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Arithmetic Milnor invariants and multiple power residue symbols in number fields: a précis

By

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Abstract

This is a research announcement of my joint work with Fumiya Amano [AM] concerning a generalization of the Legendre, power residue symbols and the Rédei triple symbol. Such a generalization was firstly discussed by the author at the conference "Algebraic number theory and related topics (2000)" [Mo2]. Afterwards, a significant progress was made by Amano on the 4-tuple quadratic residue symbol over the rationals [A2], and now we are able to introduce n-tuple m-th power residue symbols for primes of a number field.

Introduction

Our work is motivated originally by the works of Gauss, about two hundred years ago, on arithmetic of quadratic residue and topology of linking numbers [G1], [G2].

For distinct odd rational primes p_1 and p_2, the quadratic residue symbol is given by

$$\left(\frac{p_1}{p_2}\right) = \text{Frobenius over } p_2 \text{ in } \mathrm{Gal}(\mathbb{Q}(\sqrt{p_1})/\mathbb{Q}).$$

In view of the analogies between knots and primes [Mo5], it may be regarded as an arithmetic analogue of the mod 2 linking number, $\mathrm{lk}_2(p_1, p_2) \in \mathbb{Z}/2\mathbb{Z}$:

$$\left(\frac{p_1}{p_2}\right) = (-1)^{\mathrm{lk}_2(p_1,p_2)}.$$

In the 19th century, Kummer and Hilbert etc. generalized the quadratic residue symbol to higher power residue symbols in number fields. Let k be a number field

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containing a primitive m-th root of unity $(m \geq 2)$. For distinct principal prime ideals $\mathfrak{p}_1 = (\pi_1)$ and $\mathfrak{p}_2 = (\pi_2)$, prime to m, the m-th power reridue symbol is given as

$$\left(\frac{\mathfrak{p}_1}{\mathfrak{p}_2}\right)_m = \text{Frobenius over } \mathfrak{p}_2 \text{ in } \mathrm{Gal}(k(\sqrt[m]{\pi_1})/k).$$

Note that it is natural to assume $\mathfrak{p}_i$ is principal, namely, "null-homologous" in $\mathrm{Spec}(\mathcal{O}_k)$, in order to define the mod m linking number $\mathrm{lk}_m(\mathfrak{p}_1, \mathfrak{p}_2) \in \mathbb{Z}/m\mathbb{Z}$. It may be noteworthy that Hilbert already discussed the power residue symbol for non-principal ideals in § 154 of his Zahlbericht [H].

In 1939, Rédei introduced the triple symbol aiming to generalize the arithmetic of quadratic fields such as Gauss' theory of genera [R]. For distinct certain rational primes p_1, p_2 and p_3, the Rédei symbol is defined well as

$$[p_1, p_2, p_3]_{\text{Rédei}} = \text{Frobenius over } p_3 \text{ in } \mathrm{Gal}(\mathfrak{R}/\mathbb{Q}),$$

where $\mathfrak{R}$ is determined by p_1, p_2 and given concretely by

$$\mathfrak{R} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}),$$

where $\alpha = x + y\sqrt{p_1}$, $x^2 - p_1 y^2 - p_2 z^2 = 0$ $(x, y, z \in \mathbb{Z})$. Note that $\mathfrak{R}/\mathbb{Q}$ is a dihedral extension of degree 8, unramified outside p_1, p_2 and ∞ with ramification index for each p_i being 2. It might not be clear, however, why such a dihedral extension and triple symbol should be considered as a natural generalization of a quadratic field and the Legendre symbol, and it seemed that his work had been overlooked for a long time.

In the late 1990s, Kapranov and the author independently interpreted the Rédei symbol as an arithmetic analogue of a triple linking number for a link [Mi]. For example, the primes 13, 61 and 937 are linked like the Borromean ring [V].

