

Integration by parts formula for Feynman path integrals

By

Daisuke FUJIWARA*

Abstract

The aim of this paper is to present

1. Review of time slicing approximation method of Feynman path integrals introduced by Feynman [4].
2. An integration by parts formula for Feynman path integrals under suitable assumption:

$$\begin{aligned} \int_{\Omega_{x,y}} DF(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma) &= - \int_{\Omega_{x,y}} F(\gamma)\text{Div } p(\gamma)e^{i\nu S(\gamma)}\mathcal{D}(\gamma) \\ &\quad - i\nu \int_{\Omega_{x,y}} F(\gamma)DS(\gamma)[p(\gamma)]e^{i\nu S(\gamma)}\mathcal{D}(\gamma). \end{aligned}$$

This formula is an analogy to Elworthy's integration by parts formula for Wiener integrals. cf. [3]

3. An application of integration by parts formula to semiclassical asymptotic formula which holds in the case of $F(\gamma^*) = 0$. Here γ^* is the stationary point of the phase $S(\gamma)$, i.e., $\delta S(\gamma^*) = 0$.

Contents

- § 1. Path integral defined by Feynman
- § 2. Some properties of classical action
- § 3. Stationary phase method for integrals over a space of large dimension
- § 4. Convergence of Feynman path integral

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*Department of Mathematics, Gakushuin University, 1-5-1 Mejiro Toshima-ku, Tokyo 171-8588, Japan.

- § 5. An integration by parts formula
 - § 5.1. Some operators of trace class
 - § 5.2. Admissible vector field
 - § 5.3. m -smooth functional
 - § 5.4. An integration by parts formula
- § 6. Application to semiclassical asymptotic behaviour of Feynman path integrals
- References

§ 1. Path integral defined by Feynman

For simplicity we restrict ourselves to the case where the configuration space is $\mathbf{R}^1$. In this case Lagrangian function with potential $V(t, x)$ is

$$L(t, \dot{x}, x) = \frac{1}{2} \dot{x}^2 - V(t, x).$$

The case where non zero magnetic potential is present is discussed in [11]. Action of path $\gamma : [s, s'] \rightarrow \mathbf{R}$ is

$$(1.1) \quad S(\gamma) = \int_s^{s'} L(t, \dot{\gamma}(t), \gamma(t)) dt.$$

We assume throughout this paper the following assumption for potential $V(t, x)$ cf. W.Pauli [14]:

Assumption 1.1. 1. $V(t, x)$ is a real continuous function of (t, x) . If t is fixed, then it is a function of class C^∞ in x .

2. For $\forall m \geq 0$ there exists $v_m \geq 0$ such that

$$\max_{|\alpha|=m} \sup_{(t,x) \in [s,s'] \times \mathbf{R}^d} |\partial_x^\alpha V(t, x)| \leq v_m (1 + |x|)^{\max\{2-m, 0\}}.$$

We write $\mathcal{H}$ for the L^2 -Sobolev space $H^1(s, s')$ of order 1 in $[s, s']$. For any $x, y \in \mathbf{R}$ we write $\mathcal{H}_{x,y}$ for the closed subset $\{\gamma \in H^1(s, s'); \gamma(s) = y, \gamma(s') = x\}$ of $\mathcal{H}$. If $x = 0$ and $y = 0$ we write $\mathcal{H}_0$ for $\mathcal{H}_{0,0}$. It is clear that action $S(\gamma)$ (1.1) is well defined for $\gamma \in \mathcal{H}$ under the Assumption 1.1.

Proposition 1.2. *Let $\delta_0 > 0$ be so small that*

$$(1.2) \quad \frac{\delta_0^2 v_2}{8} < 1.$$

If $|s' - s| \leq \delta_0$, then for any $x, y \in \mathbf{R}$ there exists one and only path $\gamma^ \in \mathcal{H}_{x,y}$ such that*

$$S(\gamma^*) = \min_{\gamma \in \mathcal{H}_{x,y}} S(\gamma).$$

γ^* is the classical path, i.e. the first variation $\delta S(\gamma^*)$ of $S(\gamma)$ at γ^* vanishes:

$$\delta S(\gamma^*) = 0, \quad \gamma_0(s) = y, \quad \gamma_0(s') = x.$$

It is of class $C^2[s, s']$ and satisfies Euler equation:

$$\begin{aligned} \frac{d^2}{dt^2} \gamma(t) + \partial_x V(t, \gamma(t)) &= 0, \\ \gamma(s') &= x, \quad \gamma(s) = y. \end{aligned}$$

We define

$$(1.3) \quad S(s', s, x, y) = S(\gamma^*).$$

This is called classical action.

Let Δ be an arbitrary division of the interval $[s, s']$ such that

$$(1.4) \quad \Delta : s = T_0 < T_1 < \dots < T_J < T_{J+1} = s'.$$

We set $\tau_j = T_j - T_{j-1}$, $j = 1, 2, \dots, J+1$ and $|\Delta| = \max_{1 \leq j \leq J+1} \tau_j$.

Suppose that $|\Delta| \leq \delta$. We set $x_0 = y$, $x_{J+1} = x$. For all $x_j \in \mathbf{R}$, $j = 1, 2, \dots, J$, there exists one and only one piecewise classical path $\gamma_\Delta(t)$ which is the classical path for $T_{j-1} \leq t \leq T_j$ and satisfies

$$(1.5) \quad \gamma_\Delta(T_j) = x_j, \quad (j = 0, 1, 2, \dots, J+1).$$

γ_Δ may have edges at T_j . We use the symbol $\gamma_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ to express the piecewise classical path satisfying (1.5), when we want to express explicitly its dependence on $(x_{J+1}, x_J, \dots, x_1, x_0)$.

If Δ and x, y are given then we write $\Gamma_{x,y}(\Delta)$ for the totality of all piecewise classical path $\gamma_\Delta \in \mathcal{H}_{x,y}$. We write $\Gamma_0(\Delta)$ for $\Gamma_{0,0}(\Delta)$. By the map

$$(1.6) \quad \Gamma(\Delta) \ni \gamma_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) \rightarrow (x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$$

we can identify $\Gamma(\Delta)$ and $\mathbf{R}^{J+2}$. Similarly $\Gamma_{x,y}(\Delta)$ is identified with $\mathbf{R}^J$.

Given a functional $F(\gamma)$, we often abbreviate $F(\gamma_\Delta)$ as F_Δ . Once Δ is fixed, it is a function of $(x_{J+1}, x_J, \dots, x_1, x_0)$ and we denote the dependence of $F(\gamma_\Delta)$ on $(x_{J+1}, x_J, \dots, x_1, x_0)$ by writing $F(\gamma_\Delta) = F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$.

Feynman's formulation of path integral. Let $\nu = 2\pi\hbar^{-1}$, where $\hbar$ is Planck's constant. And let Ω_{xy} be the space¹ of paths starting y at time s and reaching x at time s' . Given a functional $F(\gamma)$ of $\gamma \in \Omega_{xy}$, Feynman [4] considered the following integral on finite dimensional space:

$$(1.7) \quad \begin{aligned} I[F_\Delta](\Delta; \nu, s', s, x, y) \\ = \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2} \int_{\mathbf{R}^J} F(\gamma_\Delta)(x, x_J, \dots, x_1, y) \times e^{i\nu S(\gamma_\Delta)(x, x_J, \dots, x_1, y)} \prod_{j=1}^J dx_j. \end{aligned}$$

¹In this note Ω is a symbol which expresses vaguely notion of path space.

Feynman defined his path integral by the formula:

$$(1.8) \quad \int_{\Omega_{xy}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma] = \lim_{|\Delta| \rightarrow 0} I[F_\Delta](\Delta; \nu, s', s, x, y).$$

The integral $I[F_\Delta](\Delta; \nu, s', s, x, y)$ of (1.7)² is called time slicing approximation of Feynman path integral (1.8). We say $F(\gamma)$ is "F-integrable", if the limit on the right hand side of (1.8) exists.

The main aim of Feynman's paper [4] is the statement that the path integral (1.8) with $F \equiv 1$ and s' replaced by t is the fundamental solution of Schrödinger's equation

$$(1.9) \quad \frac{i}{\nu} \partial_t u(t, x) = H(t) u(t, x) \quad (t \in (s, s')),$$

where $H(t) = \frac{1}{2}(-\frac{i}{\nu} \partial_x)^2 + V(t, x)$ is the Hamiltonian operator.

§ 2. Some properties of classical action

From now on we always assume

$$(2.1) \quad |s' - s| \leq \delta_0.$$

Calculation shows:

Proposition 2.1. *If $|s' - s| \leq \delta$, $S(s', s, x, y)$ is of the following form:*

$$S(s', s, x, y) = \frac{|x - y|^2}{2(s' - s)} + (s' - s)\phi(s', s, x, y).$$

The function $\phi(s', s, x, y)$ is a function of (s', s, x, y) of class C^1 and $\exists C > 0$ such that

$$|\phi(s', s, x, y)| \leq C(1 + |x|^2 + |y|^2).$$

Moreover, if s' and s are fixed, $\phi(s', s, x, y)$ is a C^∞ function of (x, y) and for $\forall m \geq 2$ we have

$$\max_{2 \leq |\alpha| + |\beta| \leq m} \sup_{(x, y) \in \mathbf{R}^2} |\partial_x^\alpha \partial_y^\beta \phi(s', s, x, y)| = \kappa_m < \infty.$$

In particular,

$$\kappa_2 \leq \frac{v_2}{2} \left(1 - \frac{v_2 \delta_0^2}{8} \right)^{-1}.$$

Let Δ be the division of time interval $[s, s']$ as (1.4).

We discuss time slicing approximation of path integral.

$$(2.2) \quad I[F_\Delta](\Delta; \nu, s', s, x, y) = \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2} \int_{\mathbf{R}^J} F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) e^{i\nu S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)} \prod_{j=1}^J dx_j.$$

²For fixed Δ the integral (1.7) does not converge absolutely even in the case $F(\gamma) \equiv 1$. We regard (1.7) as an oscillatory integral.

