

## ON SCHWARZ LEMMA FOR THE HALF PLANE

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**ABSTRACT.** The object of the present paper is to derive the properties for functions which are analytic in the upper half plane.

**1. INTRODUCTION.** Let  $D$  denote the upper half plane which is given as

$$D = \{z: z \in \mathbb{C}, \operatorname{Im}(z) > 0\}.$$

For  $\phi(z)$  which are analytic in  $D$ , Stankiewicz and Stankiewicz [2] (also Raducanu and Pascu [1]) have shown that

**THEOREM A.** Let  $\phi: D \longrightarrow D$  be analytic in  $D$ . If

$$(1.1) \quad \lim_{D \ni z \rightarrow \infty} (\phi(z) - z) = 0,$$

then

$$(1.2) \quad \operatorname{Im}(\phi(z)) \geq \operatorname{Im}(z) \quad (z \in D).$$

If there exists a point  $z_0 \in D$  such that  $\operatorname{Im}(\phi(z_0)) = \operatorname{Im}(z_0)$ , then  $\phi(z) = z + \alpha$ , where  $\alpha \in \mathbb{R}$  (the set of all real numbers).

In this paper, we obtain similar properties of functions  $\phi(z)$ .

**2. MAIN RESULTS.** Our main theorem of this paper is contained in

**THEOREM 1.** Let  $\phi: D \longrightarrow D$  be analytic in  $D$ . If, for all  $\varepsilon > 0$ , there exists  $r = r(\varepsilon)$  such that

$$(2.1) \quad \operatorname{Im}(\phi(z+i\varepsilon)) \geq \operatorname{Im}(z) \quad (z \in D_r^-),$$

with

$$D_r^- = \{z: \operatorname{Im}(z) \geq 0, |z| \geq r\},$$

then

$$(2.2) \quad \operatorname{Im}(\phi(z)) \geq \operatorname{Im}(z) \quad (z \in D).$$

If there exists a point  $z_0 \in D$  such that  $\operatorname{Im}(\phi(z_0)) = \operatorname{Im}(z_0)$ , then  $\phi(z) = z + \alpha$ , where  $\alpha \in \mathbb{R}$ .

PROOF. Let  $\varepsilon$  be a positive real number,  $g: D \rightarrow D$  be the function defined by  $g(z) = \phi(z+i\varepsilon)$ , and let

$$D_r^+ = \{z: \operatorname{Im}(z) \geq 0, |z| \leq r\},$$

where  $r = r(\varepsilon)$ . If  $\operatorname{Im}(g(z_0) - z_0) = \min_{z \in D_r^+} \operatorname{Im}(g(z) - z)$ , then  $z_0 \in \partial D_r^+$  and

$$(2.3) \quad \operatorname{Im}(g(z) - z) \geq \operatorname{Im}(g(z_0) - z_0) \quad (z \in D_r^+).$$

There are two possible cases:

(i)  $\operatorname{Im}(z_0) = 0$  and hence  $\operatorname{Im}(g(z) - z) \geq \operatorname{Im}(g(z_0)) > 0$ ,

(ii)  $\operatorname{Im}(z_0) > 0$ . In this case, because  $|z_0| = r$  and from the hypothesis we have  $\operatorname{Im}(g(z_0) - z_0) \geq 0$ .

Therefore, in both cases, we obtain

$$\operatorname{Im}(g(z)) \geq \operatorname{Im}(z) \quad (z \in D_r^+).$$

If  $\varepsilon \rightarrow 0$  (and hence  $r = r(\varepsilon) \rightarrow \infty$ ), this shows that  $\operatorname{Im}(\phi(z)) \geq \operatorname{Im}(z)$  in  $D$ .

If there exists a point  $z_0 \in D$  such that  $\operatorname{Im}(\phi(z_0) - z_0) = 0$ , then the function  $\operatorname{Im}(\phi(z) - z)$  is constant 0 in  $D$ . Hence  $\phi(z) - z = \alpha$ ,  $\alpha \in \mathbb{R}$ .

This completes the proof of Theorem 1.

Next we derive

**THEOREM 2.** If the function  $\phi: D \rightarrow D$  is analytic in  $D$ , and

$$(2.4) \quad \lim_{z \rightarrow \infty} \operatorname{Im}(\phi(z) - z) \geq 0 \quad (z \in D),$$

then  $\operatorname{Im}(\phi(z)) \geq \operatorname{Im}(z)$  for all  $z \in D$ . If there exists a point  $z_0 \in D$  such that  $\operatorname{Im}(\phi(z_0)) = \operatorname{Im}(z_0)$ , then  $\phi(z) = z + \alpha$ , where  $\alpha \in \mathbb{R}$ .

PROOF. Replacing  $z$  by  $z + i\varepsilon$  in (2.4), where  $\varepsilon > 0$ , we have

$$\lim_{z \rightarrow \infty} \operatorname{Im}(g(z) - z) \geq \varepsilon > 0 \quad (z \in D).$$

Applying the definition of  $g(z)$  in Theorem 1, we see the condition in Theorem 2 is satisfied.

REMARKS. (i) If  $\lim_{z \rightarrow \infty} (\phi(z) - z) = 0$  ( $z \in D$ ), then

$\lim_{z \rightarrow \infty} \operatorname{Im}(\phi(z) - z) \geq 0$  ( $z \in D$ ). Therefore, Theorem A is a consequence of Theorem 2.

(ii) Functions  $\phi_1(z) = z + \log z$  (we choose the principal branch for  $\log z$ ) and  $\phi_2(z) = z + e^{\varepsilon z} + i$  satisfy the conditions in Theorem 1, but not satisfy the conditions in Theorem A.

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## REFERENCES

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