Further the author introduced arithmetic analogues for rational primes of the Milnor invariants (higher order linking numbers) for a link in the 3-sphere [Mo1]-[Mo5]. For example, the mod 2 arithmetic Milnor invariant

$$\mu_2(12 \cdots n) \in \mathbb{Z}/2\mathbb{Z}$$

for certain rational primes $p_1, p_2, \ldots, p_n$ describes the decomposition law of p_n in a certain extension $K(n)/\mathbb{Q}$, determined by $p_1, \ldots, p_{n-1}$, which has the following property:

- $K(n)/\mathbb{Q}$ is unramified outside $p_1, \ldots, p_{n-1}$ and ∞ with ramification index for each p_i being 2, and the Galois group $\mathrm{Gal}(K(n)/\mathbb{Q}) \simeq N_n(\mathbb{F}_2)$, where $N_n(R)$ stands for the group of n by n upper-triangular unipotent matrices over a commutative ring R. In particular, we have

$$(-1)^{\mu_2(12)} = \left(\frac{p_1}{p_2}\right), \quad (-1)^{\mu_2(123)} = [p_1, p_2, p_3]_{\text{Rédei}},$$

and further

$$K(2) = \mathbb{Q}(\sqrt{p_1}), \quad K(3) = \mathfrak{R}$$

for $p_i \equiv 1 \bmod 4$ ([A1], [Mo1]-[Mo4]). This unified interpretation may tell us that Rédei's dihedral extension and triple symbol would be a natural generalization of a quadratic field and the Legendre symbol.

The idea is based on an analogy

$$\begin{array}{ccc} \pi_1^{\text{ét}}(\text{Spec}(\mathbb{Z}) \setminus \{p_1, \ldots, p_n\})(2) & \longleftrightarrow & \pi_1(\mathbb{R}^3 \setminus \{K_1, \ldots, K_n\}) \\ \text{pro-2 Galois group} & & \text{link group,} \end{array}$$

where $\pi_1^{\text{ét}}(\,\cdot\,)(l)$ means the maximal pro-l quotient for a prime number l of the étale fundamental group $\pi_1^{\text{ét}}(\,\cdot\,)$. Using this analogy, we pursue the analogies in arithmetic of the theory of Milnor invariants in link theory.

Since the Milnor invariants are defined for a link in a homology 3-sphere [T], it would be natural to ask the following

Question: Can we extend mod 2 arithmetic Milnor invariants to mod m arithmetic Milnor invariants for a finite set S of finite primes in a number field k which contains a primitive m-th root of unity and whose class group $H_k = 1$?

However, we are faced immediately with the following difficulty: There is an obstruction B_S for the analogy

$$(\star) \qquad \pi_1^{\text{ét}}(\text{Spec}(\mathcal{O}_k) \setminus S)(l) \quad \longleftrightarrow \quad \pi_1(M \setminus \mathcal{L}),$$

where l is a certain prime number and $\mathcal{L}$ is a link in a 3-manifold M. We note that the osbtruction B_S is closely related to the group of units $\mathcal{O}_k^{\times}$.

Despite of this difficulty, we have the following

Result (rough form): Suppose that m is a power of l and that k contains a primitive m-th root of unity ζ_m and the l-class group $H_k(l) = 1$. Then we can introduce mod m arithmetic Milnor invariants $\mu_m(12\cdots n) \in \mathbb{Z}/m\mathbb{Z}$ for a certain set $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of finite primes of k and the n-tuple m-th power residue symbol in the manner

$$[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m = \zeta_m^{\mu_m(1\ldots n)}.$$

The idea is simple. Namely, we enlarge S so that the obstruction vanishes and we have the analogy ($\star$), and then we show our invarians are independent of a choice of added auxiliary primes.

