Deformation of $\mathcal{N} = 4$ SYM with space-time dependent couplings

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Abstract

We study deformation of $\mathcal{N}=4$ supersymmetric Yang-Mills theory so that couplings and masses depend on space-time. Since $\mathcal{N}=4$ super Yang-Mills theory is the world-volume action of D3-branes, our deformation is corresponding to turning on non-trivial supergravity fields where a stack of D3-brane is placed. We obtain the conditions of the space-time dependent deformation while preserving part of the supersymmetry. And we find the some kind of solutions to satisfy these conditions by imposing some specific global symmetries; $ISO(1,1)\times SO(3)\times SO(3)$, $ISO(1,1)\times SO(2)\times SO(4)$, $SO(3)$, and $ISO(2)\times SO(6)$. Especially, the solutions of $ISO(1,1)\times SO(3)\times SO(3)$ case includes Janus configuration found by Gaiotto and Witten as well as the world-volume theory of D3-branes embedded in F-theory (a background with 7-branes in type IIB string theory) by limiting the parameters so that the symmetry is enhanced. We also found the general solution of the supersymmetry conditions for the cases with $ISO(1,1)\times SO(2)\times SO(4)$ symmetry. The cases with time dependent couplings and/or masses are also considered. To make these solutions have interpretations in string theory, we write down the non-Abelian world-volume action of D3-branes in general supergravity background. And we compare the parameters of deformed Yang-Mills theory and supergravity fields. As examples, we study the world-volume action of D3-branes in $[p,q]$ 5-branes background and D3-branes background. For each cases, we compare their actions with the ones obtained for $ISO(1,2)\times SO(3)\times SO(3)$ case, which is Janus configuration, and $ISO(1,1)\times SO(2)\times SO(4)$ case.

Contents

Chapter 1

Introduction

D-branes shed light on many aspects of string theory. Especially, D-brane roles a mediator of closed string theory and gauge theory. Now we know that these two theories are closely related to each other, e.g., $\mathcal{N} = 4$ super Yang-Mills theory (SYM), which is the world-volume theory of D3-branes, and Type IIB supergravity on $AdS_5 \times S^5$, which is near horizon limit of D3-branes. They are dual to each other, so we can study one by using another. Therefore it is worthy to study some unusal types of gauge theory, which are expected to exist from string theory viewpoint, and their relations with 10 dimension supergravity theory.

Quantum field theories (QFT) with broken Poincare symmetry is the one of the examples. Since D-branes break Poincare symmetry as well as supersymmetry partially, we can face with many such cases that world-volume theories on D-branes lose super-Poincare symmetry partially when another D-branes are inserted. This is the motivation of our works. We will focus on 4d supersymmetric gauge theory with broken Poincare symmetry, which is expected to D3-branes world-volume theory when the background is non-trivial BPS state.

Space-time dependent coupling sounds something unfamiliar, however, in string theory perspective, it appears naturally. For example, in the world-volume gauge theory of D3-branes, gauge coupling and theta parameter in gauge theory are related to the value of dilaton field ϕ and R-R 0-form field C_0 respectively. Since D7-branes or more generally $[p, q]$ 7-branes gives complex gauge coupling τ non-trivial monodromies around them, τ must not be constant when the D3-branes are intersected with $[p, q]$ 7-branes. Regarding τ as the modulus of a torus, we can think of uplifting the 10-dimensional space-time with 7-branes to an elliptic fibered 12-dimensional space-time, which is called F-theory [3]. Another interesting example of varying coupling is the so-called Janus configuration, in which the coupling depends on one of the spatial coordinate [4, 5].

We deform $\mathcal{N}=4$ SYM to have space-time dependence for gauge couplings. And we find the conditions to preserve some part of the supersymmetry under the deformation. It is a natural extension of the works to find curved space-times preserving SUSY, for which systematic methods using, e.g., topological twist [6] or supergravity [7] have been developed. We are of course not the first ones to consider this class of theories. Supersymmetric Janus configurations were studied in [8, 9, 10, 11, 12, 13, 14, 15]. Their generalization to the configurations with couplings depending on more than one directions were also investigated in [16, 17]. SUSY configurations in $\mathcal{N}=4$ SYM with varying couplings have been studied for example in [18, 19, 20, 21, 22, 23, 24] (see also [25, 26, 27] for earlier works). Time dependent couplings have also been considered for example in [28, 29, 30, 31, 32, 33, 34].

Our method is quite rudimentary. we deform $\mathcal{N}=4$ SYM by adding all possible renormalizable and gauge invariant operators with space-dependent coupling parameters, and let metric as well as gauge couplings be functions of space-time. And we can find the conditions to preserve SUSY by variation of this deformed action under SUSY transformation, which is generalized as adding new relevant parameters and setting SUSY parameters to have space-time dependence.

We investigate some specific cases, which are imposed special space-time symmetry as well as R-symmetry. For example, we can impose the theory to keep $ISO(1,1)$ as space-time symmetry of 4 dimension, which implies the couplings have 2 spatial direction dependence, and to keep $SO(3)\times SO(3)$ as R-symmetry. We can study the cases with other symmetries, e.g., $ISO(1,1)\times SO(2)\times SO(4)$ and $ISO(2)\times SO(6)$.

We expect that our deformed action has string theory interpretation as the world-volume theory on probe D3-branes in non-trivial BPS background. In other words, our analysis of the field theory is corresponding to surveying BPS solutions of supergravity by means of D3-branes. For example, the SUSY condition for the deformed N = 4 SYM should be related to the conditions to preserve SUSY with the D3-branes, which is Killing spinor equations of supergravity theory. We will see that in special cases we studied we can reproduce the supergravity BPS solutions as solutions of SUSY preserving conditions in field theory.

Since in non-Abelian case the world-volume action of D-branes include additional coupling terms with background fields, it is quite non-trivial non-Abelian D3 brane world-volume action in general backgrounds. The formulation for bosonic and fermionic action of non-Abelian D-brane is known, [37], [39]. We will follow these by considering the background that all possible type IIB supergravity fields are turned on.

Also we studied the situations of specic BPS backgrounds, e.g., (p,q) 5 branes, $[p,q]$ 7 branes and D3 branes background. And we compare the actions of these cases with the ones we have obtained by imposing $ISO(1,2) \times SO(3) \times SO(3)$, $ISO(1,1) \times SO(6)$, and $ISO(1,2) \times SO(2) \times SO(4)$ symmetry in the field theory analysis.

Because it is commonly faced situation when inserting probe branes in BPS backgrounds, we hope that this results of our research will be helpful and fruitful for those who wants to study supersymmetric QFT as string theory frame.

The organization of the paper is as follows. In Section 2, we determine conventions about type IIB supergravity. And we also give reviews about supersymmetric Janus solutions and non-Abelian world-volume theory of D-branes in curved background with fluxes. In Section 3, we show the conditions to preserve SUSY are derived. The details for the calculation are summarized in Appendix D. It turns out that one of the SUSY conditions become simple under a certain symmetry. Appendix E provides an explanation for this fact. In Sections 4 and 5, we demonstrate how our formalism works by examining some explicit examples. We will analyze the case with $ISO(1,1) \times SO(3) \times SO(3)$ symmetry in detail in Section 4. We show in Section 4.2.1 that the SUSY conditions reduce to two simple equations (4.2.33) and (4.2.41). As shown in Section 4.2.2 and Section 4.2.3, large classes of solutions of these equations are found. An F-theory configuration and the supersymmetric Janus configuration reviewed in 2 are obtained as special solutions. The case with $ISO(1,1) \times SO(2) \times SO(4)$ symmetry is studied in Section 5.1, in which the general solution for this case is found. Some examples with time dependence are discussed in Section 5.2. In Section 6, we present the explicit form of non-Abelian D3-branes action in general supergravity upon leading order expansion. The detail calculations to obtain the effective action of D3-branes is given in Appendix F. And then we establish a map between the background fields and the deformation parameters of the theory in Section 3. In Section 7 we apply the results of the previous section to study two cases: backgrounds of (p,q) 5-branes and backgrounds of D3-branes. And we compare results of these two cases with the results of the field theory with corresponding symmetry, 4.4 and 5.1. In Section 8 we conclude the paper with various discussions on the present work and further applications.

This ph.D thesis is based on the paper, [1] and [2]. I am grateful to my collaborators of this work, José J. Fernández-Melgarejo and Shigeki Sugimoto.

Chapter 2

Reviews and Conventions

2.1 Type IIB Supergravity

In this section, we give brief review of D-branes, and denote the convention of our paper for the afterward. In string theory, there are two types of string, closed strings and open strings.

In this paper, because our attention focus on D3-branes, we will deal with Type IIB supergravity. The bosonic action of Type IIB supergravity is,*

$$\begin{aligned} S_{\rm IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\Big\{e^{-2\phi}\left(R+4\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}|H_3|^2\right)-\frac{1}{2}\left(|F_1|^2+|\widetilde{F}_3|^2+\frac{1}{2}|\widetilde{F}_5|^2\right)\Big\} \\ &\quad +\frac{1}{4\kappa_{10}^2}\int\left(C_4+\frac{1}{2}B_2\wedge C_2\right)\wedge F_3\wedge H_3\ , \end{aligned} \tag{2.1.1}$$

where

$$H_3 = dB_2\ ,\quad F_n = dC_{n-1}\ ,\quad \widetilde{F}_n = F_n + H_3\wedge C_{n-3}\ . \tag{2.1.2}$$

The square of n-form, $|\omega_n|^2$, is defined as,

$$|\omega_n|^2 \equiv \frac{1}{n!}\overline{\omega}_{M_1\cdots M_n}\omega_{N_1\cdots N_n}g^{M_1N_1}\cdots g^{M_nN_n}\ . \tag{2.1.3}$$

κ_{10} is related by 10 dimensional Newton constant by

$$2\kappa_{10}^2 = 16\pi G_N = (2\pi)^7 l_s^8 g_s^2. \tag{2.1.4}$$

g_s is a string coupling constant. Be careful about that we have used the convention which the dilaton field ϕ become zero at the infinite.

*We have followed the convention of [37].

In addition, we have to impose the self-duality condition

$$\widetilde{F}_5 = *\widetilde{F}_5 \ . \tag{2.1.5}$$

The Hodge star $*$ is defined by

$$*(dx^{M_1} \wedge \cdots \wedge dx^{M_n}) = \frac{1}{(10-n)!}\epsilon^{M_1 \cdots M_{10}} g_{M_{n+1}N_{n+1}} \cdots g_{M_{10}N_{10}} dx^{N_{n+1}} \wedge \cdots \wedge dx^{N_{10}} \ , \tag{2.1.6}$$

where $\epsilon^{M_1 \cdots M_{10}}$ is the 10-dimensional epsilon tensor (Levi-Civita symbol) with $\epsilon^{01 \cdots 9} = 1/\sqrt{-g}$. It is useful to define $\widetilde{F}_n$ with $n > 5$ by magnetic dual of F

$$\widetilde{F}_n \equiv (-1)^{\frac{1}{2}n(n+1)+1} * \widetilde{F}_{10-n} \ . \tag{2.1.7}$$

The equations of motion and the Bianchi identities for the RR fields are written as

$$d\widetilde{F}_n + H_3 \wedge \widetilde{F}_{n-2} = 0 \ , \quad (n = 1, 3, 5, 7, 9) \ , \tag{2.1.8}$$

where $\widetilde{F}_{-1} \equiv 0$. This allows us to introduce C_{n-1} satisfying (2.1.2) for $n = 1, 3, 5, 7, 9$.

It is useful to note that type IIB supergravity action has global $SL(2, \mathbb{Z})$ symmetry. For convenience, we introduce

$$\tau \equiv g_s^{-1}(C_0 + ie^{-\phi}), \quad \mathcal{M} \equiv \frac{1}{\mathrm{Im}\tau}\begin{bmatrix} 1 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & |\tau|^2 \end{bmatrix}, \quad \mathbf{B} \equiv \begin{bmatrix} g_s^{-\frac{1}{2}} C_2 \\ g_s^{\frac{1}{2}} B_2 \end{bmatrix}, \quad \mathbf{H} \equiv d\mathbf{B} \tag{2.1.9}$$

Using Einstein frame,

$$e^{-\frac{\phi}{2}} g_{MN}^{\text{string}} = g_{MN}^{\text{Einstein}}, \tag{2.1.10}$$

we can re-write (2.1.1) with the redefined ones,

$$\begin{aligned} S_{\text{IIB}} = \quad & \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g^E}\left\{R^E - \frac{1}{12}\mathbf{H}_{MNO}^T\mathcal{M}\mathbf{H}^{MNO} + \frac{1}{4}\,\mathrm{tr}\left(\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\right) - \frac{1}{4}|\widetilde{F}_5|^2\right\} \\ & + \frac{1}{8\kappa_{10}^2}\int\left(C_4 + \frac{1}{2}B_2 \wedge C_2\right) \wedge (\epsilon_{ij}\mathbf{H}^{(i)} \wedge \mathbf{H}^{(j)}) \, . \end{aligned} \tag{2.1.11}$$

This form of type IIB action shows that this action holds $SL(2, \mathbb{Z})$ symmetry.† And transformation rules of $SL(2, \mathbb{Z})$ are,

$$\Lambda = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \in SL(2, \mathbb{Z}), \quad \mathbf{B} \to \Lambda\mathbf{B}, \quad \left(g_{MN}^{\text{Einstein}},\, C_4 + \frac{1}{2}B_2 \wedge C_2,\, \kappa_{10}\right) \ : \text{invariant},$$

$$\mathcal{M} \to (\Lambda^{-1})^T\mathcal{M}\Lambda^{-1} \ \Leftrightarrow \ \tau \to \frac{a\tau + b}{c\tau + d} \, . \tag{2.1.12}$$

†Actually $SL(2, R)$ is a symmetry of this action, but because in order to satisfy Dirac quantization with respect to R-R fields or NS-NS fields, the elements of this group also should be quantized, resulting in $SL(2, \mathbb{Z})$.

The SUSY transformation rules for gravitino ψ_M and dilatino λ are

$$\delta\psi_M = D_M\epsilon + \frac{1}{8\cdot 2!}(H_3)_{MNO}\Gamma^{NO}\otimes\sigma_3\,\epsilon + \tilde{\Omega}\Gamma_M\epsilon \qquad (2.1.13)$$

$$\delta\lambda = (\Gamma^N\partial_N\phi)\epsilon + \frac{1}{2\cdot 3!}(H_3)_{NOP}\Gamma^{NOP}\otimes\sigma_3\,\epsilon + \Gamma^N\tilde{\Omega}\Gamma_N\epsilon, \qquad (2.1.14)$$

where

$$\tilde{\Omega} = \frac{e^{\phi}}{8}\quad\big((\widetilde{F}_1)_M\Gamma^M\otimes i\sigma_2 - \tfrac{1}{3!}(\widetilde{F}_3)_{MNO}\Gamma^{MNO}\otimes\sigma_1 + \tfrac{1}{2\cdot 5!}(\widetilde{F}_5)_{MNOPQ}\Gamma^{MNOPQ}\otimes i\sigma_2\big), \qquad (2.1.15)$$

$$\Gamma^{(10)}\epsilon = -\epsilon, \quad \Gamma^{(10)}\psi_M = -\psi_M, \quad \Gamma^{(10)}\lambda = +\lambda. \qquad (2.1.16)$$

Here, the $\sigma_1, i\sigma_2, \sigma_3$ are generators of $SL(2,R)$ which act on R-symmetry index. We denote 10 dimensional gamma matrix as Γ_M, a chirality gamma matrix as $\Gamma^{(10)} = \Gamma^{\hat{0}\hat{1}\hat{2}\cdots\hat{9}}$ and use indices $\hat{M}, \hat{N}, \cdots$ for local flat indices, which can mapped from curved one by veilbeins, $e^M_{\hat{M}}$. In order to consider bosonic classical backgrounds, we can think the situation that ψ_M, λ, $\delta\psi_M$, $\delta\lambda$ vanish. By setting (2.1.13) and (2.1.14) to be zero, we can get so-called Killing spinor equations.

D_p-branes ($p = 1,3,5,7$) can be found as solitonic BPS solutions in supergravity, which break Poincare symmetry to $ISO(1,p)\times SO(9-p)$ and preserve 16 SUSY.[‡] The solution is given as,

$$ds^2 = H_p(r)^{-\frac{1}{2}}\eta_{ab}dx^a dx^b + H_p(r)^{\frac{1}{2}}\delta_{ij}dx^i dx^j, \quad e^{\phi} = H(r)_p^{(3-p)/4},$$
$$C_{p+1} = (H_p(r)^{-1}-1)dx^0\wedge dx^1\wedge\cdots\wedge dx^p, \quad C_{p'\neq p+1} = \text{constant}, \quad H_3 = 0\,, \qquad (2.1.17)$$

where a, b and i, j denote parallel and perpendicular directions to D-branes respectively, and $r = \sqrt{x^i x_i}$ is radial direction on from D-branes. $H_p(r)$ is defined as

$$H_p(r) = 1 + \left(\frac{r_p}{r}\right)^{7-p}, \quad r_p^{7-p} = l_s^{7-p}g_s n(2\sqrt{\pi})^{5-p}\Gamma\big(\frac{7-p}{2}\big). \qquad (2.1.18)$$

One can check this solution satisfying Killing spinor equations as well as equations of motion of supergravity. This solution implies that D-branes are sources of R-R fields, $\int_{S^{8-p}} *F_{p+2} = 2\pi n$. Therefore, n should be integer to satisfy Dirac quantization conditions.

For $p = 3, 7$ case, there are modifications to this form of solutions. For D3-branes case, in order to impose F_5 to be self-dual, F_5 should be

$$F_5 = dH_3(r)^{-1}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3 + *(dH_3(r)^{-1}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3)\,. \qquad (2.1.19)$$

[‡]In this paper, we do not deal with non-extremal D-branes which can be found from non-supersymmetric solutions of equations of motion in the supergravity.

For a stack of n D7-branes case, because $H_7(r)$ is a harmonic function with respect to the transverse directions of D7-branes, which satisfies

$$\partial_r H_7(r) = -\frac{g_s n}{2\pi r}, \tag{2.1.20}$$

so naively we can mend $H_7(r)$ as

$$H_7(r) = \text{constant} - \frac{g_s n}{2\pi} \log r. \tag{2.1.21}$$

However, $H_7(r)$ can be negative at large r, resulting e^{ϕ} or space direction metric to be negative. So, more re-consideration is needed.