Here $S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ is an abbreviation of $S(\gamma_\Delta)(x_{J+1}, x_J, \dots, x_1, x_0)$. We also abbreviate $S(T_j, T_{j-1}, x_j, x_{j-1})$ to $S_j(x_j, x_{j-1})$ and $\phi(T_j, T_{j-1}, x_j, x_{j-1})$ to $\phi_j(x_j, x_{j-1})$. Thus

$$S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) = \sum_{j=1}^{J+1} S_j(x_j, x_{j-1}) = \sum_{j=1}^{J+1} \left(\frac{|x_j - x_{j-1}|^2}{2\tau_j} + \tau_j \phi_j(x_j, x_{j-1}) \right).$$

Consider $J \times J$ matrix Ψ whose (j, k) element is

$$\Psi_{jk} = \partial_{x_j} \partial_{x_k} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) \quad (j, k = 1, 2, \dots, J).$$

Then we divide the matrix Ψ into two parts.

$$\Psi = H_\Delta + W_\Delta,$$

where

$$H_\Delta = \begin{pmatrix} \frac{1}{\tau_1} + \frac{1}{\tau_2} & -\frac{1}{\tau_2} & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{\tau_2} & \frac{1}{\tau_2} + \frac{1}{\tau_3} & -\frac{1}{\tau_3} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{\tau_3} & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \vdots & -\frac{1}{\tau_J} \\ 0 & 0 & 0 & \dots & 0 & -\frac{1}{\tau_J} & \frac{1}{\tau_J} + \frac{1}{\tau_{J+1}} \end{pmatrix}$$

and W_Δ is the matrix whose (j, k) element is

$$(2.3) \quad w_{jk} = \begin{cases} \partial_{x_j}^2 (\tau_j \phi_j + \tau_{j+1} \phi_{j+1}) & \text{if } j = k \\ \partial_{x_k} \partial_{x_j} \tau_j \phi_j & \text{if } k = j - 1 \\ \partial_{x_k} \partial_{x_j} \tau_{j+1} \phi_{j+1} & \text{if } k = j + 1 \\ 0 & \text{if } |j - k| \geq 2. \end{cases}$$

The matrix H_Δ is a positive definite constant matrix with determinant

$$\det H_\Delta = \frac{\tau_1 + \tau_2 + \dots + \tau_{J+1}}{\tau_1 \tau_2 \dots \tau_{J+1}} = \frac{(s' - s)}{\tau_1 \tau_2 \dots \tau_{J+1}}.$$

It has its inverse H_Δ^{-1} . Regarding W_Δ as an perturbation, we write

$$\Psi = H_\Delta (I + H_\Delta^{-1} W_\Delta).$$

Proposition 2.2. *Let $0 < \delta_1$ be so small that $\delta_1 \leq \delta_0$ and $\kappa_2 \delta_1^2 < 1$. Let $|s' - s| \leq \delta_1$. Then $\forall (x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$*

$$(1 - \kappa_2 \delta_1^2)^J \leq \det(I + H_\Delta^{-1} W_\Delta) \leq (1 + \kappa_2 \delta_1^2)^J,$$

and

$$(1 - \kappa_2 \delta_1^2)^J \frac{(s' - s)}{\tau_1 \tau_2 \dots \tau_{J+1}} \leq \det \Psi = \det(H_\Delta + W_\Delta) \leq (1 + \kappa_2 \delta_1^2)^J \frac{(s' - s)}{\tau_1 \tau_2 \dots \tau_{J+1}}.$$

Assume $|s' - s| \leq \delta_1$. Let γ^* be the unique classical path in $\mathcal{H}_{x,y}$ and let $x_j^* = \gamma^*(T_j)$ for $j = 0, 1, 2, \dots, J+1$ and $W_\Delta^* = W_\Delta \Big|_{x_j = x_j^*, 1 \leq j \leq J}$. We set

$$\begin{aligned} D(\Delta; s', s, x, y) &= \det(I + H_\Delta^{-1} W_\Delta^*) \\ &= \left(\frac{\tau_1 \tau_2 \dots \tau_{J+1}}{s' - s} \right) \det \text{Hess}_{x_J^*, x_{J-1}^*, \dots, x_1^*} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0). \end{aligned}$$

Here $\text{Hess}_{x_J^*, \dots, x_1^*} S_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)$ is the Hessian matrix at $(x_J^*, \dots, x_1^*)$ of S_Δ .

Proposition 2.3. *Suppose that $0 < |s' - s| \leq \delta_1$. Define $d(\Delta; s', s, x, y)$ by*

$$(2.4) \quad D(\Delta; s', s, x, y) = 1 + (s' - s)^2 d(\Delta; s', s, x, y).$$

Then for any $k \geq 0$

$$(2.5) \quad \sup_{|s' - s| \leq \delta_1} \sup_{\Delta} \max_{|\alpha| + |\beta| \leq k} \sup_{(x,y) \in \mathbf{R}^2} |\partial_x^\alpha \partial_y^\beta d(\Delta; s', s, x, y)| < \infty.$$

Proposition 2.4. *If $|t - s| \leq \delta_1$, then there exists the limit*

$$(2.6) \quad \lim_{|\Delta| \rightarrow 0} D(\Delta; t, s, x, y) = D(t, s, x, y).$$

Define

$$(2.7) \quad e(t, s, x, y) = \left(\frac{1}{2\pi(t - s)} \right)^{1/2} D(t, s, x, y)^{-1/2}.$$

Then this satisfies the transport equation cf. [2]:

$$\partial_t e(t, s, x, y) + \partial_x S(t, s, x, y) \partial_x e(t, s, x, y) + \frac{1}{2} \partial_x^2 S(t, s, x, y) e(t, s, x, y) = 0.$$

Let $(-\frac{d^2}{dt^2})^{-1}$ be the Green operator of Dirichlet boundary problem. Then $D(t, s, x, y)$ equals the following infinite dimensional determinant:

$$(2.8) \quad D(t, s, x, y) = \det \left(-\frac{d^2}{dt^2} - \partial_x^2 V(t, \gamma^*) \right) \left(-\frac{d^2}{dt^2} \right)^{-1}.$$

§ 3. Stationary phase method for integrals over a space of large dimension

Let $f(x_{J+1}, x_J, \dots, x_1, x_0)$ be a function of $(x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$. Let Δ be a division of interval $[s, s']$ as (1.4). Then we can regard the function f as a function defined on $\Gamma(\Delta)$, because $\mathbf{R}^{J+2}$ is identified with $\Gamma(\Delta)$. Let $0 = j_0 < j_1 < \dots < j_p < j_{p+1} = J+1$ be a subsequence of $\{0, 1, \dots, J, J+1\}$. Then

$$(3.1) \quad \Delta' : s = T_{j_0} < T_{j_1} < \dots < T_{j_p} < T_{j_{p+1}} = s'$$

is a division of the interval $[s, s']$ of which Δ is a refinement. We call a division Δ' of the interval $[s, s']$ coarser than the division Δ if Δ is a refinement of Δ' . There exists a natural embedding map $\Gamma(\Delta') \subset \Gamma(\Delta)$ and $\Gamma_{x,y}(\Delta') \subset \Gamma_{x,y}(\Delta)$. We shall write $\iota_\Delta^\Delta f : \Gamma(\Delta') \rightarrow \mathbf{C}$ for the pull back of a function $f : \Gamma(\Delta) \rightarrow \mathbf{C}$ by this embedding. If f is a function defined on

$\mathbf{R}^{J+2}$, we can define a function $\iota_{\Delta'}^{\Delta} f$ defined on $\mathbf{R}^{p+2}$ using the identifications $\mathbf{R}^{J+2} \cong \Gamma(\Delta)$ and $\mathbf{R}^{p+2} \cong \Gamma(\Delta')$.

For integers $1 \leq k < l \leq J+1$ we define

$$(3.2) \quad S_{l,j}(x_l, \dots, x_{j-1}) = S_l(x_l, x_{l-1}) + S_{l-1}(x_{l-1}, x_{l-2}) + \dots + S_j(x_j, x_{j-1}).$$

Note that $S_{J+1,1}(x_{J+1}, \dots, x_0) = S(x_{J+1}, \dots, x_0)$. We understand that $S_{j,j}(x_j, x_{j-1}) = S_j(x_j, x_{j-1})$. Suppose $1 \leq k < l \leq J+1$. For any fixed $(x_l, x_{j-1}) \in \mathbf{R}^2$ let $(x_{l-1}^*, x_{l-2}^*, \dots, x_j^*)$ be the stationary point of the function $S_{l,j}(x_l, \dots, x_{j-1})$ of (3.2). We shall write $x_k^*(x_l, x_{j-1})$ for x_k^* when we wish to express that x_k^* depends on (x_l, x_{j-1}) .

Suppose Δ' is the division given by (3.1) coarser than Δ . Then for any $f(x_{J+1}, x_J, \dots, x_0) \in \Gamma_{x,y}(\Delta)$ it is clear by definition of $\iota_{\Delta'}^{\Delta}$ that

$$\iota_{\Delta'}^{\Delta} f(x_{J+1}, x_{j_p}, \dots, x_{j_1}, x_0) = f(x_{J+1}, x_J, \dots, x_1, x_0) \Big|_{\substack{x_k = x_k^*(x_{j_n}, x_{j_{n-1}}), \\ j_{n-1} < k < j_n, n=1,2,\dots,p+1}}.$$

We write $S_{l,j}^*(x_l, x_{j-1})$ for the stationary value of $S_{l,j}(x_l, \dots, x_{j-1})$. i.e.,

$$\begin{aligned} S_{l,j}^*(x_l, x_{j-1}) &= S_l(x_l, x_{l-1}^*(x_l, x_{j-1})) \\ &\quad + \sum_{k=j+1}^{l-1} S_k(x_k^*(x_l, x_{j-1}), x_{k-1}^*(x_l, x_{j-1})) + S_j(x_j^*(x_l, x_{j-1}), x_{j-1}). \end{aligned}$$

As $S_j(x_j, x_{j-1}) = S(T_j, T_{j-1}, x_j, x_{j-1})$ is a classical action, it turns out that

$$(3.3) \quad S_{l,j}^*(x_l, x_{j-1}) = S_{l,j}(x_l, x_j).$$

Thus

$$\iota_{\Delta'}^{\Delta} S_{\Delta}(x_{J+1}, x_{j_p}, \dots, x_{j_1}, x_0) = \sum_{n=1}^{p+1} S_{j_n, j_{n-1}+1}(x_{j_n}, x_{j_{n-1}}).$$

The interval $[s, s']$ itself is a particular division of $[s, s']$, which we write $\Delta(J+1)$. Then $\iota_{\Delta(J+1)}^{\Delta} S_{\Delta}(x_{J+1}, x_0) = S(s', s, x, y)$.