Throughout this article, we shall use the following

Notations: $l :=$ a fixed prime number, $m :=$ a fixed power of l.
$k =$ a finite algebraic number field such that k contains a fixed primitive m-th root of unity ζ_m and the l-class group $H_k(l) = 1$.
$S := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ = a finite set of finite primes $\mathfrak{p}_i$ of k with $(\mathfrak{p}_i, l) = 1$. Note that $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod m$ $(1 \le i \le n)$.
$k_{\mathfrak{p}_i} :=$ the $\mathfrak{p}_i$-adic completion of k.
$S_\infty :=$ the set of infinite primes of k.
$l_S := \max\{l^e \mid \mathrm{N}\mathfrak{p}_i \equiv 1 \bmod l^e \text{ for } i = 1, \ldots, n\}$.
$k_S(l) :=$ the maximal pro-l Galois extension of k, unramified outside $S \cup S_\infty$.
$G_{k,S}(l) := \mathrm{Gal}(k_S(l)/k) = \pi_1^{\text{ét}}(\mathrm{Spec}(\mathcal{O}_k) \setminus S)(l)$.

§ 1. Obstructions and pro-l Galois groups of link type

The *obstruction* B_S is defined by the $\mathbb{F}_l$-vector space

$$B_S := \{a \in k^\times \mid (a) = \mathfrak{a}^l, a \in (k_\mathfrak{p}^\times)^l \text{ for } \mathfrak{p} \in S \cup S_\infty\}/(k^\times)^l.$$

The following theorem is due to Koch.

Theorem 1.1 ([K, 11.4]). *Assume $B_S = 1$. Then we have*

$$G_{k,S}(l) = \langle x_1, \ldots, x_n \mid x_i^{\mathrm{N}\mathfrak{p}_i - 1}[x_i, y_i] = 1 \ (i = 1, \ldots, n)\rangle,$$

where x_i is a word representiong a monodromy over $\mathfrak{p}_i$ and y_i is a pro-l word of x_i's representing a Frobenius automorphism over $\mathfrak{p}_i$.

The obstruction is closely related to the group of units $\mathcal{O}_k^\times$. We let

$$\mathcal{E}_k := \{\mathfrak{p} : \text{ finite prime of } k \mid (\mathfrak{p}, l) = 1, \mathfrak{p} \text{ is inert in } k(\sqrt[l]{\mathcal{O}_k^\times})/k\},$$

which is an infinite set by the Chebotarev density theorem.

Proposition 1.2 ([AM]). *If S contains a prime in $\mathcal{E}_k$, then $B_S = 1$.*

Example 1.3. (1) Let $k = \mathbb{Q}$ and $l = 2$. Then we have $B_S = 1$ for any S.
(2) Let $k = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ and $l = 3$. Then $B_S = 1$ if and only if S contains a prime $\mathfrak{p}$ such that $\mathrm{N}\mathfrak{p} \equiv 4$ or $7 \bmod 9$.

§ 2. Arithmetic Milnor invariants and multiple power residue symbols

By Theorem 1.1 and Proposition 1.2, we can enlarge S to $T = S \cup \{\mathfrak{p}_{n+1}, \ldots, \mathfrak{p}_t\}$ so that $B_T = 1$ and

$$G_{k,T}(l) = \langle x_1, \ldots, x_t \mid x_i^{\mathrm{N}\mathfrak{p}_i - 1}[x_i, y_i] = 1 \ (i = 1, \ldots, t) \rangle = F/N,$$

where F is the free pro-l group on the words $x_1, \ldots, x_t$ and N is the closed subgroup of F generated normally by $x_i^{\mathrm{N}\mathfrak{p}_i - 1}[x_i, y_i]$ $(i = 1, \ldots, t)$. We let

$$\Theta : F \longrightarrow \mathbb{Z}/m\mathbb{Z}\langle\langle X_1, \ldots, X_t \rangle\rangle^{\times}$$

be the mod m Magnus expansion defined by $\Theta(x_i) := 1 + X_i$ for $1 \le i \le t$. We write, for $f \in F$,

$$\Theta(f) = 1 + \sum_{1 \le i_1, \ldots, i_r \le t} \mu_m(i_1 \cdots i_r; f) X_{i_1} \cdots X_{i_r}$$

