . The set of these ρ_j is called a complete residue system mod $\mathfrak{f}^{-1}$ in $(m\mathfrak{f})^{-1}$.

Let $\xi \neq 0$ and $\xi \in (m\mathfrak{f})^{-1}$. Put

$$G(\xi, \chi) = \sum_{\beta} \chi(\beta) e^{2\pi i S(\beta \xi)},$$

where β runs over a complete residue system mod $\tilde{m}$, satisfying

$$\beta \equiv 1 \pmod{\mathfrak{f}_\infty^{(i)}} \quad (1 \leq i \leq g).$$

Further we define

$$G(\chi) = \sum_{\rho} \chi(\rho m\mathfrak{f}) e^{2\pi i S(\rho)},$$

ρ being over a complete residue system mod $\mathfrak{f}^{-1}$ in $(m\mathfrak{f})^{-1}$. Take $\eta = \eta(\xi)$ such that

$$\eta \equiv 1 \pmod{m}, \quad \eta \equiv \xi \pmod{\mathfrak{f}_\infty^{(i)}} \quad (1 \leq i \leq g),$$

then we have

$$G(\xi, \chi) = \bar{\chi}(\eta(\xi) \xi m\mathfrak{f}) G(\chi)$$

and

$$\chi(\eta(\xi)) = (\text{sgn } \xi^{(1)}) \dots (\text{sgn } \xi^{(g)}).$$

If we define the normalized Gauss sum by

$$I(\chi) = (-i)^g G(\chi) / \sqrt{N(m)},$$

then we have the following relation [7]

$$I(\chi) I(\bar{\chi}) = 1.$$

Let $\zeta(s, \chi)$ and $\zeta_k(s)$ be functions defined by

$$\sum \chi(\alpha) N\alpha^{-s} \quad \text{and} \quad \sum N\alpha^{-s}$$

for $\sigma > 1$, the summation being over all non-zero integral ideals in k . Define, for $\sigma > 1$,

$$\Gamma(s, \chi) = \int_0^\infty \int_0^\infty e^{-\sum_{p=1}^n z_p} \prod_{p=1}^n z_p^{\frac{z+a_p}{2}} \frac{dz_1 \dots dz_{r+1}}{z_1 \dots z_{r+1}},$$

where $r+1=r_1+r_2$ and

$$a_p = \begin{cases} 1 & (1 \leq p \leq g) \\ 0 & (g+1 \leq p \leq n) \end{cases}, \quad z_p = z_{p+r_2} \quad (r_1+1 \leq p \leq r_1+r_2)$$

which turns out to be

$$2^{-r_2 s} \Gamma\left(\frac{s+1}{2}\right)^g \Gamma\left(\frac{s}{2}\right)^{r_1-g} \Gamma(s)^{r_2}$$

Further define

$$\bar{\Phi}(s, \chi) = \frac{(2\pi)^{r_2}}{\sqrt{|d|}} A(\chi)^{\frac{s}{2}} \Gamma(s, \chi) \zeta(s, \chi)$$

and

$$A(\chi) = \pi^{-n} |d| N \mathfrak{m}_\chi.$$

Similarly we define

$$\Gamma(s, \chi, b) = \int \prod_{p=1}^n e^{-\sum_{p=1}^n z_p} \prod_{p=1}^n z_p^{\frac{s+a_p}{2}} \frac{dz_1 \dots dz_{r+1}}{z_1 \dots z_{r+1}}$$

$$z_p > 0, \quad z_1 \dots z_n \geq \frac{Nb^2}{A(\chi)}$$

and

$$\Psi(s, \chi) = \frac{(2\pi)^{r_2}}{\sqrt{|d|}} A(\chi)^{\frac{s}{2}} \sum \chi(b) \Gamma(s, \chi, b) N b^{-s}.$$

We know that $\Psi(s, \chi)$ is an integral function which satisfies the formula

$$\bar{\Phi}(s, \chi) = - \frac{2^{r_1+r_2} \pi^{r_2} R h}{w \sqrt{|d|}} \frac{E(\chi)}{s(1-s)} + \Psi(s, \chi) + I(\chi) \bar{\Psi}(1-s, \bar{\chi}) \quad (7)$$

where

$$E(\chi) = \begin{cases} 1 & \text{if } \mathfrak{m} = \sigma, \chi \text{ principal} \\ 0 & \text{otherwise.} \end{cases}$$

The functional equation

$$\bar{\Phi}(s, \chi) = I(\chi) \bar{\Phi}(1-s, \bar{\chi})$$

follows from (7) immediately [7].