2.2 $[p, q]$ 7-brane and (p, q) 5-brane solution

In order to focus on D7 branes solution, let's turn off B_2, C_2 and C_4 and write type IIB supergravity in Einstein frame, (2.1.11),

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left\{ R - \frac{1}{2} \frac{\partial_z \tau \partial_{\bar{z}} \bar{\tau}}{(\mathrm{Im}\tau)^2} \right\}, \tag{2.2.1}$$

where $(z, \bar{z})$ is complex coordinate on the transverse plane to D7-branes. The equation of motions are

$$\partial_z \partial_{\bar{z}} \tau - 2 \frac{\partial_z \tau \partial_{\bar{z}} \tau}{\tau - \bar{\tau}} = 0\,, \tag{2.2.2}$$

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{4(\mathrm{Im}\tau)^2} (\partial_M \tau \partial_N \bar{\tau} + \partial_N \tau \partial_M \bar{\tau} - g_{MN} \partial_P \tau \partial^P \tau) \tag{2.2.3}$$

A solution of (2.2.2) and (2.2.3) is

$$\tau(z, \bar{z}) = \tau(z), \quad g_{z\bar{z}} = \mathrm{Im}\tau(z)\, |F(z)|^2, \tag{2.2.4}$$

where $\mathrm{Im}\tau(z) > 0$ and $g_{z\bar{z}}$ is metric component of Einstein frame,

$$ds^2 = \eta_{ab} dx^a dx^b + 2 g_{z\bar{z}}(z, \bar{z}) dz d\bar{z}. \tag{2.2.5}$$

Since $SL(2, \mathbb{Z})$ transformation of $\tau(z)$ is the same form of modular transformation of complex structure moduli of torus, so we will think the elliptic fibration whose fiber has modular parameter $\tau(z)$ and $(z, \bar{z})$ plane as base space. Given the singularities and their monodromies of $\tau(z)$, we can determine its form. Let's consider elliptic fibration,

$$y^2 = x^3 + f(z)x + g(z). \tag{2.2.6}$$

$f(z)$ and $g(z)$ are polynomials of z. Using modular invariant Jacobi j-function,*

$$j(\tau) = \frac{(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_0(\tau)^8)^3}{\eta(\tau)^{24}}, \tag{2.2.8}$$

complex structure moduli τ is obtained by

$$j(\tau(z)) = 4\frac{(12f(z))^3}{4f(z)^3 + 27g(z)^2}\,. \tag{2.2.9}$$

The singularities can be found if the determinant $\Delta \equiv 4f(z)^3 + 27g(z)^2 = 0$. For D7-branes located at $(z_1, \cdots, z_N)$ case, (2.2.9) can be written nearby z_i,

$$j(\tau(z)) \sim \frac{4 \cdot 12^3 f(z_i)}{\Delta(z_i)'(z - z_i)}\,. \tag{2.2.10}$$

Since that

$$j(\tau) \sim 2^3 \exp(-2\pi i \tau) \tag{2.2.11}$$

when Im $\tau \to \infty$ in (2.2.8), we can find asymptotic behavior of τ as,

$$\tau(z) \sim \text{contant} + \frac{1}{2\pi i}\log(z - z_i)\,. \tag{2.2.12}$$

This gives the monodromies around z_i such that $\tau \to \tau + 1$ by real part and also agreement with(2.1.21) by imaginary part. This is exactly the properties what we expect for D7-branes.

For simple example, we focus on single D7-brane. Choose

$$f(z) = (z - \alpha_1)(z - \alpha_2),\ g(z) = \frac{2\beta}{3\sqrt{3}}(z - \alpha_1)(z - \alpha_2)^2,\ (\alpha_1 \neq \alpha_2) \tag{2.2.13}$$

then j-function become

$$j(\tau) = 12^3\frac{(z - \alpha_1)}{(1 + \beta^2)z - (\alpha_1 + \beta^2\alpha_2)}. \tag{2.2.14}$$

We can find this case is corresponding to a D7-brane whose τ has singularity at $z = (\alpha_1 + \beta^2\alpha_2)/(1 + \beta^2)$, and asymptotic value at infinity $\tau(\infty) = j^{-1}(12^3/(1 + \beta^2))$.

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*θ function and Dedekind η-function are,

$$\theta_2(\tau) \equiv 2e^{\frac{\pi i\tau}{4}}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})\prod_{m=1}^{\infty}(1 + e^{2\pi i m\tau})^2, \quad \theta_3(\tau) \equiv \prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})\prod_{m=1}^{\infty}(1 + e^{2\pi i(m-\frac{1}{2})\tau})^2,$$

$$\theta_3(\tau) \equiv \prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})\prod_{m=1}^{\infty}(1 - e^{2\pi i(m-\frac{1}{2})\tau})^2, \quad \eta(\tau) \equiv e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau}). \tag{2.2.7}$$

Next, let's look for the $F(z)$ in $g_{z\bar{z}}$, (2.2.4). $F(z)$, which is element of Einstein frame metric, should be invariant under $SL(2,\mathbb{Z})$ transformation and globally well defined. Under $SL(2,\mathbb{Z})$ transformation,

$$\eta(\tau) \to \omega\sqrt{c\tau + d}\,\eta(\tau), \tag{2.2.15}$$

where $\omega^{24} = 1$. So $\mathrm{Im}\tau|\eta(\tau)|^4$ is invariant under $SL(2,\mathbb{Z})$ transformation. Therefore, we can put $F(z)$ to be the form,

$$F(z) = \eta(\tau(z))^2\tilde{F}(z), \tag{2.2.16}$$

where $\tilde{F}(z)$ does not depend on $\tau(z)$ explicitly. Nearby singular point, z_i, the asymptotic value of $\eta(\tau(z))$,

$$\eta(\tau(z)) \sim \text{constant } (z - z_i)^{\frac{1}{24}} \tag{2.2.17}$$

In order to the metric to be globally well defined, $\tilde{F}(z)$ should be have the form up to constant factor,

$$\tilde{F}(z) = \prod_{i=1}^{N}(z - z_i)^{-\frac{1}{12}}. \tag{2.2.18}$$

As a result, using some constant c, the metric component $g_{z\bar{z}}$ should be

$$g_{z\bar{z}} = c\,\mathrm{Im}\tau|\eta(\tau)|^4\left|\prod_{i=1}^{N}(z - z_i)^{-\frac{1}{12}}\right|^2. \tag{2.2.19}$$

This metric has property that the defect angle when rotating $(z - z_i) \to (z - z_i)e^{2\pi i}$ is $\frac{\pi}{6}$. So, when the sum of defect angle is 4π, which corresponds to 24 7-branes, the space become compact.

We have mainly think about D7-brane, however there are other 7-branes which has singularities and $SL(2,\mathbb{Z})$ monodromies. $[p,q]$ 7-branes, which is basically the 7-brane which (p,q) string can end, can be obtained by $SL(2,\mathbb{Z})$ transformation of the one of a D7-brane. The $SL(2,\mathbb{Z})$ transformation matrix Λ in (2.1.12)that transform from a fundamental string to (p,q) string is, since charge (p,q) can be counted by the flux of magnetic dual field of $\mathbf{B}$,

$$\Lambda_{[p,q]} = \begin{bmatrix} p & b \\ q & d \end{bmatrix}, \tag{2.2.20}$$

where $pd - bq = 1$. Therefore, the monodromy for $[p,q]$ 7-brane, which can be obtained from one of a D7-brane, is

$$M_{[p,q]} = \Lambda_{[p,q]}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\Lambda_{[p,q]}^{-1} = \begin{bmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{bmatrix}. \tag{2.2.21}$$

(p,q) 5-branes solutions, which mean the bound state of p NS5-branes and q D5-branes, can be obtained from $SL(2,\mathbb{Z})$ transformation of a D5-brane solution, given by (2.1.17). The $SL(2,\mathbb{Z})$ transformation to make (p, q) 5 branes is given by replacing (a,c) to $(p,-q)$ in (2.1.12). For a D5 brane solution, axion-dilaton can be written as

$$\tau = g_s^{-1}(\chi_0 + ih(r)^{\frac{1}{2}}). \tag{2.2.22}$$

χ_0 is constant and $h(r)$ is

$$h(r) \equiv 1 + \frac{\sqrt{\rho g_s}\, l_s^2}{r^2} \ , \tag{2.2.23}$$

where $\rho \equiv p^2 g_s^{-1} + (q + p\chi_0)^2 g_s$ and r is the radius of transverse direction from a D5-brane. NS-NS field, B is zero, and R-R 3-form flux is

$$F_3 = 2\, g_s l_s^2 \epsilon_3 \ , \tag{2.2.24}$$

where ϵ_3 is volume form of unit sphere surrounding a D5-brane.

$SL(2,\mathbb{Z})$ transformed ones are,

$$\tau' = \frac{q\tau - b}{-p\tau + d} = -\frac{q}{p} - \frac{1}{2p(p\chi_0 - d)}(1 + e^{2i\psi}), \quad (F_3', H_3') = (qF_3, pF_3) \tag{2.2.25}$$

where primed parameters are transformed one, $qd - pb = 1$.† Also ψ is define by

$$\tan\psi = \frac{H^{\frac{1}{2}}}{g_s(\chi_0 - d/p)}. \tag{2.2.26}$$

Because g_s^{-1} and χ_0 imply the values of Re τ and Im τ at infinity respectively, it is better to express transformed ones,

$$g_s' = \frac{g_s}{A^2 + p^2}, \quad \chi_0' = -\frac{q}{p} - \frac{A\, g_s}{p(A^2 + p^2)}, \tag{2.2.27}$$

where $A \equiv p\chi_0 - dg_s$. Then we can write down (2.2.25) and (2.2.26) without explicit dependace of d,

$$\tau' = -\frac{q}{p} + \frac{\rho}{2p(q + p\chi_0')g_s'}(1 + e^{2i\psi}), \quad \tan\psi = \frac{ph(r)^{\frac{1}{2}}}{(q + p\chi_0')g_s'}. \tag{2.2.28}$$

String frame metric become,

$$d^2 s_{string} = e^{\frac{\phi'}{2}} d^2 s_E = \frac{p}{\sqrt{\rho g_s'}|\sin\psi|}\big[dx_\parallel^2 + (\frac{\rho}{8\pi D p^2}\tan\psi)^2 (dx_\perp)^2\big]. \tag{2.2.29}$$

† For any integer p and q, we can alway find the integer d and b such that $qd - pb = 1$, according to Bèzouts identity.

2.3 Supersymmetric Janus configuration

In this section we review of the paper, [14]. Let's consider the system that a stack of N D3-branes and NS5-brane are intersected.

$$\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \text{D3} & \text{o} & \text{o} & \text{o} & \text{o} & & & & & & \\ \text{NS5} & \text{o} & \text{o} & \text{o} & & \text{o} & \text{o} & \text{o} & & & \end{array} \tag{2.3.1}$$

This configuration preserves 8 supersymmetry, also isometry symmetry as $ISO(1,2) \times SO(3) \times SO(3)$. In this case, because of existence of an NS5-brane, the dilaton and axion, which couple with worldvolume action of D3 branes as a role of complexified gauge coupling τ, become x^3 space dependent. So, for writing down the worldvolume action of D3 branes in this configuration, it is necessary to investigate the deformation of $\mathcal{N}=4$ SYM theory to have gauge coupling position dependent as well as to preserve supersymmetry.

The steps to get this condition is quite simple.* First, we deform the $\mathcal{N}=4$ SYM action with possible gauge invariant operators preserving isometry group, $ISO(1,2) \times SO(3) \times SO(3)$, with less than dimension 4,

$$\begin{aligned} S_{GW} = \int d^4x \frac{1}{e^2} \operatorname{tr} \Bigg(&- \frac{1}{2} F_{IJ} F^{IJ} + i\overline{\Psi}\Gamma^I D_I \Psi - i\overline{\Psi}\Big(\alpha\Gamma_{012} + \beta\Gamma_{456} + \gamma\Gamma_{789}\Big)\Psi \\ &- u\epsilon^{\mu\nu\lambda}\Big(A_\mu \partial_\nu A_\lambda - \frac{2}{3} A_\mu A_\nu A_\lambda\Big) - \frac{v}{3}\epsilon^{abc} A_a [A_b, A_c] - \frac{w}{3} A_p [A_q, A_r] - \frac{r}{2} A_a A^a - \frac{\tilde{r}}{2} A_p A^p \Bigg), \end{aligned}$$

where the metric is flat one, $ds^2 = \eta_{IJ} dx^I dx^J$. The first two terms are original $\mathcal{N}=4$ Lagrangian, and the others are deformed ones. We have used 10 dimensional $\mathcal{N}=1$ gauge multiplet expression to represent 4d $\mathcal{N}=4$ gauge multiplet, $A^{4,\cdots 9}$ is nothing but scalars, and Ψ is Weyl-Majorana fermion with a chirality, $\Gamma^{01\cdots 9}\Psi = -\Psi$, which can be decomposed 4 complex Weyl fermions in 4d. Also we have used the notation of indices, $I, J = x^{0,\cdots,9}$, $a, b, c = x^{4,5,6}$ and $p, q, r = x^{7,8,9}$. Then let the coefficients of these operators as well as gauge coupling have x^3 dependent. We also make the SUSY parameters also to have x^3 dependent, and generalize the form of SUSY transformation,

$$\delta_\epsilon A_I = i\bar{\epsilon}\Gamma_I \Psi \tag{2.3.2}$$

$$\delta_\epsilon \Psi = \frac{1}{2}\Gamma^{IJ} F_{IJ}\epsilon + \frac{1}{2}\big(\Gamma^a A_a (s_1\Gamma_{456} + s_2\Gamma_{789}) + \Gamma^p A_p (t_1\Gamma_{456} + t_2\Gamma_{789})\big)\epsilon. \tag{2.3.3}$$

With this modification, one can find the condition that the deformed action should be invariant under the generalized SUSY transformation, $\delta_\epsilon S_{GW} = 0$.

* The method with usage of 3d superfields is also discussed in [14]. The final result is the same justly.

The result of this calculation can be expressed by using an undetermined function of x^3, $\psi(x^3)$,

$$\begin{aligned}\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} &= a + 4\pi D e^{-2\psi i} \ \ (\mathrm{Im}\,\tau > 0),\\ \frac{e^2\theta'}{8\pi^2} = u = -4\alpha = -2\psi',\ v &= -4\beta = \frac{2\psi'}{\cos\psi},\ w = -4\gamma = -\frac{2\psi'}{\sin\psi},\\ s_1 = 2\psi'\frac{\sin^2\psi}{\cos\psi},\ s_2 &= 2\psi'\sin\psi,\ t_1 = -2\psi'\cos\psi,\ t_2 = -2\psi'\frac{\cos^2\psi}{\sin\psi},\\ r = 2(\psi'\tan\psi)' + 2(\psi')^2,&\ \tilde{r} = -2(\psi'\cot\psi)' + 2(\psi')^2. \end{aligned} \tag{2.3.4}$$

The position dependence of SUSY parameter, ϵ, can be determined by

$$(\sin\psi\Gamma_{3456} + \cos\psi\Gamma_{3789})\epsilon = \epsilon. \tag{2.3.5}$$

We can decompose $\epsilon = \epsilon_8 \otimes \epsilon_2$, where ϵ_8 is a constant 8 dimensional spinor whose representation belongs to the double cover of $W = SO(1,2)\times SO(3)\times SO(3)$, and ϵ_2 is spinor of $SL(2,R)$ outer automorphism which characterize the space varying embedding W to $SO(1,9)$. The generators of outer automorphism $SL(2,R)$ are

$$B_0 = \Gamma_{456789} = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},\ B_1 = \Gamma_{3456} = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix},\ B_2 = \Gamma_{3789} = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.3.6}$$

Under this representation, the explicit form of ϵ in (2.3.5) is

$$\epsilon = \epsilon_8 \otimes \begin{bmatrix}\cos\psi/2\\ \sin\psi/2\end{bmatrix}. \tag{2.3.7}$$

In order to match the parameters in the field theory with the ones in the brane configuration, we compare the SUSY preserving conditions for (2.3.1). For type IIB theory, 2 kinds of SUSY parameter, ϵ_1 and ϵ_2, exist with same chirality. D3-branes solution breaks half of SUSY by†

$$\epsilon_2 = \Gamma_{0123}\epsilon_1. \tag{2.3.8}$$

Also a (p,q) 5-brane, instead of a NS5-brane in (2.3.1), breaks half of SUSY by

$$\epsilon_1 = -\Gamma_{012456}(\sin t\epsilon_1 + \cos t\epsilon_2), \tag{2.3.9}$$

where $t = \arg(p\tau + q)$. These two conditions gives

$$\epsilon_1 = (-\Gamma_{3456}\cos t + \Gamma_{3789}\sin t)\epsilon_1, \tag{2.3.10}$$

†One can derive these equations by the integrability conditions on the Killing spinor equations.

which is related to (2.3.5) when $t = \frac{\pi}{2} + \psi$. Therefore, we can derive the equations for ψ,

$$\tan(\frac{\pi}{2} + \psi) = \frac{p\,\mathrm{Im}\tau}{q + p\,\mathrm{Re}\tau} \;\Rightarrow\; \cot\psi = -\frac{4\pi D\sin 2\psi}{a + q/p + 4\pi D\cos 2\psi}\,. \tag{2.3.11}$$

Actually this equation do not constraint ψ, but only gives $a = -q/p - 4\pi D$. So, we get

$$\tau = -\frac{q}{p} - 4\pi D(1 - e^{-2\psi i}), \tag{2.3.12}$$

where $D\sin 2\psi < 0$. This is exactly same form with the solution of (p, q) 5 brane, (2.2.25). For $(p, q) = (1, 0)$, an NS5-brane case, the theta parameter as well as gauge coupling is allowed to have position dependence. However for a D5-brane case, $(p, q) = (0, 1)$, we can think $a - 4\pi D$ is fixed and a and $4\pi D$ go infinity as $-\frac{q}{p}$ goes infinity. This shows that for a D5 brane case, theta parameter is not allowed to have position dependence while the gauge couplings is allowed to.

This solution can be applied to more general cases as far as the isometry symmetry is same. For example, D3 branes ends on an NS5-brane or the different number of D3 branes end on an NS5 brane from the left side and the right side. Only difference is we should require suitable boundary conditions on the defect, intersecting surface with an NS5-brane. Other numerous cases can be analyzed by this work.

Inspired by this work, we will generalize the it to the supersymmetric gauge theory with all 4d space-time dependent couplings.

2.4 Dp-branes in curved backgrounds

In this subsection, we review the non-Abelian worldvolume action of D-branes. We will consider the worldvolume action in the curved space and when all R-R and NS-NS fluxes are non-zero. For the bosonic part we are going to follow the prescription given in [37]. For the fermionic sector we will follow [40].

2.4.1 Bosonic part

In this subsection, we review the bosonic part of the effective action on Dp-branes embedded in a curved background with fluxes following [37].

The 10-dimensional space-time coordinates are denoted as x^I ($I = 0, 1, \cdots, 9$). We choose the static gauge in which x^μ ($\mu = 0, 1, 2, 3$) are identified as the coordinates on the Dp-brane world-volume and x^i ($i = 4, \cdots, 9$) parametrize the transverse directions. The bosonic sector of the effective theory contains a $U(N)$ gauge field A_μ ($\mu = 0, \cdots, p$), $(9-p)$ scalar fields Φ^i ($i = p+1, \cdots, 9$), which belong to the adjoint representation of the gauge group $U(N)$. The reference position of the Dp-brane is chosen to be $x^i = 0$ and small deviations from it is described by the values of the scalar fields.

For constructing non-Abelian world-volume action of Dp-branes, we can start from the D9-brane theory and use T-duality to this action. T-duality on a direction, y, essentially exchange Dirichlet boundary condition and Neumann boundary condition of open strings on y direction. Naturally, a D_p-brane, which is hypersurface restricted by Dirichlet boundary condition of open strings, is transformed to to a D_{p+1}- or D_{p-1}-brane according to whether the D_p-brane is extend to y direction or not. T-duality transform type IIA supergravity with radius R of y direction to the type IIB theory with radius α'/R of y direction, or vice versa. T-duality transformation rules for NS-NS sector fields are,

$$\begin{aligned}
\tilde{g}_{yy} &= \frac{1}{g_{yy}}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu y} g_{\nu y} - B_{\mu y} B_{\nu y}}{g_{yy}}, \quad \tilde{g}_{\mu y} = \frac{B_{\mu y}}{g_{yy}}, \\
e^{2\tilde{\phi}} &= \frac{e^{2\phi}}{g_{yy}}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu y} g_{\nu y} - g_{\mu y} B_{\nu y}}{g_{yy}}, \quad \tilde{B}_{\mu y} = \frac{g_{\mu y}}{g_{yy}},
\end{aligned} \tag{2.4.1}$$

where the metric g_{MN} is written in string frame, and the letters with tilde mean the T-dualized ones. Also T-duality on R-R fields transform

$$\begin{aligned}
(\tilde{C}_n)_{\mu\cdots\nu\rho y} &= (C_{n-1})_{\mu\cdots\nu\rho} - (n-1)\frac{g_{y[\rho}(C_{n-1})_{\mu\cdots\nu]y}}{g_{yy}}, \\
(\tilde{C}_n)_{\mu\cdots\nu\rho\sigma} &= (C_{n+1})_{\mu\cdots\nu\rho\sigma y} + n(C_{n-1})_{[\mu\cdots\nu\rho}B_{\sigma]y} + n(n-1)\frac{(C_{n-1})_{[\mu\cdots\nu|y}B_{|\rho|y}g_{\sigma]y}}{g_{yy}}.
\end{aligned} \tag{2.4.2}$$

The world-volume action of D_p-brane contains either the scalar field corresponding to the T-dual direction or the gauge field component of T-dual direction. For each case, T-duality makes transformation as

$$\Phi^y \rightarrow A_y \quad \text{or} \quad A_y \rightarrow \Phi^y. \tag{2.4.3}$$

The position of Dp-branes have relation with the scalar fields as $x^i = \lambda \Phi^i$, where

$$\lambda \equiv 2\pi l_s^2. \tag{2.4.4}$$

So, when the D_p-brane extended in y direction transform the D_{p-1}-brane located at $y = 0$ by T-duality, the y dependance on a field can be interpreted via a Taylor expansion with respect to Φ^y, such as *e.g.*,

$$\widehat{\phi}(x^\mu, \lambda\Phi^y) \equiv \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \Phi^{i_1} \ldots \Phi^{i_n}\, \partial_{i_1} \ldots \partial_{i_n} \phi(x^\mu, y)\big|_{y=0} \ . \tag{2.4.5}$$

We use the hat "$\widehat{\ }$" on the background fields to indicate Taylor expansion about scalar fields.