Given a function $a_{\lambda}(x_{J+1}, x_J, \dots, x_1, x_0)$ of $(x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$ with parameter λ and a fixed division Δ , we discuss

$$(3.4) \quad \begin{aligned} I(\Delta, S_{\Delta}, a_{\lambda}, \nu)(x_{J+1}, x_0) \\ = \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2} \int_{\mathbf{R}^J} a_{\lambda}(x_{J+1}, x_J, \dots, x_1, x_0) e^{i\nu S_{\Delta}(x_{J+1}, x_J, \dots, x_1, x_0)} \prod_{j=1}^J dx_j. \end{aligned}$$

Assumption for the amplitude $a_{\lambda}(x_{J+1}, x_J, \dots, x_1, x_0)$ is the following:

Assumption 3.1. Let $m \geq 0$ be a constant. The amplitude $a_{\lambda}(x_{J+1}, x_J, \dots, x_1, x_0)$ is defined on $\mathbf{R}^{J+2}$ and may depend on a parameter λ . For any integer $K \geq 0$ there exist constants $A_K > 0$ and $X_K \geq 1$ such that

1. If $|\alpha_j| \leq K$ for all $j = 0, 1, \dots, J+1$, then $\forall (x_{J+1}, x_J, \dots, x_1, x_0) \in \mathbf{R}^{J+2}$

$$\left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) a_{\lambda}(x_{J+1}, x_J, \dots, x_1, x_0) \right| \leq A_K X_K^{J+2} (1 + |\lambda| + |x_{J+1}| + |x_J| + \dots + |x_1| + |x_0|)^m,$$

2. Let Δ' be any division defined by (3.1) coarser than Δ and let $\{\alpha_{j_k}\}$ be a sequence of indices each of which satisfies $|\alpha_{j_k}| \leq K$ for $k = 0, 1, \dots, p+1$. Then for any $(x_0, x_{j_1}, \dots, x_{j_p}, x_{J+1}) \in \mathbf{R}^{p+2}$

$$\begin{aligned} & \left| \partial_{x_0}^{\alpha_0} \partial_{x_{J+1}}^{\alpha_{J+1}} \left(\prod_{k=1}^p \partial_{x_{j_k}}^{\alpha_{j_k}} \right) (\iota_{\Delta'}^{\Delta} a_{\lambda})(x_{J+1}, x_{j_p}, \dots, x_{j_1}, x_0) \right| \\ & \leq A_K X_K^{p+2} (1 + |\lambda| + |x_{J+1}| + |x_{j_p}| + \dots + |x_{j_1}| + |x_0|)^m. \end{aligned}$$

Let $(x_{l-1}^*, \dots, x_j^*)$ be the critical point of (3.2). And let $Hess$ mean the Hessian of $S_{l,j}$ at the critical point. We define

$$(3.5) \quad D_{x_{l-1}^*, \dots, x_j^*}(S_{l,j}; x_l, x_{j-1}) = \left(\frac{\tau_l + \dots + \tau_j}{\tau_l \dots \tau_j} \right) \det Hess \left(\sum_{k=j}^l S_k(x_k, x_{k-1}) \right) \Big|_{x_k = x_k^*, j \leq k \leq l-1}.$$

For any $k = 1, 2, \dots, J+1$ we define the division

$$(3.6) \quad \Delta(k) : s = T_0 < T_k < T_{k+1} < \dots < T_{J+1} = s'.$$

$\Delta(1) = \Delta$ and $\Delta(J+1)$ is the interval itself without any intermediate dividing point. The following theorem is known [7], [10].

Theorem 3.2. *Suppose that $|s' - s| \leq \delta_1$ and $a_{\lambda}(x_{J+1}, x_J, \dots, x_1, x_0)$ satisfies Assumption 3.1. We further assume that $|\Delta||s' - s| \leq 1$. Then*

$$(3.7) \quad I(\Delta; S_{\Delta}, a_{\lambda}, \nu)(x_{J+1}, x_0) = \left(\frac{\nu}{2\pi i T} \right)^{1/2} e^{i\nu S(s', s, x_{J+1}, x_0)} k(\Delta; a_{\lambda}, \nu, s', s, x_{J+1}, x_0)$$

with

$$\begin{aligned} (3.8) \quad & k(\Delta; a_{\lambda}, \nu, s', s, x_{J+1}, x_0) \\ & = D_{x_j^*, \dots, x_1^*}(S_{J+1,1}; x_{J+1}, x_0)^{-1/2} \left(\iota_{\Delta(J+1)}^{\Delta} a_{\lambda}(x_{J+1}, x_0) + \nu^{-1}(s' - s)p(\Delta, x_{J+1}, x_0) \right) \\ & \quad + \nu^{-1}(s' - s)^2 |\Delta| q(\Delta, x_{J+1}, x_0) + \nu^{-2}(s' - s)^2 r(\Delta, \nu, x_{J+1}, x_0). \end{aligned}$$

Here

$$\begin{aligned} (3.9) \quad & p(\Delta, x_{J+1}, x_0) \\ & = -\frac{i}{2(s' - s)} \sum_{j=1}^J \frac{(T_j - s)\tau_{j+1}}{(T_{j+1} - s)} (\iota_{\Delta(J+1)}^{\Delta(j)}) \left[D_{x_{j-1}^*, \dots, x_1^*}(S_{j,1}; x_j, x_0)^{1/2} \right. \\ & \quad \left. \times \partial_{x_j}^2 (D_{x_{j-1}^*, \dots, x_1^*}(S_{j,1}, x_j, x_0)^{-1/2} \iota_{\Delta(j)}^{\Delta} a_{\lambda}) \right] (x_{J+1}, x_0). \end{aligned}$$

$q(\Delta, x_{J+1}, x_0)$ is independent of ν . And functions $q(\Delta, x_{J+1}, x_0)$ and $r(\Delta, \nu, x_{J+1}, x_0)$ satisfies the following estimate. For any $K \geq 0$ there exists an integer $M(K) \geq 0$ and a constant $C_K > 0$ independent of Δ such that

$$(3.10) \quad (1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} q(\Delta, x_{J+1}, x_0)| \leq C_K A_{M(K)} X_{M(K)}^2$$

$$(3.11) \quad (1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} r(\Delta, \nu, x_{J+1}, x_0)| \leq C_K A_{M(K)} X_{M(K)}^2,$$

if multi-indices α_0, α_{J+1} satisfies $|\alpha_0| \leq K$ and $|\alpha_{J+1}| \leq K$.

Since

$$(s' - s)^{-1} \sum_{j=1}^J \frac{(T_j - s)\tau_{j+1}}{(T_{j+1} - s)} \leq 1,$$

we have from Theorem 3.2

Corollary 3.3. *If $|\alpha_{J+1}| \leq K$ and $|\alpha_0| \leq K$, then*

$$(3.12) \quad (1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0)| \leq C_K X_{M(K)}^2 A_{M(K)}.$$

Remark 3.4. Tsuchida [16] treated the case of non-zero vector potential.

Definition 3.5. Let $p \geq 0$ and $k \geq 0$ be integers. For any function $f : \mathbf{R}^n \ni x \rightarrow \mathbf{C}$ we define a norm

$$\|f\|_{\{p,k\}} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbf{R}^n} (1 + |x|)^{-p} |\partial_x^\alpha f(x)|.$$

We write

$$\mathcal{B}_p(\mathbf{R}^n) = \{f \in C^\infty(\mathbf{R}^n) : \|f\|_{\{p,k\}} < \infty, \quad \forall k \geq 0\}.$$

$\mathcal{B}_p(\mathbf{R}^n)$ is a Fréchet space. If $p = 0$, we abbreviate $\mathcal{B}_0(\mathbf{R}^n)$ to $\mathcal{B}(\mathbf{R}^n)$.

Definition 3.6. Let $m \geq 0$ be a constant. Let $\{f_\lambda(x)\}_\lambda$ be a family of functions in $\mathcal{B}_p(\mathbf{R}^n)$. If this is a bounded set in $\mathcal{B}_p(\mathbf{R}^n)$, we write

$$f_\lambda = \mathcal{O}_{\mathcal{B}_p(\mathbf{R}^n)}(1).$$

And we write $f_\lambda = \mathcal{O}_{\mathcal{B}_p(\mathbf{R}^n)}(g)$ if $f_\lambda/g = \mathcal{O}_{\mathcal{B}_p(\mathbf{R}^n)}(1)$.

Remark 3.7. It follows from Theorem 3.2 that

$$(3.13) \quad \begin{aligned} & k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0) \\ &= D_{x_J^*, \dots, x_1^*} (S_{J+1,1}; x_{J+1}, x_0)^{-1/2} \\ &\times \left(\iota_{\Delta(J+1)}^\Delta a_\lambda(x_{J+1}, x_0) + \nu^{-1} (s' - s) p(\Delta, x_{J+1}, x_0) \right. \\ &\left. + \nu^{-1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}((s' - s)^2 |\Delta|) + \nu^{-2} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}((s' - s)^2) \right). \end{aligned}$$

Assumption 3.8 (N.Kumano-go's assumption). Suppose $a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)$ satisfies Assumption 3.1. Moreover, there exists a bounded Borel measure $\rho \geq 0$ on $[s, s']$ such that as far as $|\alpha_k| \leq K$ for $k = 0, 1, 2, \dots, J+1$

$$\begin{aligned} & \left| \left(\prod_{k=0}^{J+1} \partial_{x_k}^{\alpha_k} \right) \partial_{x_j} a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0) \right| \\ & \leq A_K X_K^{J+2} \rho([T_{j-1}, T_{j+1}]) (1 + |\lambda| + |x_{J+1}| + |x_J| + \dots + |x_1| + |x_0|)^m \quad (0 \leq \forall j \leq J+1) \end{aligned}$$

and that as far as $|\alpha_{j_k}| \leq K$ for $k = 0, 1, \dots, p+1$

$$\begin{aligned} & \left| \partial_{x_0}^{\alpha_0} \partial_{x_{J+1}}^{\alpha_{J+1}} \left(\prod_{k=1}^p \partial_{x_{j_k}}^{\alpha_{j_k}} \right) \partial_{x_{j_k}} (\iota_{\Delta'}^\Delta a_\lambda)(x_{J+1}, x_{j_p}, \dots, x_{j_1}, x_0) \right| \\ & \leq A_K X_K^{p+2} \rho([T_{j_{k-1}}, T_{j_{k+1}}]) (1 + |\lambda| + |x_{J+1}| + |x_{j_p}| + \dots + |x_{j_1}| + |x_0|)^m \quad (0 \leq \forall k \leq p+1). \end{aligned}$$