The formula (7) can be transformed in the following form:

$$\bar{\Phi}(s, \chi) = - \frac{2^{r_1+r_2} \pi^{r_2} R h}{w \sqrt{|d|}} \frac{E(\chi)}{s(1-s)} + \frac{(2\pi)^{r_2}}{\sqrt{|d|}} (\pi |d|^{-\frac{1}{n}} N \mathfrak{m}^{-\frac{1}{n}})^{\frac{s}{2}}$$

$$\sum_{\substack{\alpha \neq \sigma \\ t_p > 0 \\ N(t) \geq 1}} \int \prod_{p=1}^n (\chi(\alpha) N(t)^{\frac{s}{2}} + I(\chi) \bar{\chi}(\alpha) N(t)^{\frac{1-s}{2}}) N \alpha^{\frac{s}{n}} e^{-\pi |d|^{-\frac{1}{n}} N \alpha^{\frac{2}{n}} S(t)} \frac{dt_1 \dots dt_{r+1}}{t_1 \dots t_{r+1}}, \quad (8)$$

with the abbreviations $N(t) = t_1 \dots t_n$ and $S(t) = t_1 + \dots + t_n$.

The formula (8) is nothing but the Siegel formula in case $E(\chi) = 1$ [6].

If we take σ_0 such that

$$1 - \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \sigma_0.$$

for given $\sigma_1 \leq \sigma_2$, then it follows from (8) that

$$\Phi(s, \chi) + \frac{2^{\tau_1 + \tau_2} \pi^{\tau_2} R^{\tau_2}}{N \sqrt{|d|}} \frac{E(\chi)}{s(1-s)} \ll c(\sigma_0)$$

and

$$(s-1)^{E(\chi)} \zeta(s, \chi) \ll c(\sigma_0) e^{c|t|} N m^{\frac{\sigma_0-1}{2}} \quad (9)$$

for $\sigma_1 \leq \sigma \leq \sigma_2$.

Let $0 < \delta < 1$. Obviously

$$|\zeta(1+\delta+it, \chi)| \leq \zeta_K(1+\delta) \ll \frac{1}{\delta} \quad (10)$$

Put

$$f(s, \chi) = I(\chi) A(\chi)^{\frac{1}{2}-s} 2^{\tau_2(2s-1)} \left\{ \frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})} \right\}^{\tau_1} \left\{ \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})} \right\}^{\tau_2} \left\{ \frac{\Gamma(1-s)}{\Gamma(s)} \right\}^{\tau_2}$$

Then the functional equation can be expressed as

$$\zeta(s, \chi) = f(s, \chi) \zeta(1-s, \bar{\chi}). \quad (11)$$

With the aids of (10) and (11), we get

$$\zeta(-\delta+it, \chi) \ll c(\sigma_1, \sigma_2) N m^{\frac{1}{2}+\delta} |t|^{\frac{1}{2}+\delta} \left\{ 1 + O\left(\frac{1}{|t|}\right) \right\} \quad (12)$$

for $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \geq 1$, since

$$f(s, \chi) \ll c(\sigma_1, \sigma_2) N m^{\frac{1}{2}-\sigma} |t|^{\frac{1}{2}-\sigma} \left\{ 1 + O\left(\frac{1}{|t|}\right) \right\}.$$

Hence, from Theorem 5, we have

$$\zeta(s, \chi) \ll N m^{\frac{1}{2}(1-\sigma)} (|t|+2)^{\frac{n}{2}(1-\sigma)} \log N m (|t|+2)$$

for $0 \leq \sigma \leq 1$, provided $E(\chi)=0$. In applying the theorem we use (9) and (10), (12), putting

$$\delta = \frac{1}{\log N m (|t|+2)}.$$

More precisely we shall get the following result.

Theorem 6. In case $E(\chi)=0$, we have

$$\zeta(s, \chi) \ll \begin{cases} c(\sigma) \{ N m (|t|+2) \}^{\frac{1}{2}-\sigma} \log N m (|t|+2) & \sigma \leq 0 \\ \{ N m (|t|+2) \}^{\frac{1-\sigma}{2}} \log N m (|t|+2) & 0 \leq \sigma \leq 1 \\ \log N m (|t|+2) & 1 \leq \sigma \end{cases}$$

In case $E(\chi)=1$, we can also obtain similar results by using

$$\frac{\delta(1-\delta)}{(\delta+\sigma_0)(1-\delta+\sigma_0)} \zeta_k(\delta)$$

in place of $\zeta_k(s)$, σ_0 being determined satisfying

$$2-\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \sigma_0.$$

In particular, we have

$$\zeta\left(\frac{1}{2}+it, \chi\right) \ll \left\{ N m(|t|+2)^{\frac{1}{4}} \log N m(|t|+2) \right\}.$$

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