and set

$$\mu_m(i_1 \cdots i_a) := \mu_m(i_1 \cdots i_{a-1}; y_{i_a}) \quad (a > 1),$$

and $\mu_m(i) := 0$ $(1 \le i \le t)$. Define $\Delta_m(1 \cdots n)$ by the ideal of $\mathbb{Z}/m\mathbb{Z}$ generated by $\mu_m(J)$ (multi-indices J running over all cyclic permutations of proper subsequence of $1 \cdots n$) and the binomial coefficients $\binom{l_S}{a}$ $(1 \le a < n)$, and we let

$$\overline{\mu}_m(1 \cdots n) := \mu_m(1 \cdots n) \bmod \Delta_m(1 \cdots n).$$

Theorem 2.1 ([AM]). *Suppose $2 \le n < l_S$. Then $\overline{\mu}_m(1 \cdots n)$ is an invariant of determined by the n-tuple $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ and m, more precisely, it is independent of choices of a monodromy and an extension of Frobenius over $\mathfrak{p}_i$ $(1 \le i \le t)$ and is independent of a choice of T.*

We call $\overline{\mu}_m(1 \cdots n)$ the mod m *arithmetic Milnor invariant* of the n-tuple $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

In the following, we assume for simplicity $\binom{l_S}{a} \equiv 0 \bmod m$ $(1 \le a < n)$ and $\mu_m(j_1 \cdots j_a) = 0$ for any proper subset $\{j_1, \ldots, j_a\}$ of $\{1, \ldots, n\}$.

We then define the *n-tuple m-th power residue symbol* by

$$[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m := \zeta_m^{\mu_m(1 \cdots n)}.$$

We will discuss some properties of our multiple power residue symbols. We define a homomorphism

$$\rho_{m,n} : F \longrightarrow N_n(\mathbb{Z}/m\mathbb{Z})$$

by

$$\rho_{m,n}(f) := \begin{pmatrix} 1 & \mu_m(1;f) & \mu_m(12;f) & \cdots & \mu_m(1\cdots n-1;f) \\ 0 & 1 & \mu_m(2;f) & \cdots & \mu_m(2\cdots n-1;f) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \mu_m(n-1;f) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

It is easily shown that $\rho_{m,n}$ factors through the Galois group $G_{k,T}(l)$.

Theorem 2.2 ([AM]). *Let $K_m(n)$ be the subfield of $k_T(l)$ fixed by* $\mathrm{Ker}(\rho_{m,n})$*. Then the followings hold.*
(1) *$K_m(n)$ depends only on S.*
(2) *$K_m(n)/k$ is unramified outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n-1}$ and S_∞ with ramification index for each $\mathfrak{p}_i$ being m, and* $\mathrm{Gal}(K_m(n)/k) \simeq N_n(\mathbb{Z}/m\mathbb{Z})$*.*
(3) *$\mathfrak{p}_n$ is completely decomposed in $K_m(n)/k$ if and only if* $[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m = 1$.

In view of Theorem 2.2, the following problem is important in the arithmetic of multiple power residue symbols.

Problem 2.3. (i) Does the property (2) of Theorem 2.2 characterize $K_m(n)$ uniquely ?
(ii) Can one construct $K_m(n)$ in a concrete manner ?

Example 2.4. (1) Let $n = 2$. The classical Kummer and Hibert theory answers Problem 2.3 affirmatively and we have

$$[\mathfrak{p}_1, \mathfrak{p}_2]_m = \left(\frac{\pi_1}{\mathfrak{p}_2}\right)_m \quad \text{for } \mathfrak{p}_1 = (\pi_1).$$