Proposition 3.9. *Suppose $a_\lambda(x_{J+1}, x_J, \dots, x_1, x_0)$ satisfies Kumano-go's assumption. Then the function $k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0)$ of (3.7) is of the form*

$$(3.14) \quad k(\Delta; a_\lambda, \nu, s', s, x_{J+1}, x_0) \\ = D_{x_J^*, \dots, x_1^*}(S_{J+1,1}; x_{J+1}, x_0)^{-1/2} \left(\iota_{\Delta(J+1)}^\Delta a_\lambda(x_{J+1}, x_0) + \nu^{-1} R(\Delta, x_{J+1}, x_0) \right).$$

And for any integer $K \geq 0$ there exist C_K and $M(K)$ independent of Δ and ν such that as far as $|\alpha| \leq K, \beta \leq K$

$$(1 + |\lambda| + |x_{J+1}| + |x_0|)^{-m} |\partial_{x_{J+1}}^{\alpha_{J+1}} \partial_{x_0}^{\alpha_0} R(\Delta, x_{J+1}, x_0)| \leq C_K A_{M(K)} |s' - s| (|s' - s| + \rho([s, s'])).$$

i.e.,

$$(3.15) \quad R(\Delta, x_{J+1}, x_0) = \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(|s' - s| (|s' - s| + \rho([s, s']))).$$

§ 4. Convergence of Feynman path integral

We discuss convergence of Feynman path integral. Our discussion is valid only for those $F(\gamma)$ that have rather restrictive properties.

Assumption 4.1 (N.Kumano-go's condition). Let m be a non-negative constant and ρ be a bounded Borel measure $\rho \geq 0$ on $[s, s']$. Suppose $F(\gamma)$ is a functional defined for all piecewise classical path $\gamma \in \cup_\Delta \Gamma(\Delta)$. For any integer $K \geq 0$ there exist constants $A_K > 0$ and $X_K \geq 1$ such that for any division Δ defined by (1.4) and for any indices $\alpha_j, j = 0, 1, 2, \dots, J+1$ satisfying $|\alpha_j| \leq K$ there hold the following inequalities:

$$\left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) F(\gamma_\Delta(x_{J+1}, x_J, \dots, x_1, x_0)) \right| \leq A_K X_K^{J+2} (1 + |x_{J+1}| + |x_J| + \dots + |x_1| + |x_0|)^m, \\ \left| \left(\prod_{j=0}^{J+1} \partial_{x_j}^{\alpha_j} \right) \partial_{x_k} F(\gamma_\Delta(x_{J+1}, \dots, x_{k+1}, x_k, x_{k-1}, \dots, x_0)) \right| \\ \leq A_K X_K^{J+2} \rho([T_{k-1}, T_{k+1}]) (1 + |x_{J+1}| + |x_J| + \dots + |x_1| + |x_0|)^m.$$

Remark 4.2. $F(\gamma) \equiv 1$ clearly satisfies this assumption.

Example 4.3. Let $\rho(t)$ be a function of bounded-variation on $[s, s']$ and $f(t, x)$ be a continuous function of $(t, x) \in [s, s'] \times \mathbf{R}$ and infinitely differentiable in x . Suppose that for any α there exists a positive constant C_α such that

$$|\partial_x^\alpha f(t, x)| \leq C_\alpha (1 + |x|)^m$$

with some $m \geq 0$ independent of α and (t, x) . Then the following functional satisfies Assumptions 4.1.

$$F(\gamma) = \int_s^{s'} f(t, \gamma(t)) d\rho(t).$$

The next theorem was proved by N.Kumano-go [12], while the case $F(\gamma) \equiv 1$ had been known. [8], [11] and [6].

Theorem 4.4. *Suppose that $F(\gamma)$ satisfies Assumption 4.1 above and $|s' - s| \leq \delta_1$. Let $I[F_\Delta](\Delta; \nu, s', s, x, y)$ be the time slicing approximation defined by (2.2). We write*

$$(4.1) \quad I[F_\Delta](\Delta; \nu, s', s, x, y) = \left(\frac{\nu}{2\pi i(s' - s)} \right)^{1/2} e^{i\nu S(s', s, x, y)} k(\Delta; F_\Delta, \nu, s', s, x, y).$$

Then $k(F; \nu, s', s, x, y) = \lim_{|\Delta| \rightarrow 0} k(\Delta; F_\Delta, \nu, s', s, x, y)$ exists in the space $\mathcal{B}_m(\mathbf{R}^2)$. More precisely, for any $K \geq 0$ there exists $C_K > 0$ such that if $|\alpha| \leq K$ and $|\beta| \leq K$

$$(4.2) \quad \sup_{(x, y) \in \mathbf{R}^2} (1 + |x| + |y|)^{-m} |\partial_x^\alpha \partial_y^\beta (k(\Delta; F_\Delta, \nu, s', s, x, y) - k(F; \nu, s', s, x, y))| \\ \leq C_K A_{M(K)} X_{M(K)}^4 |\Delta| (\rho([s, s']) + |s' - s|).$$

$k(F; \nu, s', s, x, y)$ can be written as

$$(4.3) \quad k(F; \nu, s', s, x, y) = D(s', s, x, y)^{-1/2} (F(\gamma^*) + \nu^{-1} R[F](\nu, s', s, x, y))$$

and for $|\alpha| \leq K$ and $|\beta| \leq K$

$$(4.4) \quad |\partial_x^\alpha \partial_y^\beta R[F](\nu, s', s, x, y)| \leq C_K A_{M(K)} |s - s'| (|s - s'| + \rho([s, s']))(1 + |x| + |y|)^m.$$

Set

$$(4.5) \quad K[F](\nu, s', s, x, y) = \left(\frac{\nu}{2\pi i(s' - s)} \right)^{1/2} e^{i\nu S(s', s, x, y)} k(F; \nu, s', s, x, y).$$

Then

$$(4.6) \quad K[F](\nu, s', s, x, y) = \lim_{|\Delta| \rightarrow 0} I[F_\Delta](\Delta; \nu, s', s, x, y).$$

Remark 4.5. In short, $F(\gamma)$ is "F-integrable" if F satisfies Assumption 4.1. We may write

$$(4.7) \quad \int_{\Omega} e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] = K[F](\nu, s', s, x, y).$$

Remark 4.6. Equality (4.3) together with (4.4) imply semiclassical asymptotic formula.

Theorem 4.4 follows from the next proposition.

Proposition 4.7. *Let Δ^* be an arbitrary refinement of Δ . For any integer $K \geq 0$ there exist a constant C_K and an integer $M(K)$ independent of Δ, Δ^* and ν such that*

$$(4.8) \quad |\partial_x^\alpha \partial_y^\beta (k(\Delta^*; F_{\Delta^*}, \nu, s', s, x, y) - k(\Delta; F_\Delta, \nu, s', s, x, y))| \\ \leq C_K A_{M(K)} X_{M(K)}^4 |\Delta| (\rho([s, s']) + |\Delta|) (1 + |x| + |y|)^m$$

if $|\alpha| \leq k, |\beta| \leq k$.

We indicate the idea to prove Proposition 4.7. The division points of Δ^* that lies in the first subinterval $[T_0, T_1]$ of Δ make a division δ of $[T_0, T_1]$

$$(4.9) \quad \delta : s = T_0 = T_{1,0} < T_{1,1} < \cdots < T_{1,p_1+1} = T_1.$$

Let Δ_1 be the division of $[s, s']$ defined by all division points of Δ and division points of Δ^* that lies in $[T_0, T_1]$. In other words

$$(4.10) \quad \Delta_1 : s = T_0 = T_{1,0} < T_{1,1} < \cdots < T_{1,p_1+1} = T_1 < T_2 < T_3 < \cdots < T_J < T_{J+1} = s'.$$

Δ_1 is a refinement of Δ . Let $(x, y) \in \mathbf{R}^2$. For arbitrary $(y_1, \dots, y_{p_1}) \in \mathbf{R}^{p_1}$ and $(x_1, \dots, x_J)$ there exists one and only one piecewise classical path $\gamma_{\Delta_1} \in \Gamma_{x,y}(\Delta_1)$ such that

$$\begin{aligned} y_k &= \gamma_{\Delta_1}(T_{1,k}), \quad \text{for } 0 \leq k \leq p_1 + 1, \\ x_j &= \gamma_{\Delta_1}(T_j), \quad \text{for } 0 \leq j \leq J + 1, \end{aligned}$$

where we set $y_0 = x_0$ and $y_{p_1+1} = x_1$ as well as $x_{J+1} = x$, $x_0 = y$.

Proposition 4.8.

$$\begin{aligned} &k(\Delta_1; F_{\Delta_1}, \nu, s', s, x_{J+1}, x_0) - k(\Delta; F_{\Delta}, \nu, s', s, x_{J+1}, x_0) \\ &= \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_1^2) + \nu^{-1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_1^2 + \tau_1 \rho([T_0, T_1])). \end{aligned}$$

Admitting this proposition as true for the moment, we proceed in the following way. We add to dividing points of division Δ_1 all the division points of Δ^* that lie in $[T_1, T_2]$. Then we obtain a new division Δ_2 of $[s, s']$. Δ_2 is the same as Δ^* in $[T_0, T_2]$ and it is the same as Δ in $[T_2, T_{J+1}]$. We have in this case, corresponding to Proposition 4.8,

$$(4.11) \quad \begin{aligned} &k(\Delta_2; F_{\Delta_2}, \nu, s', s, x, y) - k(\Delta_1; F_{\Delta_1}, \nu, s', s, x, y) \\ &= \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_2^2) + \nu^{-1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_2^2 + \tau_2 \rho([T_1, T_2])). \end{aligned}$$

Similarly, we make Δ_3 from Δ_2 . Continuing this process $J+1$ times, we finally obtain $\Delta_{J+1} = \Delta^*$. Therefore,

$$(4.12) \quad \begin{aligned} &k(\Delta^*; F_{\Delta^*}, \nu, s', s, x, y) - k(\Delta; F_{\Delta}, \nu, s', s, x, y) \\ &= \sum_{j=1}^{J+1} (k(\Delta_j; F_{\Delta_j}, \nu, s', s, x, y) - k(\Delta_{j-1}; F_{\Delta_{j-1}}, \nu, s', s, x, y)) \\ &= \sum_{j=1}^{J+1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_j^2) + \nu^{-1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_j^2 + \tau_j \rho([T_{j-1}, T_j])) \\ &= \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(|\Delta|(s' - s)) + \nu^{-1} \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(|\Delta|(s' - s) + |\Delta| \rho([s, s'])). \end{aligned}$$

This proves Proposition 4.7.