Because D9-brane theory does not contain scalar fields at all, the generalization to non-Abelian D9-brane is straight forward, replacing to non-Abelian gauge fields and taking trace over gauge group. DBI action for non-Abelian D9-branes is,

$$S_{\rm D_9}^{\rm DBI} = -T_9 \int d^{10}x \, {\rm tr} \left\{ e^{-\phi} \sqrt{-\det(g_{\mu\nu} + B_{\mu\nu} + \lambda F_{\mu\nu})} \right\} . \tag{2.4.6}$$

Here, F is the gauge field strength living on the brane,

$$F = dA + iA \wedge A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \ , \tag{2.4.7}$$

and $T_9 = (2\pi)^{-9} l_s^{-10} g_s^{-1}$ is tension of D9-branes.

We can utilize T-duality to D9-brane world-volume action to make D_p-brane one. For convenience, we will use E_{IJ} as

$$E_{IJ} \equiv g_{IJ} + B_{IJ} \ . \tag{2.4.8}$$

Applying T-duality, (2.4.1) $\sim$ (2.4.5), regarding $x^{p,\cdots,9}$ direction, $\det(E_{IJ} + \lambda F_{IJ})$ transform to

$$\det \begin{bmatrix} \widehat{E}_{\mu\nu} - \widehat{E}_{\mu i} E^{ij} & \widehat{E}_{j\nu} + \lambda F_{\mu\nu} \widehat{E}_{\mu k} \widehat{E}^{kj} + \lambda D_\mu \Phi^j \\ -\widehat{E}^{ik} \widehat{E}_{k\nu} - \lambda D_\nu \Phi^i & \widehat{E}^{ij} + i\lambda[\Phi^i, \Phi^j] \end{bmatrix} \tag{2.4.9}$$

$$= \det \left({\rm P} \left[\widehat{E}_{\mu\nu} + \widehat{E}_{\mu i} (Q^{-1} - \delta)^{ij} \widehat{E}_{j\nu} \right] + \lambda F_{\mu\nu} \right) \det(\widehat{E}^{ij}) \det(Q^i_j) \, , \tag{2.4.10}$$

where E^{ij} is inverse of E_{ij}, and $Q^i{}_j$ is

$$Q^i{}_j \equiv \delta^i{}_j + i\lambda[\Phi^i, \Phi^k] \widehat{E}_{kj} \ . \tag{2.4.11}$$

Also we have used the pull-back, "P" , as

$${\rm P}[E_{\mu\nu}] = \widehat{E}_{\mu\nu} + \lambda \widehat{E}_{\mu i} D_\nu \Phi^i + \lambda \widehat{E}_{i\nu} D_\mu \Phi^i + \lambda^2 \widehat{E}_{ij} D_\mu \Phi^i D_\nu \Phi^j \ . \tag{2.4.12}$$

which is as different as changing partial derivative to covariant one from the Abelian one. $\det(\widehat{E}^{ij})$ factor can be absorbed by T-dual transformation of $e^{-\phi}$ as

$$e^{-\tilde{\phi}} = \frac{e^{-\phi}}{\sqrt{\det(\widehat{E}^{ij})}} \tag{2.4.13}$$

which reduce to the form of (2.4.1) when $i = j = y$.

In addition, because we have to trace over gauge indices. Ordering of fields under gauge trace become non-trivial, and we should impose symmetric trace, "Str", which average over all orderings of $F_{\mu\nu}$, $D_\mu\Phi^i$, $[\Phi^i,\Phi^j]$ and Φ^i in the Taylor expansion, (2.4.5). Then, we can conclude non-Abelian DBI action for D_p branes,

$$S^{\text{DBI}}_{\text{D}p} = -T_p\int d^{p+1}x\,\text{Str}\left\{e^{-\widehat{\phi}}\sqrt{-\det\left(\text{P}\left[\widehat{E}_{\mu\nu}+\widehat{E}_{\mu i}(Q^{-1}-\delta)^{ij}\widehat{E}_{j\nu}\right]+\lambda F_{\mu\nu}\right)\det(Q^i{}_j)}\right\}\ , \tag{2.4.14}$$

where T_p is tension of D_p brane, $T_p \equiv (2\pi)^{-p} l_s^{-p-1} g_s^{-1}$.

Next, let's consider Chern-Simons action part. Again, because the world-volume theory of D9-branes do not contain scalar fields at all, we can make D_p brane Chern-Simons action using T-duality of $x^{p+1,\cdots 9}$. Here we simply present non-Abelian generalization of the Chern-Simons action,

$$S^{\text{CS}}_{\text{D}p} = \mu_p\int \text{Str}\left\{\text{P}\left[e^{i\lambda\imath_\Phi\imath_\Phi}\left(\sum_n \widehat{C}_n\wedge e^{\widehat{B}_2}\right)\right]\wedge e^{\lambda F}\right\}\ . \tag{2.4.15}$$

$\imath_\Phi$ in (2.4.15) denotes the interior product by a vector (Φ^i), *e.g.*,

$$\imath_\Phi\imath_\Phi C_n = -\frac{1}{2(n-2)!}[\Phi^i,\Phi^j](C_n)_{ij\mu_3\cdots\mu_n}dx^{\mu_3}\wedge\cdots\wedge dx^{\mu_n}\ . \tag{2.4.16}$$

μ_p is defined by $\mu_p \equiv \pm T_p$, where the sign +(or -) appearing in μ_p, which is proportional to the R-R charge of the Dp-brane, corresponds to the case of Dp-branes (or $\overline{\text{D}p}$-branes). Note that this interior product vanish when gauge group is Abelian, which reduce (2.4.15) to Abelian one.

Let's verify this Chern-Simons action reflects T-duality well. From a D_{n+2m}-brane, let's consider T-duality on y direction included in extended direction, $x^{n+2m}=u$. For simplicity, we will restrict the metric to be $g_{\mu y}=B_{\mu y}=0$. This condition makes T-duality transformation of NS-NS field, (2.4.1), become trivial, so that we exclude B field in this discussion. The general form of each term in (2.4.15) is

$$(i\lambda)^l P\big[(\imath_\Phi\imath_\Phi)^l C_{n+2l}\big]_{a_1\cdots a_n} F_{b_1c_1}\cdots F_{b_mc_m}. \tag{2.4.17}$$

We will consider the cases that y index appear in the part involving R-R fields first and then consider the case y appear in the part involving field strength. For the case that y appears in the indices of C_{n+2l}, T-duality makes this term to

$$\frac{(i\lambda)^l}{l!}P'\big[(\imath_{\tilde\Phi}\imath_{\tilde\Phi})^l C_{n+2l})\big]_{a_1\cdots a_{n-1}} F_{b_1c_1}\cdots F_{b_mc_m}, \tag{2.4.18}$$

where the primed pull-back denotes that it does not include the pull-back of y involving $D_y\Phi^i$, and the symbol $\imath_{\tilde{\Phi}}$ denotes the interior product except for Φ^y. It is also possible that y comes from pull-back of R-R fields, contracted with $D_y\Phi^i$ which do not vanish in non-Abelian case. In this case, T-duality results to

$$2\frac{(i\lambda)^{l+1}}{l!}P'\left[(\imath_{\tilde{\Phi}}\imath_{\tilde{\Phi}})^l(\imath_{\Phi^y}\imath_{\tilde{\Phi}})C_{n+2l+1})\right]_{a_1\cdots a_{n-1}}F_{b_1c_1}\cdots F_{b_mc_m}. \tag{2.4.19}$$

Replacing l to $l+1$ in (2.4.18) and using the identity relation for a n-form field,

$$\frac{(i\lambda)^{l+1}}{(l+1)!}(\imath_\Phi\imath_\Phi)^{l+1} = \frac{(i\lambda)^{l+1}}{(l+1)!}(\imath_{\tilde{\Phi}}\imath_{\tilde{\Phi}})^{l+1} + 2\frac{(i\lambda)^{l+1}}{l!}(\imath_{\tilde{\Phi}}\imath_{\tilde{\Phi}})^l(\imath_{\Phi^y}\imath_{\tilde{\Phi}}), \tag{2.4.20}$$

now we can check that T-duality make (2.4.15) same form except pull-back, P'. This can be fixed by considering the case that y index appear in the field strength side. For this case, T-duality transform (2.4.17) to

$$m(i\lambda)^lP'\left[(\imath_\Phi\imath_\Phi)^lC_{n+2l+1}\right]_{a_1\cdots a_ny}F_{b_1c_1}\cdots F_{b_{m-1}c_{m-1}}D_{b_m}\Phi^y. \tag{2.4.21}$$

And this contribution makes the pull-back P' to be complete P in (2.4.18) if we replace to $(n+2, m-1)$ instead of (n,m). Therefore, we have verified the Chern-Simons action, (2.4.17), of a D_p-brane transform to the one of a D_{p-1}-brane with same form.

Considering T-duality, we conclude that the bosonic part of non-Abelian world-volume action of Dp-branes is

$$S^{\text{bos}}_{\text{D}p} = S^{\text{DBI}}_{\text{D}p} + S^{\text{CS}}_{\text{D}p}\ , \tag{2.4.22}$$

where $S^{\text{DBI}}_{\text{D}p}$ and $S^{\text{CS}}_{\text{D}p}$ are given by (2.4.14) and (2.4.15) respectively.

2.4.2 Fermionic part

In this subsection, we write down the fermionic part (quadratic terms with respect to the fermion fields) of the effective action on a Dp-brane embedded in any supergravity background following [40]. Here, we consider the cases with a single Dp-brane in type IIB string theory.

The action, after fixing the κ-symmetry, is given by

$$S^{\text{fermi}}_{\text{D}p} = \frac{T_p}{2}\int d^{p+1}x\, e^{-\phi}\sqrt{-\det(M_{\mu\nu})}\Big\{i\overline{\psi}\left[(M^{-1})^{\mu\nu}\Gamma_\mu\nabla^{(H)}_\nu - \Delta^{(1)}\right]\psi \\ - i\overline{\psi}\,\breve{\Gamma}^{-1}_{Dp}\left[(M^{-1})^{\mu\nu}\Gamma_\nu W_\mu - \Delta^{(2)}\right]\psi\Big\}\ , \tag{2.4.23}$$

where ψ is the fermion field (dimensional reduction of the 10-dimensional Majorana-Weyl spinor field), $\Gamma_\mu \equiv \Gamma_{\hat{I}} e^{\hat{I}}_I \partial_\mu x^I$ is the pull-back of the 10-dimensional gamma matrices, $M_{\mu\nu}$ is defined as

$$M_{\mu\nu} = \mathrm{P}[g_{\mu\nu} + B_{\mu\nu}] + \lambda F_{\mu\nu} \ , \tag{2.4.24}$$

where the field strength is the Abelian one. And other quantities are defined as follows.

The covariant derivative $\nabla^{(H)}_\nu$ is the pull-back of the 10-dimensional covariant derivative including the H-flux:

$$\nabla^{(H)}_I \equiv \partial_I + \frac{1}{4}{\omega_I}^{\hat{J}\hat{K}} \Gamma_{\hat{J}\hat{K}} + \frac{1}{4\cdot 2!} H_{IJK} \Gamma^{JK} \ , \tag{2.4.25}$$

where ${\omega_I}^{\hat{J}\hat{K}}$ is the spin connection and H_{IJK} is the field strength of the Kalb-Ramond 2-form field.

W_μ, $\Delta^{(1)}$ and $\Delta^{(2)}$ are defiend as

$$W_\mu \equiv \frac{1}{8} e^\phi \left(-F_J \Gamma^J + \frac{1}{3!} \widetilde{F}_{JKL} \Gamma^{JKL} - \frac{1}{2\cdot 5!} \widetilde{F}_{JKLMN} \Gamma^{JKLMN} \right) \Gamma_\mu \ , \tag{2.4.26}$$

$$\Delta^{(1)} \equiv \frac{1}{2} \left(\Gamma^I \partial_I \phi + \frac{1}{2\cdot 3!} H_{IJK} \Gamma^{IJK} \right) \ , \tag{2.4.27}$$

$$\Delta^{(2)} \equiv -\frac{1}{2} e^\phi \left(-F_I \Gamma^I + \frac{1}{2\cdot 3!} \widetilde{F}_{IJK} \Gamma^{IJK} \right) \ , \tag{2.4.28}$$

where F_I, $\widetilde{F}_{IJK}$ and $\widetilde{F}_{IJKLM}$ are the field strength of the R-R fields defined as

$$\begin{aligned} &F_1 = F_I dx^I \equiv dC_0 \ , \quad \widetilde{F}_3 = \frac{1}{3!} \widetilde{F}_{IJK} dx^I \wedge dx^J \wedge dx^K \equiv dC_2 + C_0 H_3 \ , \\ &\widetilde{F}_5 = \frac{1}{5!} \widetilde{F}_{IJKLM} dx^I \wedge dx^J \wedge dx^K \wedge dx^L \wedge dx^M \equiv dC_4 + H_3 \wedge C_2 \ . \end{aligned} \tag{2.4.29}$$

(See section 2.1 for our conventions.)

Finally, $\breve{\Gamma}^{-1}_{\mathrm{D}p}$ is defined by

$$\breve{\Gamma}^{-1}_{\mathrm{D}p} = (-1)^{p-2} \Gamma^{(0)}_{\mathrm{D}p} \frac{\sqrt{-g}}{\sqrt{-\det(M_{\mu\nu})}} \sum_{q\geq 0} \frac{(-1)^q}{q! 2^q} \Gamma^{\mu_1\cdots\mu_{2q}} \mathcal{F}_{\mu_1\mu_2} \cdots \mathcal{F}_{\mu_{2q-1}\mu_{2q}} \ , \tag{2.4.30}$$

where $\sqrt{-g} \equiv \sqrt{-\det \mathrm{P}[g_{\mu\nu}]}$,

$$\mathcal{F}_{\mu\nu} \equiv B_{\mu\nu} + \lambda F_{\mu\nu} \ , \tag{2.4.31}$$

and

$$\Gamma^{(0)}_{\mathrm{D}p} \equiv \frac{1}{(p+1)!} \epsilon^{\mu_1\cdots\mu_{p+1}} \Gamma_{\mu_1\cdots\mu_{p+1}} \tag{2.4.32}$$

with the Levi-Civita symbol $\epsilon^{\mu_1\cdots\mu_{p+1}}$ with $\epsilon^{01\cdots p} = 1/\sqrt{-g}$.

Chapter 3

Space-time dependent deformation of $\mathcal{N}=4$ SYM with preserving SUSY

3.1 Notations and Ansatz

In this paper, we will deform $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory to have space-time dependence on gauge coupling $g_{\rm YM}$ and theta parameter θ. Because $\mathcal{N}=4$ SYM in 4 dimension can be obtained by dimensional reduction of 10 dimensional $\mathcal{N}=1$ SYM, we use 10 dimensional notations for $\mathcal{N}=4$ SYM action, *

$$S_0=\int d^4x\sqrt{-g}\ \mathrm{tr}\left\{\frac{1}{g_{\rm YM}^2}\left(-\frac{1}{2}g^{II'}g^{JJ'}F_{IJ}F_{I'J'}+i\overline{\Psi}\Gamma^I D_I\Psi\right)+\frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}\ . \quad (3.1.1)$$

We will use 3 types letter for the indices as $I,J\cdots=0,1,\cdots,9$, $\mu,\nu,\cdots=0,1,2,3$ and $A,B,\cdots=4,\cdots,9$. We can think that the gauge field A_μ and 6 adjoint scalar fields A_A are combined into a 10-dimensional gauge field A_I, though it depends only on the 4-dimensional space-time. Also covariant derivatives are defined as,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu]\ , \quad (3.1.2)$$

$$F_{\mu A} = -F_{A\mu}\equiv\partial_\mu A_A+i[A_\mu,A_A]\equiv D_\mu A_A\ , \quad (3.1.3)$$

$$F_{AB} \equiv i[A_A,A_B]\ , \quad (3.1.4)$$

$$D_\mu\Psi \equiv \partial_\mu\Psi+i[A_\mu,\Psi]+\frac{1}{4}\omega_\mu{}^{\hat\nu\hat\rho}\Gamma_{\hat\nu\hat\rho}\Psi\ , \quad D_A\Psi\equiv i[A_A,\Psi]\ . \quad (3.1.5)$$

The 10-dimensional metric g_{IJ} is assumed to be of the form

$$ds^2=g_{IJ}dx^Idx^J=g_{\mu\nu}(x^\rho)dx^\mu dx^\nu+\delta_{AB}dx^Adx^B\ , \quad (3.1.6)$$

* Our notation is similar to (but, not exactly the same as) that used in [14].

and g^{IJ} is its inverse.[†] The indices are lowered or raised by this metric and its inverse. $\epsilon^{\mu\nu\rho\sigma}$ is the 4-dimensional epsilon tensor with $\epsilon^{0123} = 1/\sqrt{-g}$, where $\sqrt{-g} \equiv \sqrt{-\det(g_{IJ})} = \sqrt{-\det(g_{\mu\nu})}$.

The fermion field Ψ is written as a 10-dimensional negative chirality Majorana-Weyl spinor, which is equivalent to 4 Weyl spinor fields in 4-dimensional space-time. It is a real 32 component spinor satisfying

$$\Gamma^{(10)}\Psi = -\Psi \ , \tag{3.1.7}$$

where $\Gamma^{(10)} \equiv \Gamma^{\hat{0}}\Gamma^{\hat{1}} \cdots \Gamma^{\hat{9}}$ is the 10-dimensional chirality operator.[‡] Here, the gamma matrices $\Gamma^{\hat{I}}$ ($\hat{I} = 0, 1, \cdots, 9$) are 10-dimensional gamma matrices which are realized as 32×32 real matrices satisfying $\{\Gamma^{\hat{I}}, \Gamma^{\hat{J}}\} = 2\eta^{\hat{I}\hat{J}}$, where $(\eta^{\hat{I}\hat{J}}) \equiv \mathrm{diag}(-1, +1, \cdots, +1)$ is the Minkowski metric (see Appendix B for our notations and useful formulas for the gamma matrices). The hatted indices $(\hat{I}, \hat{J}, \cdots)$ are those for the local Lorentz frame which can be converted to the curved indices $(I, J, \cdots)$ by contracting them with vielbeins $e^I_{\hat{I}}$ as $\Gamma^I = e^I_{\hat{I}}\Gamma^{\hat{I}}$. The Dirac conjugate $\overline{\Psi}$ is defined by $\overline{\Psi} \equiv \Psi^T\Gamma^{\hat{0}}$. Gamma matrices with more than one indices are anti-symmetrized products of the gamma matrices defined as

$$\Gamma^{I_1 I_2 \cdots I_n} = \Gamma^{[I_1}\Gamma^{I_2} \cdots \Gamma^{I_n]} = \frac{1}{n!}\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\Gamma^{I_{\sigma(1)}}\Gamma^{I_{\sigma(2)}} \cdots \Gamma^{I_{\sigma(n)}} \ . \tag{3.1.8}$$

$\omega_\mu{}^{\hat{\nu}\hat{\rho}}$ in the covariant derivative D_μ in (3.1.5) is the spin connection:

$$\omega_{\mu\hat{\nu}\hat{\rho}} = \frac{1}{2}e^{\nu'}_{\hat{\nu}}(\partial_\mu e_{\nu'\hat{\rho}} - (\partial_{\nu'}g_{\mu\mu'})e^{\mu'}_{\hat{\rho}}) - (\hat{\nu} \leftrightarrow \hat{\rho}) \ . \tag{3.1.9}$$

In our notation, D_μ denotes the covariant derivative including gauge field A_μ, spin connection $\omega_\mu{}^{\hat{\nu}\hat{\rho}}$ and Levi-Civita connection $\Gamma^\rho_{\mu\nu}$, depending on the field it acts.