(2) Let $k = \mathbb{Q}$, $l = m = 2$ and $n = 3$. Amano [A1] answers Problem 2.3 affirmatively so that $K_2(3)$ coincides with Rédei's dihedral extension $\mathfrak{R}$, and we have

$$[p_1, p_2, p_3]_2 = [p_1, p_2, p_3]_{\text{Rédei}}.$$

(3) (Triple cubic residue symbol) This is a new case where $k = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$, $l = m = 3$ and $n = 3$. Let $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ with $\mathrm{N}\mathfrak{p}_i \equiv 1 \bmod 9$. Then there is $\pi_i \in \mathbb{Z}[\zeta_3]$ uniquely such that $\mathfrak{p}_i = (\pi_i)$ and $\pi_i \equiv 1 \bmod (1-\zeta_3)^3$. Assume that

$$\left(\frac{\pi_i}{\pi_j}\right)_3 = 1 \ \ (i \neq j).$$

By Theorem 2.2, the extension $K_3(3)/k$ is a mod 3 Heisenberg extension of degree 27, unramified outside $\mathfrak{p}_1, \mathfrak{p}_2$ with ramification index for each prime $\mathfrak{p}_i$ being 3. Then we have the following theorem.

Theorem 2.5 ([AM]). (1) *The property* (2) *of Theorem* 2.2 *with* $m = n = 3$ *characterizes* $K_3(3)$ *uniquely and we have*

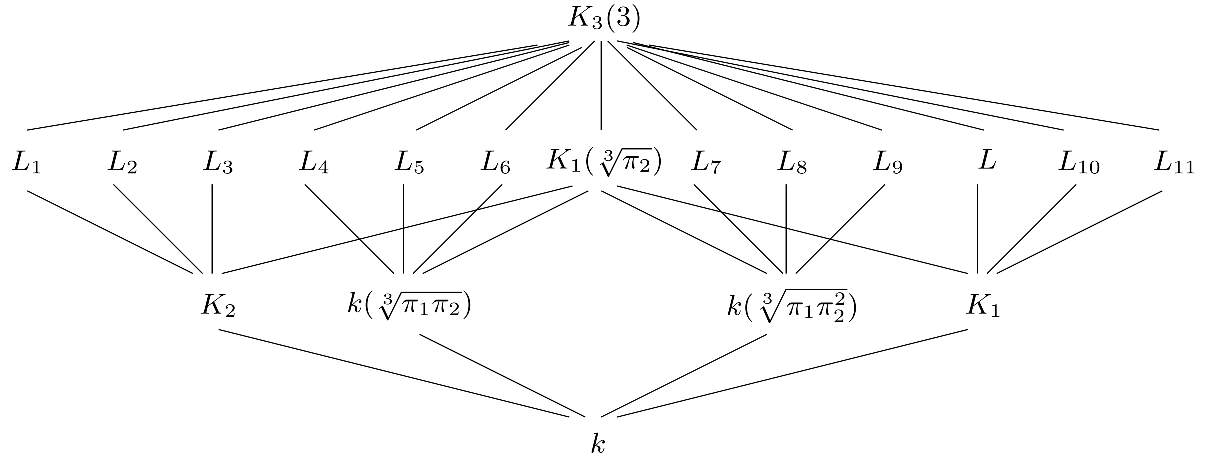
$$K_3(3) = k(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2}, \sqrt[3]{\theta}),$$