We suggest how to prove Proposition 4.8. We define

$$S_{\delta}(x_1, y_{p_1}, \dots, y_1, x_0) = \sum_{k=1}^{p_1+1} S(T_{1,k}, T_{1,k-1}; y_k, y_{k-1}).$$

Then

$$S_{\Delta_1}(x_{J+1}, \dots, x_1, y_{p_1}, \dots, y_1, x_0) = \left(\sum_{j=2}^{J+1} S_j(x_j, x_{j-1}) \right) + S_\delta(x_1, y_{p_1}, \dots, y_1, x_0).$$

By definition

(4.13)

$$\begin{aligned} I[F_{\Delta_1}](\Delta_1; \nu, s', s, x, y) &= \prod_{j=2}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2} \int_{\mathbf{R}^J} e^{i\nu \sum_{j=2}^{J+1} S_j(x_j, x_{j-1})} \prod_{j=1}^J dx_j \\ &\times \prod_{k=1}^{p_1+1} \left(\frac{\nu}{2i\pi \sigma_k} \right)^{1/2} \int_{\mathbf{R}^{p_1}} e^{i\nu S_\delta(x_1, y_{p_1}, \dots, y_1, x_0)} F_{\Delta_1}(x_{J+1}, \dots, x_1, y_{p_1}, \dots, y_1, x_0) \prod_{k=1}^{p_1} dy_k. \end{aligned}$$

We perform integration by the variables $(y_{p_1}, \dots, y_1)$ prior to integration by variables $(x_J, \dots, x_1)$. Set

(4.14)

$$\begin{aligned} &\left(\frac{\nu}{2\pi i \tau_1} \right)^{1/2} e^{i\nu S_1(x_1, x_0)} F_{\Delta/\Delta_1}(x_{J+1}, x_J, \dots, x_1, x_0) \\ &= \prod_{k=1}^{p_1+1} \left(\frac{\nu}{2i\pi \sigma_k} \right)^{1/2} \int_{\mathbf{R}^{p_1}} e^{i\nu S_\delta(x_1, y_{p_1}, \dots, y_1, x_0)} F_{\Delta_1}(x_{J+1}, \dots, x_1, y_{p_1}, \dots, y_1, x_0) \prod_{k=1}^{p_1} dy_k. \end{aligned}$$

Then (4.13) means that

$$(4.15) \quad I[F_{\Delta_1}](\Delta_1; \nu, s', s, x, y) = I[F_{\Delta/\Delta_1}](\Delta; \nu, s', s, x, y).$$

We apply Proposition 3.9 to the integration by $(y_{p_1}, \dots, y_1)$ in (4.14). Then

$$(4.16) \quad \begin{aligned} &F_{\Delta/\Delta_1}(x_{J+1}, x_J, \dots, x_1, x_0) \\ &= D(\delta; x_1, x_0)^{-1/2} \left(F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) + \nu^{-1} R_\delta[F_{\Delta_1}](\nu, x_{J+1}, x_J, \dots, x_1, x_0) \right), \end{aligned}$$

here

$$(4.17) \quad R_\delta[F_{\Delta_1}](\nu, x_{J+1}, x_J, \dots, x_1, x_0) = \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^{J+1})}(\tau_1^2 + \tau_1 \rho([T_0, T_1])).$$

On the other hand, it follows from Proposition 2.3 that $D(\delta; x_1, x_0)^{-1/2} = 1 + \mathcal{O}_{\mathcal{B}_0(\mathbf{R}^2)}(\tau_1^2)$. Combining these, we have

$$\begin{aligned} &F_{\Delta/\Delta_1}(x_{J+1}, x_J, \dots, x_1, x_0) \\ &= F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) + \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^{J+2})}(\tau_1^2 + \nu^{-1}(\tau_1^2 + \tau_1 \rho([T_0, T_1]))). \end{aligned}$$

We can show that we can apply Corollary 3.3 to the right hand side of (4.15) and that

$$\begin{aligned} &k(\Delta_1; F_{\Delta_1}, \nu, s', s, x, y) \\ &= k(\Delta; F_\Delta, \nu, s', s, x, y) + \mathcal{O}_{\mathcal{B}_m(\mathbf{R}^2)}(\tau_1^2 + \nu^{-1}(\tau_1^2 + \tau_1 \rho([T_0, T_1]))). \end{aligned}$$

This shows Proposition 4.8.

In the case $F(\gamma) \equiv 1$ we discuss the integral transformation with the kernel $K[1](\nu, t, s, x, y)$.

Definition 4.9. We define for any $\varphi \in C_0^\infty(\mathbf{R})$

$$(4.18) \quad I(\Delta; \nu, t, s)\varphi(x) = \int_{\mathbf{R}} I[1](\Delta; \nu, t, s, x, y)\varphi(y) dy,$$

$$(4.19) \quad K(\nu, t, s)\varphi(x) = \int_{\mathbf{R}} K[1](\nu, t, s, x, y)\varphi(y) dy.$$

We write $\|A\|$ for the operator norm of a linear operator A on $L^2(\mathbf{R})$. It turns out from L^2 -boundedness theorem in [1] that the following facts hold:

Proposition 4.10. *Suppose that $|t - s| \leq \delta_0$. Then there exists a positive constant C independent of ν, t and s such that*

$$(4.20) \quad \|I(\Delta; \nu, t, s)\| \leq C, \quad \|K(\nu, t, s)\| \leq C.$$

Theorem 4.11. *Suppose that $|t - s| \leq \delta_0$. Then there exists a positive constant C independent of ν, t, s such that*

$$(4.21) \quad \|I(\Delta; \nu, t, s) - K(\nu, t, s)\| \leq C(s' - s)|\Delta|.$$

Next we shall discuss the relation between Feynman path integral and propagator of Schrödinger equation.

Let $H(t)$ be the Hamiltonian operator:

$$(4.22) \quad H(t) = \frac{1}{2} (-i\nu^{-1}\partial_x)^2 + V(t, x).$$

Theorem 4.12. *Suppose that $|t - s| \leq \delta_0$. For any $f \in C_0^\infty(\mathbf{R})$ the $L^2(\mathbf{R})$ -valued function $t \rightarrow K(\nu, t, s)f$ is strongly differentiable. It satisfies*

$$(4.23) \quad i\nu^{-1} \frac{d}{dt} K(\nu, t, s)f = H(t)K(\nu, t, s)f,$$

$$(4.24) \quad s - \lim_{|t-s| \rightarrow 0} K(\nu, t, s)f = f.$$

Corollary 4.13. *$K(\nu, t, s)f(x)$ is the classical solution of Schrödinger equation*

$$(4.25) \quad i\nu^{-1} \frac{\partial}{\partial t} K(\nu, t, s)f = \left[\frac{1}{2} \left(-i\nu^{-1} \frac{\partial}{\partial x} \right)^2 + V(t, x) \right] K(\nu, t, s)f(x),$$

if $f \in C_0^\infty$.

Remark 4.14. In the case $F(\gamma) \equiv 1$, $K[1](\nu, s', s, x, y) = \int_{\Omega_{x,y}} e^{i\nu S(\gamma)} \mathcal{D}[\gamma]$ is in fact the fundamental solution of Schrödinger equation (1.9). And it has semiclassical asymptotic formula given by (4.3) and (4.4) with $F(\gamma^*) = 1$. The principal term enjoys the property shown by Proposition 2.4. cf. [2]

These main statement of Feynman's paper [4] were verified rigorously in [5], [6], [11], [8].

§ 5. An integration by parts formula

§ 5.1. Some operators of trace class

We set $s = 0$ and $s' = T$ for simplicity. Let $\mathcal{X} = L^2([0, T])$ and $\mathcal{H} = H^1([0, T])$ be the real L^2 -Sobolev space of order 1. For any $x, y \in \mathbf{R}$, we write $\mathcal{H}_{x,y} = \{\gamma \in \mathcal{H} : \gamma(0) = y, \gamma(T) = x\}$.

$x\}$. $\mathcal{H}_{x,y}$ is an infinite dimensional differentiable manifold. Its tangent space at $\gamma \in \mathcal{H}_{x,y}$ is identified with the space $\mathcal{H}_0 = H_0^1([0, T]) = \{\gamma \in \mathcal{H}; \gamma(0) = \gamma(T) = 0\}$

Let $\tilde{\rho} : \mathcal{H} \rightarrow \mathcal{X}$ be the natural embedding and $\rho : \mathcal{H}_0 \rightarrow \mathcal{X}$ be its restriction to $\mathcal{H}_0$ and $\rho^* : \mathcal{X} \rightarrow \mathcal{H}_0$ be its adjoint.

We write $(\cdot, \cdot)_{\mathcal{X}}$ for the inner product of $\mathcal{X}$. We write $\mathcal{L}(\mathcal{X})$ for the Banach space of all bounded linear operators in $\mathcal{X}$ equipped with operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{X})}$. We adopt the following inner product of $\mathcal{H}_0$:

$$(h_1, h_2)_{\mathcal{H}_0} = \int_0^T \frac{d}{dt} h_1(t) \frac{d}{dt} h_2(t) dt \quad (h_1, h_2 \in \mathcal{H}_0).$$

We write $\|h\|_{\mathcal{H}_0}$ for the norm of $h \in \mathcal{H}_0$ in $\mathcal{H}_0$. The cotangent vector $DF(\gamma)$ is identified with an element, which we also write $DF(\gamma) \in \mathcal{H}_0$, via the inner product of $\mathcal{H}_0$ by the equation $DF(\gamma)[h] = (DF(\gamma), h)_{\mathcal{H}_0}$.