When the metric is flat and the couplings (g_{YM} and θ) are constant, the action (3.1.1) is invariant under the supersymmetry (SUSY) transformation with 16 independent SUSY parameters.[§] If the metric and/or the couplings are not constant, SUSY is in general completely broken. In order to maintain a part of SUSY, we should add additional terms to the action (3.1.1) and

[†]One may introduce a non-trivial internal part of the metric $g_{AB}(x^\mu)$. However, for our purpose, we can set it to be $g_{AB}(x^\mu) = \delta_{AB}$ without loss of generality. This is because g_{AB} in the action (3.1.10) can always be replaced with δ_{AB} by redefinition of the scalar fields and the couplings a, d^{IJA}, m^{AB} and M. The off-diagonal components $g_{\mu A}$ are set to be zero for simplicity.

[‡] This should not be confused with $\Gamma^{10} = \Gamma^1\Gamma^0$.

[§]In this paper we will not consider the special conformal supersymmetry.

the SUSY transformation also should be modified accordingly. The action that we consider is

$$S = \int d^4x\, \mathcal{L} = \int d^4x \sqrt{-g}\, a \, \mathrm{tr} \bigg\{ -\frac{1}{2} g^{II'} g^{JJ'} F_{IJ} F_{I'J'} + i\,\overline{\Psi}\Gamma^I D_I \Psi + \frac{c}{4}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ - d^{IJA} F_{IJ} A_A - \frac{m^{AB}}{2} A_A A_B - i\overline{\Psi} M \Psi \bigg\} \ , \quad (3.1.10)$$

where a, c, d^{IJA}, m^{AB} are real parameters and M is a real 32×32 matrix that may depend on the space-time coordinates x^μ. a and c are related to $g_{\rm YM}$ and θ in (3.1.1) by

$$a = \frac{1}{g_{\rm YM}^2} \ , \quad c = \frac{g_{\rm YM}^2 \theta}{8\pi^2} \ . \quad (3.1.11)$$

We also use the complex coupling defined by

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\rm YM}^2} = 4\pi a (c + i) \ . \quad (3.1.12)$$

We impose the following conditions for the parameters d^{IJA} and m^{AB}, which can be done without loss of generality:¶

$$d^{IJA} = -d^{JIA} \ , \quad d^{\mu AB} = -d^{\mu BA} \ , \quad d^{ABC} = d^{[ABC]} \ , \quad m^{AB} = m^{BA} \ . \quad (3.1.13)$$

The second condition in (3.1.13) can be imposed because a term with $d^{\mu(AB)}\,\mathrm{tr}(F_{\mu A} A_B)$ can be converted to a mass term for A_A via integration by parts.

Note that $\overline{\Psi} M \Psi$ is non-vanishing only when $\Gamma^{\hat{0}} M$ is an anti-symmetric matrix that commutes with $\Gamma^{(10)}$. We can easily show that only the linear combinations of the anti-symmetric product of 3 gamma matrices are allowed. So, the most general form of such matrix is

$$M = m_{IJK} \Gamma^{IJK} \ , \quad (3.1.14)$$

where m^{IJK} is a real rank three totally anti-symmetric tensor.

The action (3.1.10) is constructed by adding operators of dimension‖ less than 4 into the leading order action (3.1.1). Although it is not the most general one,** it is general enough to

¶ Square and round brackets on indices indicate anti-symmetrization and symmetrization of the indices, respectively. For example,

$$d^{[ABC]} \equiv \frac{1}{3!}(d^{ABC} + d^{BCA} + d^{CAB} - d^{BAC} - d^{CBA} - d^{ACB}) \ ,$$
$$d^{(ABC)} \equiv \frac{1}{3!}(d^{ABC} + d^{BCA} + d^{CAB} + d^{BAC} + d^{CBA} + d^{ACB}) \ .$$

‖ Our analysis in this paper is classical and the anomalous dimensions are not taken into account.

** For example, an operator like $\mathrm{tr}(A_{(A} A_B A_{C)})$ is not included.

cancel the SUSY variations of the leading order action, as we will see in Section 3.2 and Appendix D.

The ansatz for the SUSY transformation is

$$\begin{aligned} \delta_\epsilon A_I &= i\bar{\epsilon}\,\Gamma_I \Psi \ , \\ \delta_\epsilon \Psi &= \frac{1}{2}(F_{IJ}\Gamma^{IJ} + A_A B^A)\,\epsilon \ , \\ \delta_\epsilon \overline{\Psi} &= \frac{1}{2}\bar{\epsilon}\,(-F_{IJ}\Gamma^{IJ} + A_A \overline{B}^A) \ , \end{aligned} \tag{3.1.15}$$

where B^A are real 32×32 matrices acting on the spinor indices that commute with the chirality operator $\Gamma^{(10)}$, and $\overline{B}^A \equiv -\Gamma^{\hat{0}}(B^A)^T\Gamma^{\hat{0}}$. The SUSY parameter ϵ is a 10-dimensional negative chirality Majorana-Weyl spinor. B^A and ϵ may also depend on the space-time coordinates.

3.2 SUSY conditions

The goal of this section is to determine the conditions under which the action (3.1.10) is invariant with respect to the SUSY transformations (3.1.16) for a non-zero ϵ. The approach that we follow in this work is straight. We firstly calculate the variation of the deformed action (3.1.10) with respect to the deformed transformations (3.1.16). Then, by imposing the vanishing of the variation, we obtain several constraints on the deformation parameters and the SUSY parameter ϵ.* Here, we provide the outline of the derivation and leave the details to Appendix D.

Applying the SUSY variation (3.1.16) to the action (3.1.10), we get

$$\begin{aligned} \delta_\epsilon S = \int d^4x \Bigg\{ \left(\frac{\partial \mathcal{L}}{\partial A_I} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_I)} \right) \right) \delta_\epsilon A_I + \left(\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \right) \right) \delta_\epsilon \Psi \\ + \delta_\epsilon \overline{\Psi} \left(\frac{\partial \mathcal{L}}{\partial \overline{\Psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\Psi})} \right) \right) \Bigg\} - \sqrt{-g}\, D_\nu J_\epsilon^\nu \, . \end{aligned} \tag{3.2.1}$$

The current J_ϵ^μ in the covariant total derivative term is,

$$\begin{aligned} J_\epsilon^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_I)} \delta_\epsilon A_I + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \delta_\epsilon \Psi \\ &= ia\bar{\epsilon} \left(F_{IJ} \left(\frac{3}{2} \Gamma^{IJ}\Gamma^\mu - \Gamma^{IJ\mu} + c\epsilon^{\mu\nu IJ}\Gamma_\nu \right) - 2d^{\mu IA}\Gamma_I A_A - A_A \overline{B}^A \Gamma^\mu \right) \Psi. \end{aligned} \tag{3.2.2}$$

---

* A similar analysis with a more elegant approach using supergravity à la Festuccia-Seiberg [7] has been given in [21]. However, a detailed comparison of the results still remains.

In order to be supersymmetric, we should impose some conditions this variation, (3.2.1), to be zero up to boundary terms. More detail calculations about the conditions and total derivative are written on the Appendix D.1. Imposing some conditions, the first three terms in $\{\cdots\}$ of (3.2.1) can be written as total derivatives,

$$\sqrt{-g}\,\mathrm{tr}\, D_\mu\bigg(i\bar{\epsilon}\Big(-\Gamma^{JK}\Gamma^\mu F_{JK} + a\overline{B}^A\Gamma^\mu A_A\Big)\psi\bigg). \tag{3.2.3}$$

Therefore, the Noether current for the supersymmetry, as a covariant version, is

$$\mathcal{J}^\mu_\epsilon = J^\mu_\epsilon + i\bar{\epsilon}\Big(\Gamma^{JK}\Gamma^\mu F_{JK} - a\overline{B}^A\Gamma^\mu A_A\Big)\psi. \tag{3.2.4}$$

When the space-time manifold M has boundary, the conditions to preserve supersymmetry should be imposed not only proper Dirichlet or Neumann boundary condition of the fields, but also

$$\int_M d^4x\sqrt{-g}\,D_\mu\mathcal{J}^\mu_\epsilon = \int_{\partial M} d^3x\sqrt{|\gamma|}\;\mathcal{J}^\mu_\epsilon\hat{n}_\mu = 0, \tag{3.2.5}$$

where $\hat{n}_\mu$ is unit normal vector to the boundary, and γ is determinant of induced metric on boundary.[†] However, from here we will only focus on the case when the boundary of space-time manifold are not exist.

Up to the boundary terms, the conditions to be (3.2.1) zero can be written as :

$$D_\mu\bar{\epsilon}\,\Gamma^{IJ}\Gamma^\mu = \bar{\epsilon}\left(\overline{B}^{[J}\Gamma^{I]} - \Gamma^{IJ\mu}a^{-1}\partial_\mu a + (a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} + 3\,d^{[IJK]})\Gamma_K - \Gamma^{IJ}\widetilde{M}\right)\;, \tag{3.2.7}$$

$$D_\mu(\bar{\epsilon}\,\overline{B}^A)\Gamma^\mu = \bar{\epsilon}\left(-2a^{-1}\Gamma_I D_\mu(a\,d^{I\mu A}) - m^{AB}\Gamma_B - \overline{B}^A\left(M + \frac{1}{2}\Gamma^\mu\partial_\mu\log a\right)\right)\;. \tag{3.2.8}$$

In this expression, d^{JKI} and m^{IA} can be non-zero only if $I = 4 \sim 9$, and ϵ^{IJKL} can be non-zero only for $I, J, K, L = 0 \sim 3$. Conditions (3.2.7) and (3.2.8) correspond to cancellation of terms with dimension $\frac{7}{2}$ and $\frac{5}{2}$ operators, respectively. Following [14], we call them first order and second order equations, respectively.

[†]Note that one can obtain supersymmetry charge by just choosing ∂M is space-like surface, e.g., $t = 0$. Then supercharge $\mathcal{Q}_\epsilon$ for a specific supersymmetry ϵ,

$$\mathcal{Q}_\epsilon = \int_{x=0} d^3x\sqrt{\tilde{g}}\;\mathcal{J}^0_\epsilon, \tag{3.2.6}$$

where $\tilde{g}$ is determinant of metric of spatial direction at $t = 0$.

Further algebra shows that the first order equation (3.2.7) is equivalent to the following conditions (see Appendix D.2 for the derivation):

$$
\begin{aligned}
0 &= \bar{\epsilon}\, e^{I'J'K'}\Gamma_{K'}\left(P_{I'J'}{}^{IJ} - \delta^I_{I'}\delta^J_{J'}\right) \ , && (3.2.9)\\
0 &= \bar{\epsilon}\left(\frac{1}{72}e^{IJK}\Gamma_{IJK} - \frac{1}{2}\Gamma^\mu\partial_\mu \log a - \left(\frac{1}{16}e_{\mu JK} - 3m_{\mu JK}\right)\Gamma^{\mu JK} - M\right) \ , && (3.2.10)\\
\bar{\epsilon}\,\overline{B}^A &= \bar{\epsilon}\left(F\Gamma^A + \left(-\frac{1}{4}e^{AJK} + 12m^{AJK}\right)\Gamma_{JK}\right) \ , && (3.2.11)\\
\partial_\mu\bar{\epsilon} &= \bar{\epsilon}\mathcal{A}_\mu \ , && (3.2.12)
\end{aligned}
$$

where F is a real 32×32 matrix acting on the spinor indices (see (D.2.20) or (D.2.22)), and

$$
\begin{aligned}
P_{I'J'}{}^{IJ} &\equiv \frac{1}{72}\Gamma_{I'J'}\Gamma^{IJ} + \frac{1}{4}\Gamma_{[I'}\Gamma^{[I}\delta^{J]}_{J']} \ , && (3.2.13)\\
e^{IJK} &\equiv a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} + 3\,d^{[IJK]} + 24\,m^{IJK} \ , && (3.2.14)\\
\mathcal{A}_\mu &\equiv -\frac{1}{4}\left(F\Gamma_\mu + \left(-\frac{1}{4}e_{\mu\hat{J}\hat{K}} + 12m_{\mu\hat{J}\hat{K}} - \omega_{\mu\hat{J}\hat{K}}\right)\Gamma^{\hat{J}\hat{K}}\right) \ . && (3.2.15)
\end{aligned}
$$

$P_{I'J'}{}^{IJ}$ is a projection, as it satisfies

$$
P_{IJ}{}^{I'J'}P_{I'J'}{}^{KL} = P_{IJ}{}^{KL} \ . \qquad (3.2.16)
$$

In addition, for an arbitrary G^I, we have

$$
G^{[I'}\Gamma^{J']}P_{I'J'}{}^{IJ} = G^{[I}\Gamma^{J]} \ , \quad P_{I'J'}{}^{IJ}G_{[I}\Gamma_{J]} = G_{[I'}\Gamma_{J']} \ . \qquad (3.2.17)
$$

Note that $\overline{B}^A$, F and $\mathcal{A}_\mu$ always appear in the combination $\bar{\epsilon}\overline{B}^A$, $\bar{\epsilon}F$ and $\bar{\epsilon}\mathcal{A}_\mu$, respectively, and hence we only need to determine them up to additions of matrices that vanish when $\bar{\epsilon}$ is multiplied from the left. In particular, the SUSY transformation (3.1.16) is determined once the right hand side of (3.2.11) is fixed.

The equation (3.2.12) can be solved when the integrability condition

$$
\bar{\epsilon}\mathcal{F}_{\mu\nu} = 0 \ , \qquad (3.2.18)
$$

with $\mathcal{F}_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$ is satisfied. Then, (3.2.12) can be formally solved as

$$
\bar{\epsilon}(x) = \bar{\epsilon}_0\,\mathrm{P}\exp\left(\int_{x_0}^{x} dx^\mu\mathcal{A}_\mu\right) \ , \qquad (3.2.19)
$$

where $\bar{\epsilon}_0$ is a constant spinor, "P exp" denotes the path ordered exponential (the ordering is taken from left to right) and x_0 is a fixed position. The integrability condition (3.2.18) guarantees that (3.2.19) is well-defined in a neighborhood of x_0.

(3.2.9) has a trivial solution $e^{IJK} = 0$. In fact, in all the examples we consider in the following sections, one can show that $e^{IJK} = 0$ is the only solution of (3.2.9) that is compatible with the imposed symmetry. When $e^{IJK} = 0$, (3.2.10) is simplified as

$$0 = \bar{\epsilon}\left(\frac{1}{2}\Gamma^{\mu}\partial_{\mu}\log a - 3m_{\mu JK}\Gamma^{\mu JK} + M\right) . \tag{3.2.20}$$

By definition, $e^{IJK} = 0$ is equivalent to

$$\begin{aligned} 0 &= a^{-1}\partial_{\mu}(ac)\,\epsilon^{\mu\nu\rho\sigma} + 24\,m^{\nu\rho\sigma} \ , && (3.2.21) \\ 0 &= d^{[IJA]} + 8\,m^{IJA} \ . && (3.2.22) \end{aligned}$$

Using (3.1.13), the latter is written as

$$d^{\mu\nu A} = -24\,m^{\mu\nu A} \ , \quad d^{\mu AB} = -12\,m^{\mu AB} \ , \quad d^{ABC} = -8\,m^{ABC} \ . \tag{3.2.23}$$

(3.2.21) can have a solution if and only if

$$\partial_{\nu}(\sqrt{-g}\,a\,m^{\nu\rho\sigma}) = 0 \tag{3.2.24}$$

is satisfied.

In summary, we have reduced the conditions for the SUSY invariance of the action (3.1.10) to the equations (3.2.7) and (3.2.8), where the former can be split into the equations (3.2.9)-(3.2.12). In the following section, we are going to elaborate on solutions that preserve different symmetries.

Chapter 4

Example: $ISO(1,1) \times SO(3) \times SO(3)$

In the later part, we will apply our equations to preserve SUSY, obtained from the previous section, for specific cases. First, we consider $ISO(1,1) \times SO(3) \times SO(3)$ symmetry. Here, $ISO(1,1)$ is the Poincaré group acting on $x^{0,1}$, and the first and second $SO(3)$ act as rotation of $x^{4,5,6}$ and $x^{7,8,9}$ in the 10-dimensional notation, respectively. Although our analysis is purely field theoretical, our motivation come from string theory. Consider a D3-D5-D7 system in the following table:

	0	1	2	3	4	5	6	7	8	9
D3	o	o	o	o						
D5	o	o	o		o	o	o			
D7	o	o			o	o	o	o	o	o

(4.0.1)

This configuration is 1/8 BPS configuration, which preserves part of the supersymmetry, as well as the $ISO(1,1) \times SO(3) \times SO(3)$ symmetry. If we regard the D5 and D7-branes as a supergravity background and the D3-brane as a probe embedded in it, the low energy effective theory on the D3-brane is expected the one which deformed from the $\mathcal{N} = 4$ SYM that the couplings are depend on space-time. We may replace the D5-brane and D7-branes with (p, q) 5-brane and $[p, q]$ 7-branes, respectively, to have more complicated configurations preserving SUSY.

Being absent of $[p, q]$ 7-branes, the case with D3-branes in the (p, q) 5-brane backgrounds corresponds to the supersymmetric Janus configuration considered in [14]. Also being absent of (p, q) 5-brane, D3-branes in the $[p, q]$ 7-brane backgrounds can be generalized to D3-branes in F-theory configurations, which was recently analyzed in [20, 22, 23] (see also [25, 27]). So, the case with this symmetry will combinate this two configurations by appearing as special cases in our example as we discuss separately in Sections 4.4 and 4.3, respectively.

4.1 Ansatz and SUSY conditions

We decompose the 10 dimensional coordinates in four sectors: $\alpha, \beta = 0, 1$; $i, j = 2, 3$; $a, b, c = 4, 5, 6$; $p, q, r = 7, 8, 9$. The metric and the couplings are assumed to depend only on x^i and preserve the $ISO(1,1) \times SO(3) \times SO(3)$ symmetry acting on $\{x^\alpha\} \times \{x^a\} \times \{x^p\}$. The metric (3.1.6) is of the form

$$ds^2 = e(x^i)\eta_{\alpha\beta}dx^\alpha dx^\beta + g_{ij}(x^i)dx^i dx^j + \delta_{ab}dx^a dx^b + \delta_{pq}dx^p dx^q \ . \tag{4.1.1}$$

Using the general coordinate transformation and Weyl transformation* with appropriate rescaling of the fields, we can assume $e(x^i) = 1$ and $g_{ij}(x^i) = e^{\Phi(x^i)}\delta_{ij}$ without loss of generality.