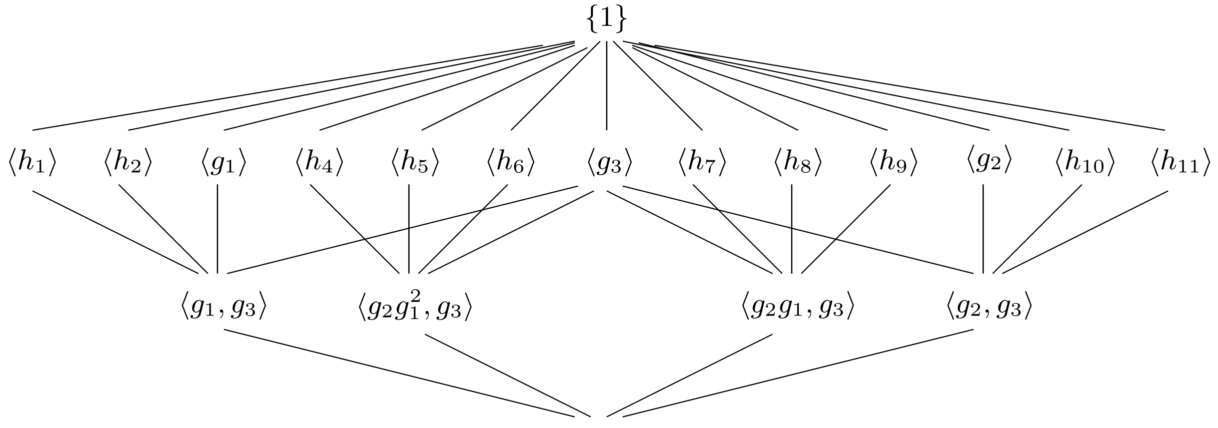
where $\theta = x + y\sqrt[3]{\pi_1} + z(\sqrt[3]{\pi_1})^2$ *is an algebraic integer in* $k(\sqrt[3]{\pi_1})$ *satisfying* $x^3 + \pi_1 y^3 + \pi_1^2 z^3 - 3\pi_1 xyz - \pi_2^3 w^2 = 0$ $(x, y, z, w \in \mathbb{Z}[\zeta_3])$. (For more detailed requirements of θ, refer to [AM; Section 4])
(2) *We have*

$$[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \frac{\mathrm{Frob}_{\mathfrak{p}_3}(\sqrt[3]{\theta})}{\sqrt[3]{\theta}},$$

where $\mathrm{Frob}_{\mathfrak{p}_3}$ *is an extension of Frobenius automorphism over* $\mathfrak{p}_3$ *in* $K_3(3)/k$.

All subgroups of $\mathrm{Gal}(K_3(3)/k)$ and the corresponding intermidiate fields are illustrated as follows.





$$N_3(\mathbb{F}_3) = \left\langle g_1, g_2, g_3 \,\middle|\, \begin{array}{l} g_1^3 = g_2^3 = g_3^3 = 1 \\ g_2 g_1 = g_3 g_2 g_1,\ g_3 g_1 = g_1 g_3,\ g_3 g_2 = g_2 g_3 \end{array} \right\rangle,$$

$$g_1 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 0\,0\,1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1\,1\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} (= [g_2, g_1]).$$

$$\begin{array}{l} g_1 : (\sqrt[3]{\pi_1},\, \sqrt[3]{\pi_2},\, \sqrt[3]{\theta_1},\, \sqrt[3]{\theta_2},\, \sqrt[3]{\theta_3}) \mapsto (\zeta_3 \sqrt[3]{\pi_1},\, \sqrt[3]{\pi_2},\, \sqrt[3]{\theta_2},\, \sqrt[3]{\theta_3},\, \sqrt[3]{\theta_1}), \\ g_2 : (\sqrt[3]{\pi_1},\, \sqrt[3]{\pi_2},\, \sqrt[3]{\theta_1},\, \sqrt[3]{\theta_2},\, \sqrt[3]{\theta_3}) \mapsto (\sqrt[3]{\pi_1}, \zeta_3 \sqrt[3]{\pi_2},\, \sqrt[3]{\theta_1}, \zeta_3^2 \sqrt[3]{\theta_2}, \zeta_3 \sqrt[3]{\theta_3}), \\ g_3 : (\sqrt[3]{\pi_1},\, \sqrt[3]{\pi_2},\, \sqrt[3]{\theta_1},\, \sqrt[3]{\theta_2},\, \sqrt[3]{\theta_3}) \mapsto (\sqrt[3]{\pi_1},\, \sqrt[3]{\pi_2}, \zeta_3^2 \sqrt[3]{\theta_1}, \zeta_3^2 \sqrt[3]{\theta_2}, \zeta_3^2 \sqrt[3]{\theta_3}), \end{array}$$

where $\theta_1 := \theta, \theta_2 := g_1(\theta), \theta_2 := g_1^2(\theta)$.