Let $\omega = \pi T^{-1}$ and let $e_n(t) = \sqrt{\frac{2}{T}} \sin n\omega t$. Then $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system of $\mathcal{X}$. We can choose a complete orthogonal system $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{H}_0$ such that $\rho\varphi_n = (n\omega)^{-1}e_n$, i.e., $\rho\varphi_n(t) = (n\omega)^{-1} \sqrt{\frac{2}{T}} \sin n\omega t$. It is clear that $\rho^*e_n = (n\omega)^{-1}\varphi_n$. Therefore, ρ and ρ^* are Hilbert Schmidt operators and

$$(5.1) \quad \rho\rho^*e_n = (n\omega)^{-2}e_n, \quad \rho^*\rho\varphi_n = (n\omega)^{-2}\varphi_n \quad (n = 1, 2, 3, \dots).$$

It turns out that

$$(5.2) \quad -\frac{d^2}{dt^2} \rho\rho^*e_n(t) = e_n(t), \quad e_n(0) = e_n(T) = 0 \quad (n = 1, 2, \dots).$$

Proposition 5.1. *cf. Kato [15]. Suppose that $B : \mathcal{X} \rightarrow \mathcal{X}$ is a bounded linear operator with operator norm $\|B\|_{\mathcal{L}(\mathcal{X})}$. Both of linear operators $\rho^*B\rho : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ and $\rho\rho^*B : \mathcal{X} \rightarrow \mathcal{X}$ are of trace class. Their traces are equal:*

$$\text{tr} \rho^*B\rho = \text{tr} \rho\rho^*B.$$

Since $\rho\rho^*B$ is in trace class, it has the kernel function $\exists k(s, t) \in L^2([0, T] \times [0, T])$, i.e.,

$$(5.3) \quad \rho\rho^*Bf(s) = \int_0^T k(s, t)f(t)dt \quad (\forall f \in \mathcal{X}).$$

In particular, the kernel function of $\rho\rho^*$ is the Green operator for the Dirichlet boundary value problem.

Proposition 5.2. *$k(s, t)$ has the properties:*

1. *If each $s \in [0, T]$ is fixed, then the function $k_s : [0, T] \ni t \rightarrow k(s, t)$ is a well-defined function in $\mathcal{X}$ of t .*
2. *$[0, T] \ni s \rightarrow k_s \in \mathcal{X}$ is a strongly continuous mapping from $[0, T]$ to $\mathcal{X}$.*
3. *The function $[0, T] \ni s \rightarrow k(s, t)$ regarded as a function of s is in the image of the map ρ if t is fixed for almost all $t \in [0, T]$.*

Proposition 5.3. *The value $k(t, t)$ is well-defined for almost all $t \in [0, T]$ and*

$$\int_0^T |k(t, t)|^2 dt < \infty.$$

$$\text{tr} \rho \rho^* B = \int_0^T k(t, t) dt.$$

§ 5.2. Admissible vector field

Let p be a C^1 map $p : \mathcal{H}_{xy} \ni \gamma \rightarrow p(\gamma) \in \mathcal{H}_0$. Then $p(\gamma)$ is a tangent vector field on $\mathcal{H}_{xy}$. We write as usual $p(\gamma, s) = \rho p(\gamma)(s)$. We have $\partial_s p(\gamma, s) \in \mathcal{X}$.

Definition 5.4 (Admissible vector field). We say that $p(\gamma)$ is an admissible vector field if $p(\gamma)$ has the following properties:

1. There exists a C^1 map $q : \mathcal{H} \rightarrow \mathcal{X}$ such that

$$(5.4) \quad p(\gamma) = \rho^* q(\gamma), \quad (\gamma \in \mathcal{H}_{x,y}).$$

2. If $\gamma \in \mathcal{H}_{x,y}$, then there exists a bounded linear map $B(\gamma) \in \mathcal{L}(\mathcal{X})$ such that the Fréchet differential $Dq(\gamma) : \mathcal{H}_0 \ni h \rightarrow Dq(\gamma)[h] \in \mathcal{X}$ is given by

$$(5.5) \quad Dq(\gamma)[h] = B(\gamma)\rho h \quad (h \in \mathcal{H}_0).$$

Remark 5.5. Suppose $p(\gamma)$ is an admissible vector field. Then we often write $\frac{\delta q(\gamma)}{\delta \gamma}$ for $B(\gamma)$. It follows from (5.4) and (5.5) that

$$Dp(\gamma)[h] = \rho^* B(\gamma)\rho h \quad (\gamma \in \mathcal{H}_{x,y}, h \in \mathcal{H}_0).$$

That is, for all $\gamma \in \mathcal{H}_{x,y}$ and $h_1, h_2 \in \mathcal{H}_0$,

$$(Dp(\gamma)[h_1], h_2)_{\mathcal{H}_0} = (B(\gamma)\rho h_1, \rho h_2)_{\mathcal{X}}.$$

Definition 5.6 (Divergence of a vector field). Suppose that $p(\gamma)$ is an admissible vector field. We define its divergence $\text{Div } p(\gamma)$ at $\gamma \in \mathcal{H}_{x,y}$ by the following equality:

$$\text{Div } p(\gamma) = \text{tr} \rho^* B(\gamma)\rho = \text{tr} \rho^* \frac{\delta q(\gamma)}{\delta \gamma} \rho.$$

Remark 5.7 (Another expression of divergence). Let $p(\gamma)$ be an admissible vector field. Since $p(\gamma, s) = (\rho p(\gamma))(s)$ for $s \in [0, T]$, $p(\gamma, s) = (\rho \rho^* q(\gamma))(s)$. Therefore, it follows from (5.5) that

$$Dp(\gamma, s)[h] = (\rho \rho^* Dq(\gamma)[h])(s) = (\rho \rho^* B(\gamma)\rho h)(s).$$

Let $k_\gamma(s, t)$ be the integral kernel function of the trace class operator $\rho \rho^* B(\gamma)$. Then

$$(5.6) \quad Dp(\gamma, s)[h] = \int_0^T k_\gamma(s, t)(\rho h)(t) dt.$$

We often write $\frac{\delta p(\gamma, s)}{\delta \gamma(t)}$ for $k_\gamma(s, t)$, i.e.,

$$(5.7) \quad Dp(\gamma, s)[h] = \int_0^T \frac{\delta p(\gamma, s)}{\delta \gamma(t)} \rho h(t) dt.$$

The next Proposition follows from Proposition 5.3.

Proposition 5.8. *Assume $p(\gamma)$ is an admissible vector field. Then*

$$\text{Div } p(\gamma) = \int_0^T \frac{\delta p(\gamma, t)}{\delta \gamma(t)} dt.$$

The notion of admissible vector field defined above is an analogy to infinitesimal version of "admissible transformation" in the case of Wiener integral. cf.[13].

§ 5.3. m -smooth functional

We use the following notation : Let $\mathcal{Y}$ be a Banach space with norm $\|\cdot\|_{\mathcal{Y}}$. Let Δ be a division of $[0, T]$, γ_{Δ} and $\{x_{J+1}, x_J, \dots, x_1, x_0\}$ be as before. Assume that $F(\gamma_{\Delta})$ is a map $F : \Gamma(\Delta) \ni \gamma_{\Delta} \rightarrow F(\gamma_{\Delta}) \in \mathcal{Y}$ and is infinitely differentiable with respect to $(x_{J+1}, \dots, x_0)$. Let K be a nonnegative integer, m be a nonnegative constant and $X \geq 1$ be a constant. Then we define a norm of $F(\gamma_{\Delta})$ defined on $\Gamma(\Delta)$:

$$(5.8) \quad \|F(\gamma_{\Delta})\|_{\{\mathcal{Y}; \Delta, m, K, X\}} = \max_{\substack{0 \leq \alpha_j \leq K, \\ j=0, 1, \dots, J+1}} \sup_{(x_{J+1}, \dots, x_0) \in \mathbf{R}^{J+1}} (1 + |x_{J+1}| + \dots + |x_0|)^{-m} \left\| \prod_{j=0}^{J+1} X^{-|\alpha_j|} \partial_{x_j}^{\alpha_j} F(\gamma_{\Delta}) \right\|_{\mathcal{Y}}.$$

Moreover if $F(\gamma)$ is defined on $\mathcal{H}$, then we define

$$(5.9) \quad \|F\|_{\{\mathcal{Y}; m, K, X\}} = \sup_{\Delta} \|F\|_{\{\mathcal{Y}; \Delta, m, K, X\}},$$

where sup is taken over all divisions Δ of $[0, T]$. If $\mathcal{Y} = \mathbf{R}$ or $\mathbf{C}$, we simply write $\|F\|_{\{\Delta, m, K, X\}}$ and $\|F\|_{\{m, K, X\}}$.

Suppose that a functional $F(\gamma) : \mathcal{H}_{x,y} \rightarrow \mathbf{C}$ is Fréchet differentiable at γ . Then $DF(\gamma)$ denotes its differential. For $h \in \mathcal{H}_0$,

$$DF(\gamma)[h] = (DF(\gamma), h)_{\mathcal{H}_0} \quad (h \in \mathcal{H}_0).$$

Moreover, if there exists a density $f_{\gamma}(s) \in \mathcal{X}$ such that $DF(\gamma) = \rho^* f_{\gamma}$, i.e.,

$$(5.10) \quad DF(\gamma)[h] = \int_0^T f_{\gamma}(s) \rho h(s) ds \quad (h \in \mathcal{H}_0),$$

then we often write $\frac{\delta F(\gamma)}{\delta \gamma(s)}$ or $\delta F(\gamma)(s)$ for $f_{\gamma}(s)$.

Definition 5.9. Let $m \geq 0$ be a constant. We call $F(\gamma)$ an m -smooth functional if $F(\gamma)$ satisfies the following conditions.

F-I $F(\gamma)$ is an infinitely differentiable map from $\mathcal{H}$ to $\mathbf{C}$.

F-2 $\forall x, \forall y \in \mathbf{R}$ and $\gamma \in \mathcal{H}_{xy}$ the differential $DF(\gamma)$ has its density $\frac{\delta F(\gamma)}{\delta \gamma(s)}$, that is, $\forall \gamma \in \mathcal{H}_{x,y} \forall h \in \mathcal{H}_0$

$$DF(\gamma)[h] = \int_0^T \frac{\delta F(\gamma)}{\delta \gamma(s)} \rho h(s) ds,$$

F-3 Functional $\frac{\delta F(\gamma)}{\delta \gamma(s)}$ is a continuous functional of $\mathcal{H} \times [0, T] \ni (\gamma, s) \rightarrow \mathbf{C}$. It is infinitely differentiable with respect to $\gamma \in \mathcal{H}_{x,y}$ if s is fixed.