M can be written as a general form to respect this symmetry,

$$\begin{aligned} M &= 6\left(m_{01i}\Gamma^{01i} + m_{456}\Gamma^{456} + m_{789}\Gamma^{789}\right) \\ &\equiv \alpha_i\Gamma^i\Gamma^{01} + \beta\Gamma^{456} + \gamma\Gamma^{789} \ . \end{aligned} \tag{4.1.2}$$

The non-zero components of d^{IJA} and m^{AB} are

$$d^{abc} = \frac{v}{3}\epsilon^{abc} \ , \quad d^{pqr} = \frac{w}{3}\epsilon^{pqr} \ , \quad m^{ab} = r\delta^{ab} \ , \quad m^{pq} = \widetilde{r}\delta^{pq} \ . \tag{4.1.3}$$

We also define

$$q_i \equiv \partial_i \log a \ . \tag{4.1.4}$$

The non-trivial components of e^{IJK} defined in (3.2.14) are

$$e^{01i} = a^{-1}\partial_j(ac)\,\epsilon^{ji} - 4\alpha^i \ , \quad e^{abc} = (v + 4\beta)\epsilon^{abc} \ , \quad e^{pqr} = (w + 4\gamma)\epsilon^{abc} \ . \tag{4.1.5}$$

In this case, one can show that the condition (3.2.9) implies $e^{IJK} = 0$ (see Appendix E.1) and hence

$$a^{-1}\partial_j(ac)\,\epsilon^{ji} = 4\alpha^i \ , \quad v = -4\beta \ , \quad w = -4\gamma \ . \tag{4.1.6}$$

The integrability condition (3.2.24) for the first equation of (4.1.6) is

$$g^{kj}\partial_k(a\alpha_j) = 0 \ . \tag{4.1.7}$$

The non-zero components of the spin connection are

$$\omega_i{}^{\hat{2}\hat{3}} = -\omega_i{}^{\hat{3}\hat{2}} = \frac{1}{2}\epsilon_i{}^j\partial_j\Phi \ , \tag{4.1.8}$$

*See appendix C.1.

and $\mathcal{A}_\mu$ defined in (3.2.15) is

$$\mathcal{A}_0 = -\frac{1}{4}\left(F\Gamma_0 + 4\alpha_i\Gamma^{1i}\right) \ , \quad \mathcal{A}_1 = -\frac{1}{4}\left(F\Gamma_1 - 4\alpha_i\Gamma^{0i}\right) \ ,$$
$$\mathcal{A}_i = -\frac{1}{4}\left(F\Gamma_i + 4\alpha_i\Gamma^{01} - \epsilon_i{}^j\partial_j\Phi\Gamma^{\hat{2}\hat{3}}\right) \ . \tag{4.1.9}$$

The condition (3.2.12) with $\mu = 0, 1$ implies

$$0 = \bar{\epsilon}(F + 4\alpha_i\Gamma^{01i}) \ . \tag{4.1.10}$$

Using this the equation, $\mathcal{A}_i$ in (3.2.12) with $\mu = i = 2, 3$ can be replaced with

$$\mathcal{A}_i = \epsilon_i{}^j\left(-\alpha_j\Gamma^{01} + \frac{1}{4}\partial_j\Phi\right)\Gamma^{\hat{2}\hat{3}} \ , \tag{4.1.11}$$

and (3.2.20), (3.2.11) and (3.2.12) become

$$0 = \bar{\epsilon}\left(\Gamma^i q_i - 4\alpha_i\Gamma^{01i} + 2\beta\Gamma^{456} + 2\gamma\Gamma^{789}\right) \ , \tag{4.1.12}$$
$$\bar{\epsilon}\overline{B}^a = \bar{\epsilon}\left(-4\alpha_i\Gamma^{01ia} + 2\beta\epsilon^{abc}\Gamma_{bc}\right) \ , \tag{4.1.13}$$
$$\bar{\epsilon}\overline{B}^p = \bar{\epsilon}\left(-4\alpha_i\Gamma^{01ip} + 2\gamma\epsilon^{pqr}\Gamma_{qr}\right) \ , \tag{4.1.14}$$
$$\partial_i\bar{\epsilon} = \bar{\epsilon}\,\epsilon_i{}^j\left(-\alpha_j\Gamma^{01} + \frac{1}{4}\partial_j\Phi\right)\Gamma^{\hat{2}\hat{3}} \ . \tag{4.1.15}$$

The integrability condition (3.2.18) for (4.1.15) is

$$0 = \bar{\epsilon}\, g^{kj}\left(4\partial_k\alpha_j\Gamma^{01} - \partial_k\partial_j\Phi\right) \ . \tag{4.1.16}$$

The second order equation (3.2.8) for this case is

$$D_i(\bar{\epsilon}\overline{B}^a)\Gamma^i = \bar{\epsilon}\left(-r\Gamma^a - \overline{B}^a\left(M + \frac{1}{2}\Gamma^i q_i\right)\right) \ , \tag{4.1.17}$$
$$D_i(\bar{\epsilon}\overline{B}^p)\Gamma^i = \bar{\epsilon}\left(-\widetilde{r}\Gamma^p - \overline{B}^p\left(M + \frac{1}{2}\Gamma^i q_i\right)\right) \ . \tag{4.1.18}$$

Using (4.1.13)–(4.1.15), one can show that (4.1.17) and (4.1.18) are equivalent to

$$0 = r + \widetilde{r} + g^{ij}q_i q_j + 2g^{ij}\partial_i q_j - 4(\beta^2 + \gamma^2) \ , \tag{4.1.19}$$
$$0 = \bar{\epsilon}\left(r - \widetilde{r} + 4\partial_i\beta\Gamma^{i456} - 4\partial_i\gamma\Gamma^{i789} + 8\alpha_i\Gamma^{01i}\left(\beta\Gamma^{456} - \gamma\Gamma^{789}\right)\right) \ . \tag{4.1.20}$$

Therefore, equations that we have to solve are (4.1.7), (4.1.12), (4.1.16) and (4.1.20). Once we find a, α_j, β, γ, Φ and $r-\widetilde{r}$ satisfying these equations, the other parameters can be easily obtained by (4.1.6), (4.1.13), (4.1.14) and (4.1.19). It can be easily checked that these SUSY conditions reduce to those given in [14] when the symmetry is enhanced to $ISO(1,2) \times SO(3) \times SO(3)$.

4.2 Solutions of the SUSY conditions

In this section we are going to elaborate on a prescription for the SUSY parameter ϵ which simplifies the study of solutions. In addition, we give two examples of solutions, where the latter is a generalization of the former.

4.2.1 More on SUSY conditions

Let us first try to solve (4.1.16). Decomposing $\bar{\epsilon}$ as

$$\bar{\epsilon} = \bar{\epsilon}_+ + \bar{\epsilon}_- \ , \tag{4.2.1}$$

with $\bar{\epsilon}_\pm \Gamma^{01} = \pm\bar{\epsilon}_\pm$, (4.1.16) can be written as

$$0 = \bar{\epsilon}_+ g^{kj}\left(4\partial_k\alpha_j - \partial_k\partial_j\Phi\right) - \bar{\epsilon}_- g^{kj}\left(4\partial_k\alpha_j + \partial_k\partial_j\Phi\right) \ . \tag{4.2.2}$$

If $g^{kj}\partial_k\partial_j\Phi = 0$ and $g^{kj}\partial_k\alpha_j = 0$, both $\bar{\epsilon}_+$ and $\bar{\epsilon}_-$ can be non-zero. However, if $g^{kj}\partial_k\partial_j\Phi \neq 0$, it has a non-trivial solution only if

$$g^{kj}\partial_k(\partial_j\Phi \mp 4\alpha_j) = 0 \ , \quad \bar{\epsilon}_\mp = 0 \tag{4.2.3}$$

are satisfied. Therefore, when Φ is not a harmonic function, the unbroken SUSY is inevitably chiral in 2-dimensions. In the following, we impose (4.2.3), though we do not assume $g^{kj}\partial_k\partial_j\Phi \neq 0$. The general solution for the case with $g^{kj}\partial_k\partial_j\Phi = 0$ can be easily obtained by taking a linear combination of a solution with $\bar{\epsilon} = \bar{\epsilon}_+$ and that with $\bar{\epsilon} = \bar{\epsilon}_-$.

(4.2.3) can be solved (at least locally) if and only if there exists a function $\varphi_\pm$ satisfying

$$\pm\alpha_j - \frac{1}{4}\partial_j\Phi = \epsilon_j{}^i\partial_i\varphi_\pm \ . \tag{4.2.4}$$

Then, the solution of (4.1.15) is

$$\bar{\epsilon} = \bar{\epsilon}^0_\pm e^{\varphi_\pm\Gamma^{\hat{2}\hat{3}}} = \bar{\epsilon}^0_\pm\left(\cos\varphi_\pm + \Gamma^{\hat{2}\hat{3}}\sin\varphi_\pm\right) \ , \tag{4.2.5}$$

where $\bar{\epsilon}^0_\pm$ is a constant spinor satisfying

$$\bar{\epsilon}^0_\pm\,\Gamma^{01} = \pm\bar{\epsilon}^0_\pm \ , \quad \bar{\epsilon}^0_\pm\,\Gamma^{(10)} = \bar{\epsilon}^0_\pm \ . \tag{4.2.6}$$

The second condition in (4.2.6) follows from the chirality condition (3.1.7).

This $\bar{\epsilon}^0_\pm$ belongs to the 8-dimensional Majorana-Weyl representation of the $SO(8)$ subgroup of the 10-dimensional Lorentz group. Since the operators acting on $\bar{\epsilon}$ in (4.1.12) and (4.1.20) commute with $SO(3)\times SO(3)$ generators Γ^{ab} and Γ^{pq}, it is convenient to decompose it to two $(2,2)$ representations of the $SO(3)\times SO(3)$ as

$$\bar{\epsilon}^0_{\pm} = \sum_{v=1}^{4}(f^0_v \langle\, 0; v\,| + f^1_v \langle\, 1; v\,|) \ , \tag{4.2.7}$$

where $v = 1,2,3,4$ is an index labeling the 4-dimensional representation ($(2,2)$ representation) of $SO(3)\times SO(3)$, and f^0_v and f^1_v ($v = 1,2,3,4$) are real parameters. The set $\{\langle\, 0; v\,|, \langle\, 1; v\,|\ |\ v = 1,2,3,4\}$ is the basis of the spinors satisfying (4.2.6). They can be constructed explicitly as follows. Let us define

$$C_1 \equiv \frac{1}{2}(\Gamma^{\hat{2}} + \Gamma^{456}) \ , \quad C_2 \equiv \frac{1}{2}(\Gamma^{\hat{3}} \mp \Gamma^{789}) \ , \tag{4.2.8}$$

which satisfy

$$\{C_s, C_t^\dagger\} = \delta_{st} \ , \quad \{C_s, C_t\} = 0 \ , \quad (s,t = 1,2) \ . \tag{4.2.9}$$

Let $\langle\, 0; v\,|$ be a spinor satisfying

$$\langle\, 0; v\,| C_s^\dagger = 0 \ , \quad (s = 1,2) \tag{4.2.10}$$

and $\langle\, 1; v\,|$ is defined as

$$\langle\, 1; v\,| \equiv \langle\, 0; v\,| C_1 C_2 \ . \tag{4.2.11}$$

Since all the operators acting on $\bar{\epsilon}^0_{\pm}$ in (4.1.12) and (4.1.20) do not mix the spaces with different index v, we can assume

$$\bar{\epsilon}^0_{\pm} = f^0_v \langle\, 0; v\,| + f^1_v \langle\, 1; v\,| \tag{4.2.12}$$

with fixed v. (The general solution is just a linear combination of this type.) Note that $\Gamma^{\hat{2}\hat{3}} = (C_1 + C_1^\dagger)(C_2 + C_2^\dagger)$ and

$$\langle\, 0; v\,| \Gamma^{\hat{2}\hat{3}} = \langle\, 1; v\,| \ , \quad \langle\, 1; v\,| \Gamma^{\hat{2}\hat{3}} = -\langle\, 0; v\,| \ . \tag{4.2.13}$$

Then, (4.2.12) can be written as

$$\bar{\epsilon}^0_{\pm} = \eta \langle\, 0; v\,| e^{\xi\, \Gamma^{\hat{2}\hat{3}}} \ , \tag{4.2.14}$$

where $\eta = \sqrt{(f^0_v)^2 + (f^1_v)^2}$ and $\xi = \arctan(f^1_v/f^0_v)$. As a consequence, the number of remaining SUSY is 4 in general, corresponding to the choice of $v = 1,2,3,4$. This agrees with what we expect from the brane configuration (4.0.1). If the phase ξ can be chosen freely without changing the action, the number of SUSY is enhanced to 8. As mentioned above, if Φ satisfies the Laplace equation $g^{kj}\partial_k\partial_j\Phi = 0$ and both $\bar{\epsilon}_+$ and $\bar{\epsilon}_-$ are allowed, the number of SUSY is doubled. We will see some examples with the SUSY enhancement later.

Since ξ can be absorbed by the shift of $\varphi_\pm$ in (4.2.5), we set $\xi = 0$ in the following. Then, using (4.2.4), (4.2.5) and (4.2.14), one can show that the SUSY condition (4.1.12) is equivalent to

$$\begin{aligned}
\beta &= e^{-\Phi}a^{-1/2}\left(\partial_2\left(e^{\frac{1}{2}\Phi}a^{-1/2}\cos(2\varphi_\pm)\right) + \partial_3\left(e^{\frac{1}{2}\Phi}a^{-1/2}\sin(2\varphi_\pm)\right)\right) , &(4.2.15)\\
\gamma &= \mp e^{-\Phi}a^{1/2}\left(-\partial_2\left(e^{\frac{1}{2}\Phi}a^{-1/2}\sin(2\varphi_\pm)\right) + \partial_3\left(e^{\frac{1}{2}\Phi}a^{-1/2}\cos(2\varphi_\pm)\right)\right) , &(4.2.16)
\end{aligned}$$

and (4.1.20) is equivalent to

$$\partial_2\left(e^{\frac{1}{2}\Phi}(-\beta\cos(2\varphi_\pm) \pm \gamma\sin(2\varphi_\pm))\right) + \partial_3\left(e^{\frac{1}{2}\Phi}(\mp\gamma\cos(2\varphi_\pm) - \beta\sin(2\varphi_\pm))\right) = -\frac{1}{4}(r-\widetilde{r})e^{\Phi} , \quad (4.2.17)$$

$$\partial_2\left(e^{\frac{1}{2}\Phi}(\pm\gamma\cos(2\varphi_\pm) + \beta\sin(2\varphi_\pm))\right) + \partial_3\left(e^{\frac{1}{2}\Phi}(-\beta\cos(2\varphi_\pm) \pm \gamma\sin(2\varphi_\pm))\right) = 0 . \quad (4.2.18)$$

It is convenient to write these equations using a complex coordinate $z \equiv \frac{1}{\sqrt{2}}(x^2 + ix^3)$:

$$\beta \pm i\gamma = \sqrt{2}\, e^{-\Phi}a^{1/2}\partial_z\left(e^{\frac{1}{2}\Phi + i2\varphi_\pm}a^{-1/2}\right) , \quad (4.2.19)$$

$$\begin{aligned}
\frac{1}{4}(r-\widetilde{r})e^{\Phi} &= \sqrt{2}\,\partial_z\left(e^{\frac{1}{2}\Phi+i2\varphi_\pm}(\beta \pm i\gamma)\right)\\
&= 2\partial_z\left(e^{-\frac{1}{2}\Phi+i2\varphi_\pm}a^{1/2}\partial_z\left(e^{\frac{1}{2}\Phi+i2\varphi_\pm}a^{-1/2}\right)\right) , &(4.2.20)
\end{aligned}$$

where we have used (4.2.19) in the last step. β, γ and $(r-\widetilde{r})$ are determined by (4.2.19) and the real part of (4.2.20). The imaginary part of (4.2.20) gives a non-trivial constraint:

$$\mathrm{Im}\left[\partial_z\left(e^{-\frac{1}{2}\Phi+i2\varphi_\pm}a^{1/2}\partial_z\left(e^{\frac{1}{2}\Phi+i2\varphi_\pm}a^{-1/2}\right)\right)\right] = 0 . \quad (4.2.21)$$

This equation can be solved if there exists a real function $f(z,\bar{z})$ satisfying

$$\begin{aligned}
\partial_{\bar{z}}f &= e^{-\frac{1}{2}\Phi+i2\varphi_\pm}a^{1/2}\partial_z\left(e^{\frac{1}{2}\Phi+i2\varphi_\pm}a^{-1/2}\right)\\
&= e^{i4\varphi_\pm}\left(i2\partial_z\varphi_\pm + \frac{1}{2}\partial_z(\Phi - \log a)\right) . &(4.2.22)
\end{aligned}$$

This is equivalent to

$$e^{-i2\varphi_\pm+\frac{1}{2}(\Phi-\log a)}\partial_{\bar{z}}e^f = \partial_z\left(e^{i2\varphi_\pm+\frac{1}{2}(\Phi-\log a)}\right)e^f . \quad (4.2.23)$$

The complex conjugate of this equation is

$$\partial_{\overline{z}} \left(e^{-i2\varphi_\pm + \frac{1}{2}(\Phi - \log a)} \right) e^f = e^{i2\varphi_\pm + \frac{1}{2}(\Phi - \log a)} \partial_z e^f \ . \tag{4.2.24}$$

The sum of these two equations gives

$$\partial_{\overline{z}} \left(e^{-i2\varphi_\pm + \frac{1}{2}(\Phi - \log a)} e^f \right) = \partial_z \left(e^{i2\varphi_\pm + \frac{1}{2}(\Phi - \log a)} e^f \right) \ , \tag{4.2.25}$$

which is equivalent to

$$\mathrm{Im} \left[\partial_z \left(e^{i2\varphi_\pm + \frac{1}{2}(\Phi - \log a) + f} \right) \right] = 0 \ . \tag{4.2.26}$$

This equation can be solved if there exists a real function $g(z, \overline{z})$ satisfying

$$e^{i2\varphi_\pm + \frac{1}{2}(\Phi - \log a)} = e^{-f} \partial_{\overline{z}} g \ . \tag{4.2.27}$$

Inserting this into (4.2.23), we obtain

$$\partial_z g \partial_{\overline{z}} f + \partial_z f \partial_{\overline{z}} g = \partial_z \partial_{\overline{z}} g \ . \tag{4.2.28}$$

If we are able to find real functions g and f satisfying this relation, $\varphi_\pm$ and $(\Phi - \log a)$ are obtained by (4.2.27). (4.2.19) and (4.2.20) are

$$\begin{aligned} \beta \pm i\gamma &= \sqrt{2} e^{-\Phi} a^{1/2} \partial_z (e^{-f} \partial_{\overline{z}} g) = \sqrt{2} e^{-\Phi} a^{1/2} e^{-f} \partial_z g \partial_{\overline{z}} f \ , & (4.2.29) \\ r - \widetilde{r} &= 8 e^{-\Phi} \partial_z \partial_{\overline{z}} f \ . & (4.2.30) \end{aligned}$$

Note that (4.2.28) can also be written as

$$\mathrm{Re}[\partial_z (e^{-2f} \partial_{\overline{z}} g)] = 0 \ . \tag{4.2.31}$$

This equation can be solved if there exists a real function $h(z, \overline{z})$ satisfying

$$e^{-2f} \partial_z g = i \partial_z h \ , \tag{4.2.32}$$

which implies

$$\partial_z g \partial_{\overline{z}} h + \partial_{\overline{z}} g \partial_z h = 0 \ . \tag{4.2.33}$$

It shows that the gradient of g and h are orthogonal to each other. Conversely, if we are able to find real functions g and h satisfying (4.2.33), f is obtained as

$$e^{-2f} = i \frac{\partial_z h}{\partial_z g} = -i \frac{\partial_{\overline{z}} h}{\partial_{\overline{z}} g} \ . \tag{4.2.34}$$

(4.2.27) can also be written as

$$e^{i4\varphi_\pm+\Phi-\log a} = -i\partial_{\overline{z}}h\partial_{\overline{z}}g \ . \tag{4.2.35}$$

In addition to these equations, we should also solve (4.1.7). Using (4.2.4), (4.1.7) can be written as

$$g^{ij}\partial_j(a\partial_i\Phi) + 4\epsilon^{jk}\partial_j a\partial_k\varphi_\pm = 0 \ , \tag{4.2.36}$$

which is equivalent to

$$\mathrm{Re}\left[\partial_{\overline{z}}(a\,\partial_z(\Phi+4i\varphi_\pm))\right] = 0 \ . \tag{4.2.37}$$

This equation can be solved if there exists a real function $k(z,\overline{z})$ satisfying

$$a\,\partial_z(\Phi-\log a+4i\varphi_\pm)+\partial_z a = a\,\partial_z(\Phi+4i\varphi_\pm) = i\partial_z k \ . \tag{4.2.38}$$

From the first equation of (4.1.6) and (4.2.4), we see that this k is proportional to the theta parameter as

$$k = \mp ac = \mp\frac{\theta}{8\pi^2} \ , \tag{4.2.39}$$

up to an additive constant.