$$\begin{array}{c} h_1 = g_1 g_3,\ \ h_2 = g_1 g_3^2,\ \ h_4 = g_2 g_1^2,\ \ h_5 = g_2 g_1^2 g_3,\ \ h_6 = g_2 g_1^2 g_3^2, \\ h_7 = g_2 g_1,\ \ h_8 = g_2 g_1 g_3,\ \ h_9 = g_2 g_1 g_3^2,\ \ h_{10} = g_2 g_3,\ \ h_{11} = g_2 g_3^2. \end{array}$$

§ 3. Massey products

It is known that Milnor invariants of a link are interpreted as Massey products in the cohomology of the link complement [T]. Similarly, we can interpret our multiple power residue symbols as Massy products in the étale cohomology of the complement $\mathfrak{X} := \mathrm{Spec}(\mathcal{O}_k) \setminus T$, where T is, as in §2, a finite set containing S such that $B_T = 1$. This is a generalization of the cup product interpretation of the power residue symbol [S]. We keep the same notations as in Section 2.

In order to define the Massey product structure in étale cohomology, we use Verdier's construction of étale cohomology presented in [AGV; Exposé V]. Let $U_\bullet$ be a hypercovering on $\mathfrak{X}_{\text{ét}}$ and let $C^j := C^j(U_\bullet, \mathbb{Z}/m\mathbb{Z})$ be the Čech j-cochains ($j \geq 0$) associated to $U_\bullet$ with coefficients in the constant sheaf $\mathbb{Z}/m\mathbb{Z}$. Consider the differential graded algebra ($C^\bullet := \oplus_{j \geq 0} C^j, d^\bullet$) equipped with multiplication given by the Alexander-Whiteney cup product $\cup$. Then, by the general procedure to define Massey products [Ma], we have the Massey product structure on the étale cohomology $H^\bullet(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}) = \varinjlim H^*(C^\bullet(U_\bullet, \mathbb{Z}/m\mathbb{Z}))$, where the limit is taken over the homotopy category of hypercoverings $U_\bullet$.

Let us explain concretely the triple Massey product concerning Example 2.4 (3). For the general case, we refer to [AM; Section 5]. Let $\chi_i \in H^1(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ be the Kronecker dual to the monodromies $x_j \in G_{k,T}(3)$, namely, $\chi_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq \#T$. By the assumption $\left(\dfrac{\pi_i}{\pi_j}\right)_3 = 1 \;\; (i \neq j)$, we have $\chi_{12}, \chi_{23} \in C^1$ such that

$$\chi_1 \cup \chi_2 = d\chi_{12}, \;\; \chi_2 \cup \chi_3 = d\chi_{23}.$$

Then the triple Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle \in H^2(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ is defined by the cohomology class of the 2-cocycle

$$\chi_1 \cup \chi_{23} + \chi_{12} \cup \chi_3.$$

On the other hand, let $\delta_3 \in H_2(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$ be the image of the canonical generator of $H_2(\mathrm{Spec}(k_{\mathfrak{p}_3})_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$, which represents a "boundary of a tubular neighborhood" of $\mathrm{Spec}(\mathcal{O}_k/\mathfrak{p}_3)$, under the natural homomorphism $H_2(\mathrm{Spec}(k_{\mathfrak{p}_3})_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}) \to H_2(\mathfrak{X}_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$.

Theorem 3.1 ([AM]). $[\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3]_3 = \zeta_3^{-\langle \chi_1, \chi_2, \chi_3 \rangle (\delta_3)}$.

Finally we present an important problem left unsolved:

Problem (reciprocty law). Study the behavior of $[\mathfrak{p}_1, \ldots, \mathfrak{p}_n]_m$ under permutations of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

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