F-4 For any integer $K \geq 0$ there are constants $A_K > 0$ and $X_K \geq 1$ such that $\forall K = 0, 1, 2, \dots$,

$$(5.11) \quad A_K = \sup_{\gamma \in \mathcal{H}} \left(\|F(\gamma)\|_{\{m, K, X_K\}} + \sup_{s \in [0, T]} \left\| \frac{\delta F(\gamma)}{\delta \gamma(s)} \right\|_{\{m, K, X_K\}} \right) < \infty.$$

Remark 5.10. Let δ_2 be so small that $v_2 \delta_2^2 < 4$ and $v_2 \delta_2 < 1$. If $T \leq \delta_2$, then a m -smooth functional satisfies condition of N. Kumano-go 4.1 and it is "F-integrable".

§ 5.4. An integration by parts formula

Definition 5.11. Let m be a nonnegative number. We say that the vector field $p(\gamma)$ is an m -admissible vector field if it has all the following properties:

P1 p is an infinitely differentiable map $p : \mathcal{H} \ni \gamma \rightarrow p(\gamma) \in \mathcal{H}_0$ of which the restriction to $\mathcal{H}_{x,y}$ is an admissible vector field for any fixed $x, y \in \mathbf{R}$, that is, there are C^∞ maps $q : \mathcal{H} \rightarrow \mathcal{X}$ and $B : \mathcal{H} \rightarrow \mathcal{L}(\mathcal{X})$ such that $p(\gamma) = \rho^* q(\gamma)$ and that for $\gamma \in \mathcal{H}_{x,y}$ and all $h \in \mathcal{H}_0$, $Dq(\gamma)[h] = B(\gamma)\rho h$.

P2 The map $\mathcal{H} \ni \gamma \rightarrow B(\gamma) \in \mathcal{L}(\mathcal{X})$ is infinitely differentiable. For any integer $K \geq 0$ there exists a constant $Y_K \geq 1$ such that

$$(5.12) \quad B_K = \sup_{\gamma \in \mathcal{H}} \left(\|q(\gamma)\|_{\{\mathcal{X}, m, K, Y_K\}} + \|B(\gamma)\|_{\{\mathcal{L}(\mathcal{X}), m, K, Y_K\}} \right) < \infty.$$

We often write $\frac{\delta q(\gamma)}{\delta \gamma}$ for $B(\gamma)$.

Let δ_0 be as in (1.2). Our main theorem is the following cf.[9]:

Theorem 5.12 (Integration by parts). *Let $T \leq \delta_0$. Suppose that $F(\gamma)$ is an m -smooth functional and that $p(\gamma)$ is an m' -admissible vector field. We further assume that two of $DF(\gamma)[p(\gamma)]$, $F(\gamma)\text{Div}p(\gamma)$ and $F(\gamma)DS(\gamma)[p(\gamma)]$ are F-integrable. Then the rest is also F-integrable and the following equality holds.*

$$(5.13) \quad \begin{aligned} & \int_{\Omega_{x,y}} DF(\gamma)[p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}(\gamma) \\ &= - \int_{\Omega_{x,y}} F(\gamma) \text{Div} p(\gamma) e^{i\nu S(\gamma)} \mathcal{D}(\gamma) - i\nu \int_{\Omega_{x,y}} F(\gamma) DS(\gamma)[p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}(\gamma). \end{aligned}$$

Remark 5.13. cf. N.Kumano-go [12]. If $p(\gamma, s)$ is independent of γ , i.e., $p(\gamma, s) = h(s)$ then $\text{Div}p(\gamma) = 0$ and the above formula reduces to

$$(5.14) \quad \int_{\Omega_{x,y}} DF(\gamma)[h] e^{i\nu S(\gamma)} \mathcal{D}(\gamma) = -i\nu \int_{\Omega_{x,y}} F(\gamma) DS(\gamma)[h] e^{i\nu S(\gamma)} \mathcal{D}(\gamma).$$

We explain the idea of proof. We use the abbreviation:

$$N(\Delta) = \prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i \tau_j} \right)^{1/2},$$

and set $y_{\Delta,j} = p(\gamma_{\Delta}, T_j) = \rho p(\gamma_{\Delta})(T_j)$ for $j = 0, 1, \dots, J+1$, in particular $y_0 = 0 = y_{J+1}$. It is clear from definition of oscillatory integrals on $\mathbf{R}^J$ that

$$\int_{\mathbf{R}^J} \sum_{j=1}^J \frac{\partial}{\partial x_j} \left(F(\gamma_{\Delta}) y_{\Delta,j} e^{i\nu S(\gamma_{\Delta})} \right) \prod_{j=1}^J dx_j = 0.$$

It follows from this that

$$\begin{aligned} (5.15) \quad N(\Delta) \int_{\mathbf{R}^J} \sum_{j=1}^J \partial_{x_j} (F(\gamma_{\Delta})) y_{\Delta,j} e^{i\nu S(\gamma_{\Delta})} \prod_{j=1}^J dx_j \\ = -N(\Delta) \int_{\mathbf{R}^J} F(\gamma_{\Delta}) \sum_{j=1}^J \partial_{x_j} (y_{\Delta,j}) e^{i\nu S(\gamma_{\Delta})} \prod_{j=1}^J dx_j \\ - i\nu N(\Delta) \int_{\mathbf{R}^J} F(\gamma_{\Delta}) \sum_{j=1}^J y_{\Delta,j} \partial_{x_j} S(\gamma_{\Delta}) e^{i\nu S(\gamma_{\Delta})} \prod_{j=1}^J dx_j. \end{aligned}$$

Theorem 5.12 follows from the next Proposition.

Proposition 5.14.

$$\begin{aligned} (5.16) \quad \lim_{\Delta \rightarrow 0} N(\Delta) \int_{\mathbf{R}^J} F(\gamma_{\Delta}) \sum_{j=1}^J y_{\Delta,j} \partial_{x_j} S(\gamma_{\Delta}) e^{i\nu S(\gamma_{\Delta})} \prod_{j=1}^J dx_j \\ = \int_{\Omega} F(\gamma) DS(\gamma) [p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}(\gamma), \end{aligned}$$

$$\begin{aligned} (5.17) \quad \lim_{\Delta \rightarrow 0} N(\Delta) \int_{\mathbf{R}^J} \sum_{j=1}^J \partial_{x_j} (F(\gamma_{\Delta})) y_{\Delta,j} e^{i\nu S(\gamma_{\Delta})} \prod_{j=1}^J dx_j \\ = \int_{\Omega} DF(\gamma) [p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}(\gamma), \end{aligned}$$

$$\begin{aligned} (5.18) \quad \lim_{\Delta \rightarrow 0} N(\Delta) \int_{\mathbf{R}^J} F(\gamma_{\Delta}) \sum_{j=1}^J \partial_{x_j} (y_{\Delta,j}) e^{i\nu S(\gamma_{\Delta})} \prod_{j=1}^J dx_j \\ = \int_{\Omega} F(\gamma) \text{Div } p(\gamma) e^{i\nu S(\gamma)} \mathcal{D}(\gamma). \end{aligned}$$

Proof of (5.16). Since $\gamma_{\Delta}(t)$ is a piecewise classical path with edges at $t = T_j$ for $j = 1, 2, \dots, J$, integration by parts gives

$$\begin{aligned} (5.19) \quad DS(\gamma_{\Delta})[p(\gamma_{\Delta})] &= \int_0^T \left(\frac{d}{dt} \gamma_{\Delta}(t) \frac{d}{dt} p(\gamma_{\Delta}, t) - \partial_x V(t, \gamma_{\Delta}(t)) p(\gamma_{\Delta}, t) \right) dt \\ &= \sum_{j=1}^{J+1} \frac{d}{dt} \gamma_{\Delta}(T_j - 0) p(\gamma_{\Delta}, T_j) - \frac{d}{dt} \gamma_{\Delta}(T_{j-1} + 0) p(\gamma_{\Delta}, T_{j-1}) = \sum_{j=1}^J \partial_{x_j} S(\gamma_{\Delta}) y_{\Delta,j}. \end{aligned}$$

(5.16) is proved.

Idea of proof of (5.17). Since $F(\gamma)$ is m -smooth, $\delta F(\gamma) = \delta F(\gamma)/\delta\gamma \in \mathcal{X}$. We express the right hand side of (5.17) as a limit of time slicing approximation. Then we have only to prove that

$$(5.20) \quad \lim_{\Delta \rightarrow 0} N(\Delta) \int_{\mathbf{R}^J} \left(\sum_{j=1}^J \partial x_j(F(\gamma_\Delta)) y_{\Delta,j} - DF(\gamma_\Delta)[p(\gamma_\Delta)] \right) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^J dx_j = 0.$$

Let $\zeta_{\Delta,j}(t) = \partial x_j \gamma_\Delta(t)$ for $t \in [0, T]$. Then $\partial x_j F(\gamma_\Delta) = (\delta F(\gamma_\Delta), \zeta_{\Delta,j})_{\mathcal{X}}$. It is clear that $\zeta_{\Delta,j}(t) = 0$ if $t \notin [T_{j-1}, T_{j+1}]$ and that

$$(5.21) \quad \frac{d^2}{dt^2} \zeta_{\Delta,j}(t) + \partial_x^2 V(t, \gamma_\Delta(t)) \zeta_{\Delta,j}(t) = 0 \quad (t \in (T_{j-1}, T_j) \cup (T_j, T_{j+1})),$$

and that $\zeta_{\Delta,j}(T_{j-1}) = 0 = \zeta_{\Delta,j}(T_{j+1})$ and $\zeta_{\Delta,j}(T_j) = 1$. It is a piecewise C^1 continuous function.