Using (4.2.27), (4.2.38) becomes

$$2a\partial_z(-f+\log\partial_{\overline{z}}g)+\partial_z a = i\partial_z k \ , \tag{4.2.40}$$

which can also be written as

$$\partial_z(a\partial_{\overline{z}}h\partial_{\overline{z}}g) = i\partial_{\overline{z}}h\partial_{\overline{z}}g\partial_z k \ . \tag{4.2.41}$$

In summary, the SUSY conditions are now reduced to a problem of finding real functions h, g, a and k satisfying (4.2.33) and (4.2.41). Then, the real function f is obtained by (4.2.34) and other parameters are determined by (4.2.27) (or (4.2.35)), (4.2.29), (4.2.30) and (4.2.39). Although we have not been able to find the general solution of the SUSY conditions (4.2.33) and (4.2.41), a lot of non-trivial solutions have been found. In the following subsections, we show some of the explicit solutions.

4.2.2 Solution 1

First, we introduce new coordinates (y^1, y^2) defined as

$$y^1 \equiv \bar{l}(\bar{z}) + l(z) \ , \quad y^2 \equiv i(\bar{l}(\bar{z}) - l(z)) \ , \tag{4.2.42}$$

where $l(z) \in \mathbb{C}$ is a holomorphic function of $z = \frac{1}{\sqrt{2}}(x^2 + ix^3)$ and $\bar{l}(\bar{z})$ is its complex conjugate. They are related to the original coordinates (x^2, x^3) by a conformal transformation on the 2-dimensional plane. Note that our ansatz explained in Section 4.1 is compatible with the conformal transformation* and hence our results in the previous subsection are valid in the coordinates (y^1, y^2) as well. In fact, it is easy to see that the equations (4.2.33) and (4.2.41) are invariant under the conformal transformation. Once one finds a solution, one can generate new solutions by the conformal transformations. In order to emphasize this point, we write down the solutions that work for any choice of the holomorphic function $l(z)$, rather than using this degrees of freedom to simplify the equations.

Since the condition (4.2.33) is equivalent to the statement that the gradient of g and h are orthogonal to each other, it is clear that it can be solved when g and h are of the form:

$$g(z, \bar{z}) = G_1(y^1) \ , \quad h(z, \bar{z}) = H_2(y^2) \ , \tag{4.2.43}$$

where G_1 and H_2 are real functions. The subscripts of these functions suggest which of the coordinates (y^1 or y^2) they depend on. Inserting these into (4.2.41), we obtain

$$\partial_z(aH_2'G_1') = iH_2'G_1'\partial_z k \ , \tag{4.2.44}$$

where the prime denotes the derivative, e.g., $G_1' \equiv \partial_{y^1} G_1(y^1)$.

One can check that the following ansatz gives a solution of (4.2.44):

$$a(z, \bar{z}) = \frac{L(y^1, y^2)}{G_1'(y^1)H_2'(y^2)} \ , \quad k(z, \bar{z}) = K_1(y^1) + K_2(y^2) \ , \tag{4.2.45}$$

where K_1 and K_2 are real functions satisfying

$$G_1'K_1' = \kappa_1 \ , \quad H_2'K_2' = \kappa_2 \ , \tag{4.2.46}$$

with real constants κ_i $(i = 1, 2)$ and A is a real function defined as

$$L(y^1, y^2) \equiv \kappa_2 G_1(y^1) - \kappa_1 H_2(y^2) \ . \tag{4.2.47}$$

* Because we set $g_{ij}(x^i) = e^{\Phi(x^i)}\delta_{ij}$, general coordinate transformation of (x^2, x^3) is not compatible with our ansatz. The conformal transformation on the (x^2, x^3)-plane keeps this form invariant.

Then, using (4.1.19), (4.2.29), (4.2.30), (4.2.35) and (4.2.39), we obtain the rest of parameters:

$$
\begin{aligned}
c = & \mp \frac{G_1' H_2'}{L}(K_1 + K_2) \ , && (4.2.48)\\
\Phi = & \log\left(L\partial_z l \partial_{\bar z}\bar l\right) \ , && (4.2.49)\\
\varphi_\pm = & \frac{1}{2}\mathrm{Im}[\log\partial_{\bar z}\bar l] \ , && (4.2.50)\\
\beta \pm i\gamma = & \frac{1}{\sqrt{2L}}\left((\log G_1')' - i(\log H_2')'\right) \ , && (4.2.51)\\
r - \widetilde r = & \frac{4}{L}\left((\log G_1')'' - (\log H_2')''\right) \ , && (4.2.52)\\
r + \widetilde r = & \frac{4}{L}\left((\log G_1')'' + (\log H_2')''\right) + \frac{2}{L^3}(\kappa_2^2 G_1'^2 + \kappa_1^2 H_2'^2) \ . && (4.2.53)
\end{aligned}
$$

4.2.3 Solution 2

Let us generalize the solutions in the previous subsection by considering the following ansatz:

$$
g(z,\bar z) = G_1(y^1) + G_2(y^2) \ , \quad h(z,\bar z) = H_1(y^1) + H_2(y^2) \ , \tag{4.2.54}
$$

where, G_i and H_i $(i = 1,2)$ are real functions. Then, (4.2.33) can be solved when these functions satisfy

$$
G_1' H_1' = -c_0 \ , \quad G_2' H_2' = c_0 \ , \tag{4.2.55}
$$

where c_0 is a real constant. Note that (4.2.34) implies

$$
e^{-2f} = \frac{H_2'}{G_1'} \ , \tag{4.2.56}
$$

which makes sense when $H_2' G_1' > 0$.

Then, (4.2.41) can be written as

$$
e^{i\sigma}\partial_z A = \partial_z B \ , \tag{4.2.57}
$$

where we have defined

$$
\begin{aligned}
e^{i\sigma} \ \equiv & \ \frac{W + 2c_0 i}{V} \ , && (4.2.58)\\
W \ \equiv & \ G_1' H_2' + G_2' H_1' = G_1' H_2' - \frac{c_0^2}{G_1' H_2'} \ , && (4.2.59)\\
V \ \equiv & \ \sqrt{W^2 + 4c_0^2} = G_1' H_2' + \frac{c_0^2}{G_1' H_2'} \ , && (4.2.60)\\
A \ \equiv & \ aW + 2c_0 k \ , && (4.2.61)\\
B \ \equiv & \ aV \ . && (4.2.62)
\end{aligned}
$$

Some useful identities are

$$G_1' H_2' = c_0 \cot\left(\frac{\sigma}{2}\right) \ , \quad W = 2c_0 \cot\sigma \ , \quad V = \frac{2c_0}{\sin\sigma} \ . \tag{4.2.63}$$

Note that in the Taylor expansion of (4.2.57) with respect to c_0, the leading term is trivially satisfied and the $\mathcal{O}(c_0)$ term reproduces (4.2.44).

Here, we try to solve (4.2.57) using the following ansatz:

$$A(y^1,y^2) = A_1(y^1) + A_2(y^2) \ , \quad B(y^1,y^2) = B_1(y^1) + B_2(y^2) \ , \tag{4.2.64}$$

where A_i and B_i are real functions. Inserting this ansatz into (4.2.57), we obtain

$$G_1' H_2'(A_1' - B_1') - \frac{c_0^2}{G_1' H_2'}(A_1' + B_1') \;=\; -2c_0 A_2' \ , \tag{4.2.65}$$

$$G_1' H_2'(A_2' - B_2') - \frac{c_0^2}{G_1' H_2'}(A_2' + B_2') \;=\; 2c_0 A_1' \ . \tag{4.2.66}$$

These equations can be solved when

$$A_1' + B_1' = 2\kappa_2 G_1' \ , \quad A_1' - B_1' = \frac{2c_0\kappa_1}{G_1'} \ , \quad A_2' + B_2' = -2\kappa_1 H_2' \ , \quad A_2' - B_2' = \frac{2c_0\kappa_2}{H_2'} \ , \tag{4.2.67}$$

where κ_i $(i=1,2)$ are real constants. Then, we obtain

$$A \;=\; \kappa_2 G_1 - \kappa_1 H_2 + c_0(K_1 + K_2) + a_0 \ , \tag{4.2.68}$$

$$B \;=\; \kappa_2 G_1 - \kappa_1 H_2 - c_0(K_1 + K_2) + b_0 \ , \tag{4.2.69}$$

where $K_i = K_i(y^i)$ $(i=1,2)$ are real functions satisfying (4.2.46), and a_0 and b_0 are real constants. (We can set $a_0 = b_0 = 0$ by absorbing them in the constant parts of G_1, H_2 and K_i, but we will keep them for convenience.)

Then, by the definition of A and B, we get

$$a = \frac{B}{V} = \frac{B}{2c_0}\sin\sigma \ , \tag{4.2.70}$$

$$k = \frac{1}{2c_0}\left(A - B\frac{W}{V}\right) = \frac{1}{2c_0}\left(A - B\cos\sigma\right) \ . \tag{4.2.71}$$

Other parameters are obtained by using (4.1.19), (4.2.29), (4.2.30), (4.2.35) and (4.2.39):

$$
\begin{aligned}
c =&\ \pm \frac{1}{2c_0}\left(W - V\frac{A}{B}\right) = \pm\left(\cot\sigma - \frac{A}{B\sin\sigma}\right) \ , && (4.2.72)\\
\Phi =&\ \log\left(B\partial_z l \partial_{\bar z}\bar l\right) \ , && (4.2.73)\\
\varphi_\pm =&\ \frac{\sigma}{4} + \frac{1}{2}\mathrm{Im}[\log\partial_{\bar z}\bar l] \ , && (4.2.74)\\
\beta \pm i\gamma =&\ \frac{e^{-\frac{\sigma}{2}i}}{\sqrt{2B}}\left((\log G_1')' - i(\log H_2')'\right) \ , && (4.2.75)\\
r - \widetilde{r} =&\ \frac{4}{B}\left((\log G_1')'' - (\log H_2')''\right) \ , && (4.2.76)\\
r + \widetilde{r} =&\ \frac{4}{B}\frac{W}{V}\left((\log G_1')'' + (\log H_2')''\right) + \frac{2}{B^3}\left(\kappa_2^2 G_1'^2 + \kappa_1^2 H_2'^2 + c_0^2(K_1'^2 + K_2'^2)\right)\\
&+ \frac{24}{B}\frac{c_0^2}{V^2}\left(((\log G_1')')^2 + ((\log H_2')')^2\right)\\
&+ \frac{4}{B^2}\left(\frac{W}{V} - 1\right)(\kappa_2 G_1'' - \kappa_1 H_2'') - \frac{4c_0}{B^2}\left(\frac{W}{V} + 1\right)(K_1'' + K_2'')\\
=&\ \frac{4}{B}\cos\sigma\left((\log G_1')'' + (\log H_2')''\right) + \frac{2}{B^3}\left(\kappa_2^2 G_1'^2 + \kappa_1^2 H_2'^2 + c_0^2(K_1'^2 + K_2'^2)\right)\\
&+ \frac{6}{B}\sin^2\sigma\left(((\log G_1')')^2 + ((\log H_2')')^2\right)\\
&- \frac{8}{B^2}\sin^2\left(\frac{\sigma}{2}\right)(\kappa_2 G_1'' - \kappa_1 H_2'') - \frac{8c_0}{B^2}\sin^2\left(\frac{\sigma}{2}\right)(K_1'' + K_2'') \ . && (4.2.77)
\end{aligned}
$$

One can check that this solution reduces to the one given in the previous subsection in the $c_0 \to 0$, $\sigma \to 0$ limit.

4.3 F-theory configuration: $ISO(1,1) \times SO(6)$

As an explicit example of the solutions obtained in Section 4.2.2, let us consider

$$
G_1 = y^1 \ , \quad H_2 = y^2 \ , \quad K_1 = \kappa_1 y^1 \ , \quad K_2 = \kappa_2 y^2 \ . \tag{4.3.1}
$$

In this case, (4.2.45) gives

$$
a = \kappa_2 y^1 - \kappa_1 y^2 \ , \quad k = \kappa_1 y^1 + \kappa_2 y^2 \ , \tag{4.3.2}
$$

and hence

$$
a + ik = 2(\kappa_2 + i\kappa_1) l(z) \tag{4.3.3}
$$

is a holomorphic function of z. This implies that the complex coupling $\tau \propto i(a \pm ik)$ defined in (3.1.12) is holomorphic or anti-holomorphic depending on the chirality of the unbroken SUSY. Another immediate consequence is that the combination $\Phi - \log a \propto \mathrm{Re}(\log \partial_z l)$ is a harmonic function satisfying the Laplace equation in 2-dimensions:

$$\partial_z \partial_{\bar{z}} (\Phi - \log a) = 0 \ . \tag{4.3.4}$$

Actually, in this case, (4.2.48)–(4.2.53) imply $\beta = \gamma = 0$ and $r = \widetilde{r}$, and the symmetry $SO(3) \times SO(3)$ is enhanced to $SO(6)$.

Though this solution is a special solution, the properties (4.3.3) and (4.3.4) hold for general solutions with $ISO(1,1) \times SO(6)$ symmetry. In fact, it is not difficult to find the general solutions for the cases with $ISO(1,1) \times SO(6)$ symmetry. If we impose $\beta = \gamma = 0$, (4.2.29) implies $g =$ constant or $f =$ constant. However, in order to have non-singular solution of (4.2.27), g cannot be a constant. Then, f has to be a constant and (4.2.28) implies that g is a harmonic function satisfying $\partial_z \partial_{\bar{z}} g = 0$. Then, (4.2.27) and (4.2.40) imply that

$$\Phi - \log a - 4i\varphi_{\pm} \tag{4.3.5}$$

and $a + ik$ are holomorphic functions. Therefore the general solution can be written as

$$a = \bar{l}(\bar{z}) + l(z) \ , \quad k = i(\bar{l}(\bar{z}) - l(z)) \ , \tag{4.3.6}$$

$$\Phi = \log(\bar{l}(\bar{z}) + l(z)) + \overline{m}(\bar{z}) + m(z) \ , \quad \varphi_{\pm} = -\frac{i}{4}(\overline{m}(\bar{z}) - m(z)) \tag{4.3.7}$$

with holomorphic functions $l(z)$ and $m(z)$.

Note that all the parameters in the Lagrangian are invariant under a constant shift of the imaginary part of $m(z)$, but $\varphi_{\pm}$ is shifted. This shift induces a shift of ξ in (4.2.14) through (4.2.5) and hence we can choose any value for ξ without changing the action. Therefore, as explained below (4.2.14), the number of preserved SUSY is enhanced to 8. And this solution agrees with (2.2.4). Further analysis when the singularities and monodromies of τ is given is the same with the one in review 2.2.

4.4 Gaiotto-Witten solution: $ISO(1,2) \times SO(3) \times SO(3)$

The supersymmetric Janus configuration found by Gaiotto-Witten in [14] can be obtained as a special solution of the solutions obtained in Section 4.2.3. It is a solution with $ISO(1,2) \times SO(3) \times SO(3)$ symmetry and all the deformation parameters depend only on one coordinate x^3.

To see this, let us consider the case with $\kappa_i = 0$, $K_i = 0$ $(i = 1, 2)$ and $H_2 = y^2$. In this case, Φ is a harmonic function and, as discussed in Section 4.2.1, we may have solutions with both $\bar{\epsilon}_+$ and $\bar{\epsilon}_-$ being non-zero. To obtain such solutions, we should make sure that the parameters in the Lagrangian are consistent with the SUSY conditions for both $\bar{\epsilon}_+$ and $\bar{\epsilon}_-$ simultaneously. To this end, we choose

$$c_0 = \pm\frac{1}{D} \ , \quad \sigma = \pm 2\psi \tag{4.4.1}$$

for the solution in (4.2.73)∼(4.2.77). Then, we have the relation $\cot\psi = DG_1'(y^1)$ and

$$\begin{aligned}
a &= \frac{b_0 D}{2}\sin(2\psi) \ , &(4.4.2)\\
c &= \cot(2\psi) - \frac{a_0}{b_0\sin(2\psi)} \ , &(4.4.3)\\
\Phi &= \log\left(b_0\partial_z l\partial_{\bar{z}}\bar{l}\right) \ , &(4.4.4)\\
\varphi_\pm &= \pm\frac{\psi}{2} + \frac{1}{2}\mathrm{Im}[\log\partial_{\bar{z}}\bar{l}] \ , &(4.4.5)\\
\beta \pm i\gamma &= \frac{e^{\mp i\psi}}{\sqrt{2b_0}}(\log\cot\psi)' \ , &(4.4.6)\\
r - \widetilde{r} &= \frac{4}{b_0}(\log\cot\psi)'' \ , &(4.4.7)\\
r + \widetilde{r} &= \frac{4}{b_0}\cos(2\psi)(\log\cot\psi)'' + \frac{6}{b_0}\sin^2(2\psi)((\log\cot\psi)')^2 \ . &(4.4.8)
\end{aligned}$$

This agrees with the supersymmetric Janus solution, (2.3.4) and (2.3.5) in 2.2 when $l(z) = -iz/\sqrt{2}$ and $b_0 = 2$. In this configuration, 8 supersymmetries are preserved.

Chapter 5

Other examples

5.1 $ISO(1,1) \times SO(2) \times SO(4)$

Similar to the case with $ISO(1,1)\times SO(3)\times SO(3)$ symmetry considered in the previous section, we expect to have SUSY preserving configurations with $ISO(1,1) \times SO(2) \times SO(4)$ symmetry, as it can be realized in a D3-D3(-D7) system in the following table:

	0	1	2	3	4	5	6	7	8	9
D3	o	o	o	o						
D3	o	o			o	o				
D7	o	o			o	o	o	o	o	o

(5.1.1)

In this case, we regard the D3-brane extended along $x^{0\sim 3}$ directions as a probe embedded in the supergravity background corresponding to the other D3 and D7-branes. The 4 dimensional gauge theory with varying couplings is realized on the world-volume of the probe D3-brane.

5.1.1 Ansatz for deformation

First, we set the metric and other deformation parameters consistent with the global symmetry. The ansatz for the metric (3.1.6) is the same as that used in Section 4.1:

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta + e^{\Phi(x^i)}\delta_{ij}dx^i dx^j + \delta_{ab}dx^a dx^b + \delta_{pq}dx^p dx^q \ , \tag{5.1.2}$$

where the indices are $\alpha, \beta = 0, 1$; $i, j = 2, 3$; $a, b = 4, 5$ and $p, q = 6, 7, 8, 9$. The non-trivial components of the spin connection are given in (4.1.8). Parameters M, d^{IJA} and m^{AB} consistent with the symmetry are of the form

$$M = 6\left(m_{01i}\Gamma^{01i} + m_{i45}\Gamma^{i45}\right) \equiv \alpha_i \Gamma^i \Gamma^{01} + \beta_i \Gamma^i \Gamma^{45} \ , \tag{5.1.3}$$

$$d^{iab} = -d^{aib} = v^i \epsilon^{ab} \ , \quad m^{ab} = r\delta^{ab}, \quad m^{pq} = \widetilde{r}\delta^{pq} \ , \tag{5.1.4}$$

where ϵ^{ab} is the epsilon tensor satisfying $\epsilon^{45} = -\epsilon^{54} = 1$. Here, α_i, β_i, v^i, r and $\widetilde{r}$ are functions of x^i.