We compare $\zeta_{\Delta,j}(t)$ with the piecewise linear function $e_{\Delta,j}(t)$ such that for $1 \leq j \leq J$

$$(5.22) \quad e_{\Delta,j}(t) = \begin{cases} 0 & \text{if } t \notin [T_{j-1}, T_{j+1}], \\ (t - T_{j-1})\tau_j^{-1} & \text{if } t \in [T_{j-1}, T_j], \\ (T_{j+1} - t)\tau_{j+1}^{-1} & \text{if } t \in [T_j, T_{j+1}]. \end{cases}$$

$e_{\Delta,0}(t)$ and $e_{\Delta,J+1}(t)$ are defined in such a way that

$$(5.23) \quad \sum_{j=0}^{J+1} e_{\Delta,j}(t) = 1 \quad (t \in [0, T]).$$

Then it turns out that for any α, β

$$(5.24) \quad |\partial_{x_{j-1}}^\alpha \partial_{x_j}^\beta (\zeta_{\Delta,j}(t) - e_{\Delta,j}(t))| = \mathcal{O}(\tau_j^2) \quad (t \in [T_{j-1}, T_j])$$

$$(5.25) \quad |\partial_{x_j}^\alpha \partial_{x_{j+1}}^\beta (\zeta_{\Delta,j}(t) - e_{\Delta,j}(t))| = \mathcal{O}(\tau_{j+1}^2) \quad (t \in [T_j, T_{j+1}]).$$

Therefore,

$$\begin{aligned} DF(\gamma_\Delta)[p(\gamma_\Delta)] - \sum_j \partial x_j F(\gamma_\Delta) y_{\Delta,j} &= DF(\gamma_\Delta)[p(\gamma_\Delta)] - \sum_j y_{\Delta,j} (\delta F(\gamma_\Delta), \zeta_{\Delta,j})_{\mathcal{X}} \\ &= \sum_j (\delta F(\gamma_\Delta), (\rho p(\gamma_\Delta) - y_{\Delta,j}) e_{\Delta,j})_{\mathcal{X}} - \sum_j y_{\Delta,j} (\delta F(\gamma_\Delta), (e_{\Delta,j} - \zeta_{\Delta,j}))_{\mathcal{X}}, \end{aligned}$$

Using (5.24), we can show

$$(5.26) \quad \sum_j y_{\Delta,j} (\delta F(\gamma_\Delta), (e_{\Delta,j} - \zeta_{\Delta,j}))_{\mathcal{X}} = \mathcal{O}(|\Delta|T).$$

Since $p(\gamma)$ is m' -admissible, $\rho p(\gamma_\Delta)(t) = \rho \rho^* q(\gamma_\Delta)$ is in $C^1([0, T])$. As $\rho p(\gamma_\Delta)(T_j) = y_{\Delta,j}$ and $e_{\Delta,j}$ vanishes outside $[T_{j-1}, T_{j+1}]$, we can show

$$(\rho p(\gamma_\Delta)(t) - y_{\Delta,j}) e_{\Delta,j}(t) = \mathcal{O}(\tau_j + \tau_{j+1}) \quad (t \in [0, T]).$$

Hence

$$(5.27) \quad \sum_j (\delta F(\gamma_\Delta), (\rho p(\gamma_\Delta)(t) - y_{\Delta,j}) e_{\Delta,j})_{\mathcal{X}} = \mathcal{O}(|\Delta|T).$$

It follows from (5.26), (5.27) and Theorem 3.2 that

$$(5.28) \quad N(\Delta) \int_{\mathbf{R}^J} \left(\sum_{j=1}^J \partial x_j (F(\gamma_\Delta)) y_{\Delta,j} - DF(\gamma_\Delta)[p(\gamma_\Delta)] \right) e^{i\nu S(\gamma_\Delta)} \prod_{j=1}^J dx_j = \mathcal{O}(T|\Delta|).$$

This shows (5.20).

Similarly, we can show (5.18).

§ 6. Application to semiclassical asymptotic behaviour of Feynman path integrals

We always assume $T < \delta$. Let $F(\gamma)$ be an m -smooth functional. Then semiclassical asymptotic formula (4.3) was proved by Kumano-go [12]. The principal part of (4.3) is $F(\gamma^*)$, the value of F at the classical path γ^* .

What happens if $F(\gamma^*) = 0$? Integration by parts formula enables us to get a sharper information even in this case.

Assumption 6.1. 1. $F(\gamma)$ is a real valued m -smooth functional. For fixed $\gamma \in \mathcal{H}_{x,y}$, $\frac{\delta F(\gamma)}{\delta \gamma(s)}$ is a $\mathcal{X}$ -valued function, which we write $\frac{\delta F(\gamma)}{\delta \gamma}$. The map $\mathcal{H}_{x,y} \ni \gamma \rightarrow \frac{\delta F(\gamma)}{\delta \gamma} \in \mathcal{X}$ is a C^∞ map. There exists a C^∞ map $\mathcal{H}_{x,y} \ni \gamma \rightarrow A(\gamma) \in B(\mathcal{X})$ such that for any $h \in \mathcal{H}_0$,

$$(6.1) \quad D \frac{\delta F(\gamma)}{\delta \gamma} [h] = A(\gamma) \rho h.$$

2. Linear operator $A(\gamma)$ has the integral kernel $k_\gamma(s, t)$ which is continuous in $(s, t) \in [0, T] \times [0, T]$ and we have for any $K = 0, 1, 2, \dots$

$$(6.2) \quad \sup_{(s,t)} \|k_\gamma(s, t)\|_{\{m, K, X_K\}} < \infty.$$

Suppose $F(\gamma)$ satisfies the above conditions and $F(\gamma^*) = 0$. Let $\gamma_\theta = \theta\gamma + (1 - \theta)\gamma^*$ for $0 \leq \theta \leq 1$. We define an element $\zeta(\gamma) \in \mathcal{X}$ by

$$(6.3) \quad \zeta(\gamma, t) = \int_0^1 \frac{\delta F(\gamma)}{\delta \gamma(t)} \Big|_{\gamma=\gamma_\theta} d\theta.$$

Let $\tilde{W}(\gamma)$ be the multiplication operator in $\mathcal{X}$ defined by

$$(6.4) \quad \mathcal{X} \ni g(s) \rightarrow \tilde{W}(\gamma, s)g(s) \quad (g \in \mathcal{X}),$$

where

$$(6.5) \quad \tilde{W}(\gamma, s) = \int_0^1 \partial_x^2 V(s, \gamma_\theta(s)) d\theta.$$

Since $T < \delta$, $(I - \tilde{W}(\gamma)\rho\rho^*)^{-1} \in \mathcal{L}(\mathcal{X})$. We define a vector field

$$(6.6) \quad p(\gamma) = \rho^*(I - \tilde{W}(\gamma)\rho\rho^*)^{-1}\zeta(\gamma).$$

Proposition 6.2. *If $F(\gamma)$ satisfies our assumptions and $F(\gamma^*) = 0$, then $p(\gamma)$ is an m -admissible vector field and*

$$(6.7) \quad DS(\gamma)[p(\gamma)] = F(\gamma).$$

Thus $DS(\gamma)[p(\gamma)]$ is F -integrable. The following equality holds:

$$(6.8) \quad \int_{\Omega_{xy}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma] = \int_{\Omega_{xy}} DS(\gamma)[p(\gamma)] e^{i\nu S(\gamma)} \mathcal{D}[\gamma].$$

We can apply the integration by parts theorem 5.12 and obtain

Theorem 6.3. *Suppose $F(\gamma)$ is an m -smooth functional with some $m \geq 0$ and it satisfies the additional assumption 6.1. Assume further that $F(\gamma^*) = 0$. Define $\zeta(\gamma, t)$ and $p(\gamma)$ as above. Then we have*

$$(6.9) \quad \int_{\Omega_{xy}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma] = -(i\nu)^{-1} \int_{\Omega_{xy}} \text{Div} p(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma].$$

Apply Kumano-go's theorem of semiclassical asymptotics to (6.9), we have the following theorem.

Theorem 6.4. [12]. *Under the same assumption as in Theorem 6.3 the following asymptotic formula holds:*

$$\begin{aligned} & \int_{\Omega_{xy}} F(\gamma) e^{i\nu S(\gamma)} \mathcal{D}[\gamma] \\ &= \left(\frac{-i\nu}{2\pi T} \right)^{1/2} D(T, 0, x, y)^{-1/2} e^{i\nu S(\gamma^*)} \left(-(i\nu)^{-1} \text{Div} p(\gamma^*) + \nu^{-2} r(\nu, T, 0, x, y) \right). \end{aligned}$$

For $\forall \alpha, \beta$ there exists a constant $C_{\alpha\beta} > 0$ such that

$$(6.10) \quad \left| \partial_x^\alpha \partial_y^\beta r(\nu, T, 0, x, y) \right| \leq C_{\alpha\beta} (1 + |x| + |y|)^m.$$

Let $G_{\gamma^*}(t, s)$ be the Green function of differential equation of Jacobi field:

$$(6.11) \quad - \left(\frac{d^2}{dt^2} + \partial_x^2 V(t, \gamma^*(t)) \right) u(t) = f(t), \quad u(0) = 0 = u(T).$$

Calculation shows:

Theorem 6.5. *Under the same assumption as in Theorem 6.4*

$$\begin{aligned} \text{Div} p(\gamma^*) &= \frac{1}{2} \int_0^T \int_0^T \frac{\delta}{\delta \gamma(t)} (G_{\gamma^*}(t, s) \frac{\delta F(\gamma^*)}{\delta \gamma(s)}) ds dt \\ &= \frac{1}{2} \int_0^T \int_0^T \frac{\delta G_{\gamma^*}(t, s)}{\delta \gamma(t)} \frac{\delta F(\gamma^*)}{\delta \gamma(s)} ds dt + \frac{1}{2} \int_0^T \int_0^T G_{\gamma^*}(t, s) \frac{\delta^2 F(\gamma^*)}{\delta \gamma(s) \delta \gamma(t)} ds dt. \end{aligned}$$

Remark 6.6 (The 2nd moment of Feynman path integral). Let

$$F(\gamma) = \int_0^T \int_0^T (\gamma(s) - \gamma^*(s))(\gamma(t) - \gamma^*(t))a(s, t) dsdt.$$

Then

$$\begin{aligned} & \int_{\Omega_{x,y}} e^{i\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] \\ &= \left(\frac{\nu}{2\pi iT} \right)^{1/2} D(T, 0, x, y)^{-1/2} e^{i\nu S(\gamma^*)} \\ & \quad \times \left(-(i2\nu)^{-1} \int_0^T \int_0^T G_{\gamma^*}(s, t) a(s, t) dsdt + \nu^{-2} r(\nu, T, 0, x, y) \right). \end{aligned}$$

Here $r(\nu, T, 0, x, y)$ satisfies (6.10) with $m = 2$.

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