5.1.2 Solutions of the SUSY conditions

First, let us apply the above ansatz to the first order equations (3.2.9)–(3.2.12). It is easy to show that all the components of e^{IJK} vanish. The components of e^{IJK} that are allowed by the symmetry are

$$\begin{aligned} e^{01i} &= a^{-1}\partial_j(ac)\epsilon^{ji} - 4\alpha^i, & (5.1.5)\\ e^{i45} &= 4\beta^i + 2v^i. & (5.1.6) \end{aligned}$$

The equation (3.2.9) for $(I, J) = (6, 7)$ implies

$$0 = \bar{\epsilon}\, e^{I'J'K'}\Gamma_{K'}\Gamma_{I'J'} = 6\,\bar{\epsilon}(e^{01i}\Gamma_{01i} + e^{i45}\Gamma_{i45}) \ . \tag{5.1.7}$$

From this equation, together with (3.2.9) for $(I, J) = (4, 6)$, we obtain

$$\bar{\epsilon}\, e^{01i}\Gamma_i = \bar{\epsilon}\, e^{i45}\Gamma_i = 0 \ . \tag{5.1.8}$$

Because $\Gamma^{\hat{2}\hat{3}}$ does not have a real eigenvalue, we conclude that $e^{01i} = e^{i45} = 0$ to have non-zero $\bar{\epsilon}$. Therefore, we obtain

$$-2\beta^i = v^i \ , \quad a^{-1}\partial_j(ac)\epsilon^{ji} = 4\alpha^i \ . \tag{5.1.9}$$

In this case, (3.2.20) gives

$$0 = \bar{\epsilon}\ \left(\Gamma^i q_i - 4\alpha_i\Gamma^{01i}\right) \ , \tag{5.1.10}$$

where $q_i \equiv \partial_i \log a$. When we decompose $\bar{\epsilon}$ as $\bar{\epsilon} = \bar{\epsilon}_+ + \bar{\epsilon}_-$ with $\bar{\epsilon}_\pm\Gamma^{01} = \pm\bar{\epsilon}_\pm$, (5.1.10) becomes

$$0 = \bar{\epsilon}_+(q_i - 4\alpha_i)\Gamma^i + \bar{\epsilon}_-(q_i + 4\alpha_i)\Gamma^i \ . \tag{5.1.11}$$

This is satisfied if and only if

$$q_i = \pm 4\alpha_i \quad \text{and} \quad \bar{\epsilon}_\mp = 0 \ , \tag{5.1.12}$$

or

$$q_i = \alpha_i = 0 \ . \tag{5.1.13}$$

Note that the second equation of (5.1.9) implies that the complex coupling (3.1.12) satisfies *

$$\partial_z \tau = 4\pi a\, i(4\alpha_z + q_z) \ , \quad \partial_{\bar{z}} \tau = 4\pi a\, i(-4\alpha_{\bar{z}} + q_{\bar{z}}) \ . \tag{5.1.14}$$

From this we find that the complex coupling τ is holomorphic or anti-holomorphic when $q_i = 4\alpha_i$ or $q_i = -4\alpha_i$, respectively. This is the same situation as that observed in section 4.3. For the case with (5.1.13), the gauge coupling and theta parameter are constant.

From (3.2.15), we obtain the components

$$\begin{aligned} \mathcal{A}_0 &= -\frac{1}{4}\left(F\Gamma_0 + 4\alpha_i \Gamma^{1i}\right) \ , \quad \mathcal{A}_1 = -\frac{1}{4}\left(F\Gamma_1 - 4\alpha_i \Gamma^{0i}\right) \ , \\ \mathcal{A}_i &= -\frac{1}{4}\left(F\Gamma_i + 4\alpha_i \Gamma^{01} + 4\beta_i \Gamma^{45} - \epsilon_i{}^j \partial_j \Phi \Gamma^{\hat{2}\hat{3}}\right) \ . \end{aligned} \tag{5.1.15}$$

We find that (4.1.10) is also valid in this case. Then, (3.2.11) implies

$$\bar{\epsilon}\overline{B}^a = -4\,\bar{\epsilon}\,(\alpha_i \Gamma^{i01} - \beta_i \Gamma^{i45})\Gamma^a \ , \quad \bar{\epsilon}\overline{B}^p = -4\,\bar{\epsilon}\,\alpha_i \Gamma^{i01}\Gamma^p \ , \tag{5.1.16}$$

and (3.2.12) with $\mu = i = 2, 3$ can be written as

$$\partial_i \bar{\epsilon} = \bar{\epsilon}\left(-\frac{1}{4}\partial_j \Phi \Gamma^j{}_i + \alpha_j \Gamma^{01}\Gamma^j{}_i - \beta_i \Gamma^{45}\right) \ . \tag{5.1.17}$$

The integrability condition (3.2.18) for (5.1.17) is equivalent to

$$0 = \bar{\epsilon}\left(-\frac{1}{4}g^{ij}\partial_i \partial_j \Phi + g^{ij}\partial_i \alpha_j \Gamma^{01} + \epsilon^{ij}\partial_i \beta_j \Gamma^{\hat{2}\hat{3}45}\right) \ . \tag{5.1.18}$$

In order to solve this equation, we decompose $\bar{\epsilon}$ into the eigenspaces of Γ^{01} and $\Gamma^{\hat{2}\hat{3}45}$ as

$$\bar{\epsilon} = \bar{\epsilon}_+^+ + \bar{\epsilon}_+^- + \bar{\epsilon}_-^+ + \bar{\epsilon}_-^- \ , \tag{5.1.19}$$

where $\bar{\epsilon}_t^s$ ($s = \pm$, $t = \pm$) satisfy

$$\bar{\epsilon}_t^s\,\Gamma^{01} = t\,\bar{\epsilon}_t^s \ , \quad \bar{\epsilon}_t^s\,\Gamma^{\hat{2}\hat{3}45} = s\,\bar{\epsilon}_t^s \ , \quad \bar{\epsilon}_t^s\,\Gamma^{(10)} = \bar{\epsilon}_t^s \ . \tag{5.1.20}$$

(5.1.18) implies that $\bar{\epsilon}_t^s$ can be non-zero only if

$$0 = -\frac{1}{4}g^{ij}\partial_i \partial_j \Phi + t g^{ij}\partial_i \alpha_j + s\epsilon^{ij}\partial_i \beta_j \tag{5.1.21}$$

*The notation for complex coordinates is given in Appendix A.

is satisfied. This equation can be solved when there exists a function φ_t^s satisfying

$$-\frac{1}{4}\partial_j\Phi + t\alpha_j + s\epsilon_j{}^i\beta_i = \epsilon_j{}^i\partial_i\varphi_t^s \ . \tag{5.1.22}$$

Then, the solution of (5.1.17) is given by

$$\bar{\epsilon}_t^s = \bar{\epsilon}_t^{0s} e^{\varphi_t^s \Gamma^{\hat{2}\hat{3}}} \ , \tag{5.1.23}$$

where $\bar{\epsilon}_t^{0s}$ is a constant spinor satisfying the conditions in (5.1.20).

Then we distinguish the following four cases:

- (C1): $\alpha_j \neq 0$ and $\epsilon^{ij}\partial_i\beta_j \neq 0$.
  (5.1.12) and (5.1.21) imply that only one combination of the signs (t, s) can have non-zero $\bar{\epsilon}_t^s$.

- (C2): $\alpha_j \neq 0$ and $\epsilon^{ij}\partial_i\beta_j = 0$.
  (5.1.12) implies that only one sign for t is allowed, but both $\bar{\epsilon}_t^+$ and $\bar{\epsilon}_t^-$ can be non-zero. In this case (5.1.12) and (5.1.21) imply

$$g^{ij}\partial_i\partial_j(\Phi - \log a) = 0 \ , \tag{5.1.24}$$

  and there exist functions φ_1 and φ_2 satisfying

$$-\frac{1}{4}\partial_j(\Phi - \log a) = \epsilon_j{}^i\partial_i\varphi_1 \ , \quad \beta_j = \partial_j\varphi_2 \ . \tag{5.1.25}$$

  Then, $\varphi_t^s \equiv \varphi_1 + s\varphi_2$ satisfies (5.1.22).

- (C3): $\alpha_j = 0$ and $\epsilon^{ij}\partial_i\beta_j \neq 0$.
  (5.1.21) implies that only one sign for s is allowed, but both $\bar{\epsilon}_+^s$ and $\bar{\epsilon}_-^s$ can be non-zero.

- (C4): $\alpha_j = 0$ and $\epsilon^{ij}\partial_i\beta_j = 0$.
  All the sixteen components of $\bar{\epsilon}$ can be non-zero. (5.1.21) implies $g^{ij}\partial_i\partial_j\Phi = 0$.

Next, let us consider the second order equation (3.2.8). In this case, it can be written as

$$D_i(\bar{\epsilon}\overline{B}^a)\Gamma^i = \bar{\epsilon}\left(-2(D_iv^i + q_iv^i)\epsilon^{ab}\Gamma_b - r\Gamma^a - \overline{B}^a\left(M + \frac{1}{2}\Gamma^i q_i\right)\right) \ , \tag{5.1.26}$$

$$D_i(\bar{\epsilon}\overline{B}^p)\Gamma^i = \bar{\epsilon}\left(-\widetilde{r}\Gamma^a - \overline{B}^p\left(M + \frac{1}{2}\Gamma^i q_i\right)\right) \ . \tag{5.1.27}$$

Inserting (5.1.16) into these equations and using (5.1.17), we obtain

$$0 = \bar{\epsilon}\left(r + \frac{1}{2}g^{ij}q_iq_j + g^{ij}\partial_iq_j - 8g^{ij}\beta_i\beta_j + 4\partial_i\beta_j\epsilon^{ij}\Gamma^{\hat{2}\hat{3}45}\right) , \tag{5.1.28}$$

$$\widetilde{r} = -\frac{1}{2}g^{ij}q_iq_j - g^{ij}\partial_iq_j , \tag{5.1.29}$$

where we have used the relation in (5.1.9),

$$0 = \bar{\epsilon}\left(q_i + 4\alpha_i\Gamma^{01}\right) , \tag{5.1.30}$$

that is valid for both cases (5.1.12) and (5.1.13). Then, from (5.1.28) we obtain

$$r = -\frac{1}{2}g^{ij}q_iq_j - g^{ij}\partial_iq_j + 8g^{ij}\beta_i\beta_j - 4s\partial_i\beta_j\epsilon^{ij} , \tag{5.1.31}$$

where we have used (5.1.20) in order to have non-zero $\bar{\epsilon}^s_t$.

In summary, we can construct a generic solution of the SUSY conditions by the following steps. First, pick a holomorphic (or anti-holomorphic) function $\tau(z)$ such that $\mathrm{Im}\,\tau > 0$. Then, a and c are obtained from (3.1.12) and α_i is given by (5.1.12). Next, choose arbitrary real functions Φ and φ^s_t. Then, β_i is determined by (5.1.22). If one wants to find a solution with $\epsilon^{ij}\partial_i\beta_j = 0$, (the case (C2) above), Φ is determined by solving (5.1.24). More explicitly, the solutions of (5.1.24) are obtained by choosing a holomorphic function $l(z)$ and setting

$$\Phi = \log a + l + \bar{l} . \tag{5.1.32}$$

φ_1 in (5.1.25) is given by

$$\varphi_1 = \frac{i}{4}(l - \bar{l}) . \tag{5.1.33}$$

Other parameters v, r and $\widetilde{r}$ are obtained by solving equations (5.1.9), (5.1.31) and (5.1.29), respectively.

The number of unbroken SUSY is 4 for the generic case (C1), 8 for the cases (C2) and (C3), and 16 for (C4). The case (C4) is a trivial solution that is related to the undeformed $\mathcal{N} = 4$ SYM by a coordinate transformation and a field redefinition. In fact, because Φ is harmonic, it can be eliminated by a conformal transformation on the z-plane. β_i is a pure gauge configuration and it can be eliminated by a local $SO(6)$ rotation (see Appendix C.2). One can guess that the cases (C1), (C2) and (C3) correspond to the D3-D3-D7, D3-D7, D3-D3 systems, respectively [2].

5.2 Time dependent solutions

5.2.1 $ISO(3)$

Let us consider the cases in which the couplings depend on time. First, as a trial, let us assume that spatial translational and rotational symmetry $ISO(3)$ are preserved. It turns out that the time dependent solutions of the SUSY conditions with this symmetry can always be mapped to a system with a constant gauge coupling and theta parameter by general coordinate transformations and field redefinitions.*

Note that the metric (3.1.6) can be chosen to be flat. This is because the general form of metric preserving $ISO(3)$ symmetry (flat FLRW metric) can be written as

$$ds^2 = e^{\Phi(\eta)}(-d\eta^2 + \delta_{ij}dx^i dx^j) + \delta_{AB}dx^A dx^B \ , \tag{5.2.1}$$

where $i, j = 1, 2, 3$ and η is the conformal time defined by $d\eta = e^{-\Phi/2}dt$, and the overall factor e^{Φ} in the 4-dimensional metric can be eliminated by the Weyl transformation† ($g_{\mu\nu} \to e^{-\Phi}g_{\mu\nu}$) without loss of generality.

Then, it is particularly easy to show that the gauge coupling cannot depend on time to preserve SUSY. To see this, note that (3.2.10) may be written as

$$\bar{\epsilon}\big(\partial_0 \log a\big) = \bar{\epsilon}\left(\frac{1}{36}e^{IJK}\Gamma_{IJK}\Gamma_0 - \left(\frac{1}{8}e_{\mu JK} - 6m_{\mu JK}\right)\Gamma^{\mu JK}\Gamma_0 - 2M\Gamma_0\right) . \tag{5.2.2}$$

Because $\Gamma_{IJK}\Gamma_0$ is an anti-symmetric matrix for all $I, J, K = 0, \ldots, 9$, the right hand side of this equation is $\bar{\epsilon}$ times a real anti-symmetric matrix. However, since a real anti-symmetric matrix cannot have a real non-zero eigenvalue, the only possibility is that both left and right hand sides are zero:

$$\partial_0 a = 0 \ , \quad \bar{\epsilon}\left(\frac{1}{36}e^{IJK}\Gamma_{IJK} - \left(\frac{1}{8}e_{\mu JK} - 6m_{\mu JK}\right)\Gamma^{\mu JK} - 2M\right) = 0 \ . \tag{5.2.3}$$

Therefore, the gauge coupling has to be time independent.

As explained in Appendix E.2, all the components of e^{IJK} vanish for the cases with $ISO(3)$ symmetry. Because non-trivial components of m^{IJK} are m^{ABC}, m^{0AB} and m^{123}, the second equation of (5.2.3) becomes

$$\bar{\epsilon}(-12\, m^{123}) = \bar{\epsilon}\, m_{ABC}\Gamma^{ABC}\Gamma^{123} \ , \tag{5.2.4}$$

*We have not been able to exclude the possibility that mass parameters have non-trivial time dependence, despite the gauge coupling and theta parameter are constant.

†See Appendix C.1.

which means that m^{123} is proportional to the eigenvalue of $m_{ABC}\Gamma^{ABC}\Gamma^{123}$. However, since $m_{ABC}\Gamma^{ABC}\Gamma^{123}$ is a real anti-symmetric matrix, the eigenvalue cannot take a non-zero real value. The only possibility is

$$m^{123} = 0 \ , \quad \bar{\epsilon}\, m_{ABC}\Gamma^{ABC} = 0 \ . \tag{5.2.5}$$

Then, (3.2.21) implies $\partial_0(ac) = 0$ and, as a consequence, the theta parameter (3.1.11) is also a constant.

Furthermore, (3.2.12) with $\mu = 1, 2, 3$ and (3.2.15) implies $0 = \bar{\epsilon}\mathcal{A}_i \propto \bar{\epsilon}F\Gamma_i$ and we have

$$\partial_0\bar{\epsilon} = \bar{\epsilon}\mathcal{A}_0 = -3\,\bar{\epsilon}\, m_{0AB}\Gamma^{AB} \ . \tag{5.2.6}$$

We can set $m_{0AB} = 0$ by the local $SO(6)$ transformation[‡] and then the SUSY parameter $\bar{\epsilon}$ is also constant.

5.2.2 $ISO(2) \times SO(6)$

To get a solution with time dependent gauge coupling, we consider the cases that the couplings can depend on x^0 and x^1. We impose the $ISO(2)$ symmetry that acts on $x^{2,3}$ directions and $SO(6)_R$ symmetry to simplify the analysis. The ansatz for the metric is

$$ds^2 = e^{\Phi(x^\alpha)}\eta_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ij}dx^i dx^j + \delta_{AB}dx^A dx^B \ , \tag{5.2.7}$$

where $\alpha, \beta = 0, 1$; $i, j = 2, 3$ and $A, B = 4, 5, \cdots, 9$. The non-trivial components of the spin connection are

$$\omega_{0\hat{0}\hat{1}} = -\frac{\partial_1\Phi}{2} \ , \quad \omega_{1\hat{0}\hat{1}} = -\frac{\partial_0\Phi}{2} \ . \tag{5.2.8}$$

Because of the $ISO(2) \times SO(6)$ symmetry, all the components of d^{IJK} are zero and the form of M consistent with the symmetry is

$$M = 6m_{\alpha 23}\Gamma^{\alpha 23} \equiv \alpha_\alpha\Gamma^{\alpha 23} \ . \tag{5.2.9}$$

Then, $e^{\mu AI}$ ($\mu = 0, \cdots, 3$ and $A = 4, \cdots, 9$) are all zero and we can use the argument given in Appendix E.1 to conclude that all the components of e^{IJK} are zero. In addition, (3.2.21) and (3.2.24) imply

$$a^{-1}\partial_0(ac) = -4\alpha_1 \ , \quad a^{-1}\partial_1(ac) = -4\alpha_0 \ , \quad g^{\alpha\beta}\partial_\alpha(a\alpha_\beta) = 0 \ , \tag{5.2.10}$$

[‡]See Appendix C.2.

and (3.2.20) becomes

$$0 = \bar{\epsilon}\left(q_\alpha \Gamma^\alpha - 4\alpha_\alpha \Gamma^{\alpha 23}\right) , \tag{5.2.11}$$

where $q_\alpha = \partial_\alpha \log a$. (5.2.11) is equivalent to

$$0 = \bar{\epsilon}_\pm \left((q_0 \pm q_1) - 4(\alpha_0 \pm \alpha_1)\Gamma^{23}\right) , \tag{5.2.12}$$

where $\bar{\epsilon}$ is decomposed as $\bar{\epsilon} = \bar{\epsilon}_+ + \bar{\epsilon}_-$ with $\bar{\epsilon}_\pm \Gamma^{\hat{0}\hat{1}} = \pm \bar{\epsilon}_\pm$. Since Γ^{23} does not have a real eigenvalue, this equation implies

$$q_0 = \mp q_1 , \quad \alpha_0 = \mp \alpha_1 \quad \text{and} \quad \bar{\epsilon}_\mp = 0 , \tag{5.2.13}$$

or a trivial solution

$$q_\alpha = \alpha_\alpha = 0 . \tag{5.2.14}$$

In the following we focus on the case (5.2.13). (5.2.13) and (5.2.10) imply

$$(\partial_0 \pm \partial_1) a = 0 , \quad (\partial_0 \pm \partial_1)(ac) = 0 , \quad (\partial_0 \pm \partial_1)\alpha_\alpha = 0 , \tag{5.2.15}$$

and the solution can be written as

$$\alpha_0 = \mp \alpha_1 = h'_\mp(x^\mp) , \quad ac = k_\mp(x^\mp) , \quad a = \pm \frac{k'_\mp(x^\mp)}{4h'_\mp(x^\mp)} , \tag{5.2.16}$$

or

$$\alpha_\alpha = 0 , \quad c = 0 \quad \text{and} \quad a = a_\mp(x^\mp) , \tag{5.2.17}$$

where $h_\mp$, $k_\mp$ and $a_\mp$ are arbitrary real functions of $x^\mp \equiv x^0 \mp x^1$ and prime denotes the derivative.[§]

From (3.2.15), $\mathcal{A}_\mu$ is given by

$$\begin{aligned}
\mathcal{A}_0 &= -\frac{1}{4}\left(F\Gamma_0 + 4\alpha_0\Gamma^{23} + \partial_1\Phi\Gamma^{\hat{0}\hat{1}}\right) , && (5.2.18)\\
\mathcal{A}_1 &= -\frac{1}{4}\left(F\Gamma_1 + 4\alpha_1\Gamma^{23} + \partial_0\Phi\Gamma^{\hat{0}\hat{1}}\right) , && (5.2.19)\\
\mathcal{A}_i &= -\frac{1}{4}\left(F\Gamma_i + \epsilon_{ij}4\alpha_\alpha\Gamma^{j\alpha}\right) . && (5.2.20)
\end{aligned}$$

For $\mu = i$, the condition (3.2.12) is $0 = \partial_i \bar{\epsilon} = \bar{\epsilon}\mathcal{A}_i$ and implies

$$\bar{\epsilon}\, F = -4\, \bar{\epsilon}\, \alpha_\alpha \Gamma^{\alpha 23} = 0 , \tag{5.2.21}$$

[§] The solution (5.2.17) was discussed in [16].

where we have used (5.2.13). Then, from (3.2.11) and (3.2.12) we obtain, respectively,

$$\bar{\epsilon}\,\overline{B}^{A} = 0 \ , \tag{5.2.22}$$

and

$$\partial_0\bar{\epsilon} = -\bar{\epsilon}\left(\alpha_0\Gamma^{23} + \frac{1}{4}\partial_1\Phi\Gamma^{\hat{0}\hat{1}}\right) \ , \quad \partial_1\bar{\epsilon} = -\bar{\epsilon}\left(\alpha_1\Gamma^{23} + \frac{1}{4}\partial_0\Phi\Gamma^{\hat{0}\hat{1}}\right) \ . \tag{5.2.23}$$

The integrability condition (3.2.18) for (5.2.23) is

$$0 = \bar{\epsilon}\left((\partial_1\alpha_0 - \partial_0\alpha_1)\Gamma^{23} + \frac{1}{4}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Phi\Gamma^{\hat{0}\hat{1}}\right) \ , \tag{5.2.24}$$

which implies

$$\partial_1\alpha_0 - \partial_0\alpha_1 = 0 \ , \quad g^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = 0 \ . \tag{5.2.25}$$

The solutions (5.2.16) and (5.2.17) satisfy the first equation of (5.2.25) and the general solution of the second equation is

$$\Phi = f_+(x^+) + f_-(x^-) \ , \tag{5.2.26}$$

where $f_\pm$ are arbitrary real functions of $x^\pm$. Then, the equations (5.2.23) can be integrated as

$$\bar{\epsilon}_\pm = \bar{\epsilon}^0_\pm \, e^{\mp\frac{1}{4}(f_+ - f_-) - h_\mp\Gamma^{23}}, \tag{5.2.27}$$

where $\bar{\epsilon}^0_\pm$ is a constant spinor satisfying $\bar{\epsilon}_\pm\Gamma^{\hat{0}\hat{1}} = \pm\bar{\epsilon}_\pm$. Using the above results, the second order equation (3.2.8) becomes simply

$$0 = \bar{\epsilon}\, m^{AB}\Gamma_B \ . \tag{5.2.28}$$

Multiplying this equation by $m_{AC}\Gamma^C$, we find $m_{AB} = 0$.

Chapter 6

D3-branes in curved backgrounds

In this section, we study the case of D3-branes in curved backgrounds and try to relate the couplings in the action (3.1.10) with the supergravity fields in (2.4.22) and (2.4.23). To this end, we expand the D3-brane effective action with respect to λ and keep only the terms that suvive in the $l_s \to 0$ limit, assuming that the background fields are of $\mathcal{O}(\lambda^0)$. We have used chirality condition for Weyl-Majonara fermion,

$$\Gamma^{(10)}\Psi = +\Psi, \tag{6.1}$$

which is different from the previous section, (3.1.7). We assume, for simplicity, that (μ, i) components of the metric $g_{\mu i}$ vanish everywhere, and all the components of the Kalb-Ramond 2-form fields and all the R-R fields, except the R-R 0-form C_0, vanish at $x^i = 0$ $(i = 4, 5, \cdots, 9)^*$:

$$g_{\mu i} = 0\ , \quad B_2\big|_{x^i=0} = 0\ , \quad C_n\big|_{x^i=0} = 0\ , \quad (n \neq 0)\ . \tag{6.2}$$

Similar analyses have been done in the context of flux compactification,† in which Poincare symmetry in 4-dimensional space-time is assume. The effective action of D3-branes in a background corresponding to 7-branes (F-theory) intersecting with them was analyzed in [26], [25], [27]. We generalize such analyses to the cases with relatively mild asuumptions (6.2).

Here, we simply state the results first, and explain the detail calculations in the appendix.

Here, we simply state the results and leave the details to the appendix F. Neglecting the $\mathcal{O}(\lambda)$

---

* It is generically possible to choose a gauge such that $B_2|_{x^i=0} = 0$ and $C_n|_{x^i=0} = 0$ $(n \neq 0)$ (at least locally) provided the components $H_{\mu\nu\rho}$ and $F_{\mu\nu\rho}$ vanish at $x^i = 0$. Obviously, the reason for considering a non-vanishing C_0 is that we want to capture the theta parameter θ in the SYM action (3.1.10).

† See, e.g., [38].

terms in the action (2.4.22) and (2.4.23), we obtain:

$$S_{\rm D3}^{\rm boson} = \frac{T_3\lambda^2}{2}\int d^4x\sqrt{-g}\,{\rm tr}\Big\{ -\frac{e^{-\phi}}{2}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} \pm \frac{C_0}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$- e^{-\phi}g^{\mu\nu}g_{ij}D_\mu\Phi^i D_\nu\Phi^j + \frac{e^{-\phi}}{2}g_{ii'}g_{jj'}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}] - 2V(\Phi^i)$$

$$\pm (G^R_\pm)_i{}^{\mu\nu}\Phi^i F_{\mu\nu} \mp \frac{i}{3}(G^R_\pm)_{ijk}\Phi^i[\Phi^j,\Phi^k] \mp (*_4\widetilde{F}_5)^\mu{}_{ij}\Phi^i D_\mu\Phi^j\Big\}\ , \tag{6.3}$$

$$S_{\rm D3}^{\rm fermi} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-g}\,e^{-\phi}\,{\rm tr}\left\{ i\left(\overline{\Psi}\Gamma^\mu D_\mu\Psi + \overline{\Psi}\Gamma_k i[\Phi^k,\Psi]\right) - i\overline{\Psi}\left(M_\pm - \frac{1}{4}\omega_{\mu\hat{i}\hat{j}}\Gamma^{\mu\hat{i}\hat{j}}\right)\Psi\right\}\ , \tag{6.4}$$

where the upper (lower) signs correspond to the case of D3- ($\overline{\rm D3}$-) branes, D_μ denotes the 4 dimensional covariant derivative defined in (3.1.5) and (3.1.3), and $\omega_{\mu\hat{i}\hat{j}}$ is the $(\mu,\hat{i},\hat{j})$ component of the spin connection given by

$$\omega_{\mu\hat{i}\hat{j}} = \frac{1}{2}(e^k_{\hat{i}}\partial_\mu e_{k\hat{j}} - e^k_{\hat{j}}\partial_\mu e_{k\hat{i}})\ . \tag{6.5}$$

The quantity $M_\pm$ in (6.4) is given by

$$M_\pm \equiv \mp\frac{e^\phi}{8}\left(\frac{1}{3}(*_4F_1)_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - (G^R_\pm)_{i\mu\nu}\Gamma^{i\mu\nu} + \frac{1}{3}(G^R_\pm)_{ijk}\Gamma^{ijk} - (*_4\widetilde{F}_5)_{\mu ij}\Gamma^{\mu ij}\right)\ , \tag{6.6}$$

where the field strengths are given by

$$\begin{aligned} G_3 &\equiv F_3 + (C_0 + ie^{-\phi})H_3 = \widetilde{F}_3 + ie^{-\phi}H_3\ , & (6.7)\\ (G^R_\pm)_i{}^{\mu\nu} &\equiv {\rm Re}((*_4G_3)_i{}^{\mu\nu} \pm iG_i{}^{\mu\nu}) = (*_4\widetilde{F}_3)_i{}^{\mu\nu} \mp e^{-\phi}H_i{}^{\mu\nu}\ , & (6.8)\\ (G^R_\pm)_{ijk} &\equiv {\rm Re}((*_6G_3)_{ijk} \pm iG_{ijk}) = (*_6\widetilde{F}_3)_{ijk} \mp e^{-\phi}H_{ijk}\ , & (6.9)\end{aligned}$$

and

$$(*_4F_1)_{\nu\rho\sigma} \equiv \epsilon^\mu{}_{\nu\rho\sigma}\partial_\mu C_0\ , \quad (*_4\widetilde{F}_5)^\mu{}_{ij} \equiv \frac{1}{3!}\epsilon^{\nu\rho\sigma\mu}\widetilde{F}_{\nu\rho\sigma ij}\ . \tag{6.10}$$

The potential $V(\Phi^i)$ in (6.3) has two contributions:

$$V(\Phi^i) \equiv \lambda^{-2}(V_{\rm DBI}(\Phi^i) \pm V_{\rm CS}(\Phi^i))\ , \tag{6.11}$$

where

$$\begin{aligned} V_{\rm DBI}(\Phi^i) &\equiv e^{-\phi}\left(1 + \lambda v_i^{\rm DBI}\Phi^i + \frac{\lambda^2}{2}m_{ij}^{\rm DBI}\Phi^i\Phi^j\right)\ , & (6.12)\\ V_{\rm CS}(\Phi^i) &\equiv \lambda v_i^{\rm CS}\Phi^i + \frac{\lambda^2}{2}m_{ij}^{\rm CS}\Phi^i\Phi^j\ , & (6.13)\end{aligned}$$

with

$$v_i^{\rm DBI} \equiv \, - \partial_i\phi + \frac{1}{2}g^{\mu\nu}\partial_i g_{\mu\nu} = \partial_i \log\left(\sqrt{-g}\, e^{-\phi}\right) \ , \tag{6.14}$$

$$\begin{aligned} m_{ij}^{\rm DBI} &\equiv v_i^{\rm DBI}v_j^{\rm DBI} - \partial_i\partial_j\phi + \frac{1}{2}\left(g^{\mu\nu}\partial_i\partial_j g_{\nu\mu} - g^{\mu\mu'}\partial_i g_{\mu'\nu'}g^{\nu'\nu}\partial_j g_{\nu\mu} - g^{\mu\mu'}g^{\nu\nu'}H_{i\mu'\nu'}H_{j\nu\mu}\right) \\ &= \frac{1}{\sqrt{-g}\,e^{-\phi}}\partial_i\partial_j\left(\sqrt{-g}\,e^{-\phi}\right) + \frac{1}{2}H_i{}^{\mu\nu}H_{j\mu\nu} \ , \end{aligned} \tag{6.15}$$

$$v_i^{\rm CS} \equiv \, - (*_4\widetilde{F}_5)_i \ , \tag{6.16}$$

$$m_{ij}^{\rm CS} \equiv \, - \frac{1}{2}\left(\partial_i(*_4\widetilde{F}_5)_j + \frac{1}{2}(*_4\widetilde{F}_3)_j{}^{\mu\nu}H_{\mu\nu i} + (i\leftrightarrow j)\right) \ , \tag{6.17}$$

and

$$(*_4\widetilde{F}_5)_i \equiv \frac{1}{4!}\widetilde{F}_{\mu\nu\rho\sigma i}\epsilon^{\mu\nu\rho\sigma} \ . \tag{6.18}$$

All the supergravity fields and their derivatives in the action (6.3) and (6.4) are evaluated at $x^i = 0$. The first term in $V_{\rm DBI}$ (6.12) can be discarded in the field theory analysis, because it doesn't depend on Φ^i. If we require $\Phi^i = 0$ and $A_\mu = 0$ to be a solution of the equations of motion, the linear term in (6.11) has to vanish:

$$0 = e^{-\phi}v_i^{\rm DBI} \pm v_i^{\rm CS} \ , \tag{6.19}$$

which is the condition that the force due to NS-NS and R-R fields cancel each other. In this case, we can safely take the $l_s \to 0$ limit.

Since the metric used in the action (3.1.10) is assumed to be of the form (3.1.6), we introduce a new metric

$$\bar{g}_{\mu\nu} \equiv e^{2\omega}g_{\mu\nu} \ , \quad \bar{g}_{AB} \equiv \delta_{AB} \ , \quad \bar{g}_{\mu A} \equiv 0 \ , \tag{6.20}$$

where $\mu,\nu = 0,\dots,3$; $A,B = 4,\dots,9$ and ω is a real function. Here, we put a facotr $e^{2\omega}$ in the 4-dimensional metric, because it is often convenient to make a Weyl transformation.[‡] In addition, we redefine the scalar fields as

$$A^A \equiv e^{-\omega}e_i^A\Phi^i \ , \tag{6.21}$$

where e_i^A is the vielbein for the transverse space $e_i^{\hat{i}}$ with the identification $A = \hat{i} = 4,\dots,9$. so that the kinetic term can be written as in (3.1.10) with the metric (6.20).

[‡]We have wrote down the useful formula for Weyl transformation in Appendix C.1.

Then, discarding the total derivative terms, the bosonic part of the D3 brane action (6.3) becomes

$$
\begin{aligned}
S_{\rm D3}^{\rm boson} = \frac{T_3\lambda^2}{2}\int d^4x\sqrt{-\bar g}\,{\rm tr}\bigg\{ &-\frac{e^{-\phi}}{2}\bar g^{IJ}\bar g^{KL}F_{IK}F_{JL} \pm \frac{C_0}{4}\bar\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \pm e^{-3\omega}(G_\pm^R)_A^{\ \mu\nu}A^AF_{\mu\nu} \\
&\mp \frac{i}{3}e^{-\omega}(G_\pm^R)_{ABC}A^A[A^B,A^C] + \Big(\mp(*_4\widetilde F_5)_{\mu AB} + 2e^{-\phi}\omega_{\mu AB}\Big)\,\bar g^{\mu\nu}A^AD_\nu A^B \\
&-\widehat m_{AB}A^AA^B - 2e^{-4\omega}V(\Phi^i)\bigg\}\ ,
\end{aligned} \tag{6.22}
$$

where $\bar\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol with $\bar\epsilon^{0123} = 1/\sqrt{-\bar g}$, supergravity fields with indices A, B, C are defined as $(G_\pm^R)_{ABC} \equiv (G_\pm^R)_{ijk}e^i_Ae^j_Be^k_C$, etc., and $\widehat m_{AB}$ is defined as

$$
\widehat m_{AB} \equiv \frac{1}{2}\bigg(e^{-\phi}\bar g^{\mu\nu}E_{\mu A'A}E_{\nu\ \ B}^{\ A'} - \frac{1}{\sqrt{-\bar g}}\partial_\nu\left(\sqrt{-\bar g}\,e^{-\phi}\bar g^{\mu\nu}E_{\mu AB}\right) \pm e^{-2\omega}(*_4\widetilde F_5)^\mu_{\ \ AA'}E_{\mu\ \ B}^{\ A'}\bigg) + (A\leftrightarrow B) \tag{6.23}
$$

with

$$
E_{\mu AB} \equiv e^{-\omega}e_{iA}\partial_\mu(e^\omega e^i_B) = \frac{1}{2}\partial_\mu g^{ij}e_{iA}e_{jB} + \partial_\mu\omega\delta_{AB} + \omega_{\mu AB}\ . \tag{6.24}
$$

The fermionic part (6.4) is rewritten as

$$
S_{\rm D3}^{\rm fermi} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-\bar g}\,e^{-\phi}\,{\rm tr}\left\{i(\overline{\widehat\Psi}\widehat\Gamma^\mu D_\mu\widehat\Psi + \overline{\widehat\Psi}\widehat\Gamma_Ai[A^A,\widehat\Psi]) - i\overline{\widehat\Psi}\left(\widehat{M_\pm} - \frac{1}{4}\omega_{\mu AB}\widehat\Gamma^{\mu AB}\right)\widehat\Psi\right\}\ , \tag{6.25}
$$

where we have defined

$$
\begin{aligned}
\widehat{M_\pm} &\equiv e^{-\omega}M_\pm \\
&= \mp\frac{e^\phi}{8}\left(\frac{e^{2\omega}}{3}(*_4F_1)_{\mu\nu\rho}\widehat\Gamma^{\mu\nu\rho} - e^\omega(G_\pm^R)_{A\mu\nu}\widehat\Gamma^{A\mu\nu} + \frac{e^{-\omega}}{3}(G_\pm^R)_{ABC}\widehat\Gamma^{ABC} - (*_4\widetilde F_5)_{\mu AB}\widehat\Gamma^{\mu AB}\right)\ ,
\end{aligned} \tag{6.26}
$$

and

$$
\widehat\Gamma^\mu \equiv e^{-\omega}\Gamma^\mu\ ,\quad \widehat\Gamma^A \equiv e^A_i\Gamma^i\ ,\quad \widehat\Psi \equiv e^{-\frac{3}{2}\omega}\Psi\ . \tag{6.27}
$$

Here, $\widehat\Gamma^I$ $(I = 0,1,\dots,9)$ are the gamma matrices satisfying

$$
\{\widehat\Gamma^I,\widehat\Gamma^J\} = 2\bar g^{IJ}\ . \tag{6.28}
$$

Now, we can readily find the correspondence between the couplings and the supergravity fields. By comparing the action (3.1.10) with (6.22) and (6.25), assuming (6.19), we obtain

$$a = \frac{T_3\lambda^2}{2}e^{-\phi} \ , \quad c = \pm e^{\phi}C_0 \ , \quad d^{\mu\nu A} = \mp e^{-3\omega+\phi}(G^R_{\pm})^{A\mu\nu} \ , \tag{6.29}$$

$$d_{\mu AB} = \frac{1}{2}\left(\mp e^{\phi}(*_4\widetilde{F}_5)_{\mu AB} + 2\omega_{\mu AB}\right) \ , \quad d^{ABC} = \pm\frac{e^{-\omega+\phi}}{3}(G^R_{\pm})_{ABC} \ , \tag{6.30}$$

$$m_{\mu\nu\rho} = \mp\frac{e^{2\omega+\phi}}{24}(*_4F_1)_{\mu\nu\rho} \ , \quad m_{\mu\nu A} = \pm\frac{e^{\omega+\phi}}{24}(G^R_{\pm})_{A\mu\nu} \ , \tag{6.31}$$

$$m_{\mu AB} = \pm\frac{e^{\phi}}{24}(*_4\widetilde{F}_5)_{\mu AB} - \frac{1}{12}\omega_{\mu AB} \ , \quad m_{ABC} = \mp\frac{e^{-\omega+\phi}}{24}(G^R_{\pm})_{ABC} \ , \tag{6.32}$$

and

$$m_{AB} = 2(\widehat{m}_{AB} + e^{-2\omega}m^{\rm DBI}_{AB} \pm e^{-2\omega+\phi}m^{\rm CS}_{AB}) \ . \tag{6.33}$$

Note that the first equation of (6.31) can be written as

$$m_{\mu\nu\rho} = \mp\frac{e^{2\omega+\phi}}{24}\epsilon^{\mu}{}_{\nu\rho\sigma}\partial_\mu C_0 = \mp\frac{e^{\phi}}{24}\bar{\epsilon}^{\mu}{}_{\nu\rho\sigma}\partial_\mu C_0 \ . \tag{6.34}$$

Then, the relations (6.29)-(6.33) imply (3.2.21) and (3.2.23), which is equivalent to the condition $e^{IJK} = 0$ that solves one of the SUSY condition (3.2.7) as discussed in the previous section.

In the following section we are going to check these identifications by explicitly inserting some particular backgrounds in the effective action for D3-branes and comparing with the supersymmetric deformations of the SYM theory in the formalism of section 3.2.

Chapter 7

Comparison with specific examples

We apply our results to find the non-Abelian action of D3-branes with specific backgrounds, (p, q) 5-branes and D3-branes BPS solutions. The one with $[p, q]$ 7-branes background is studied in [39] .

7.1 (p, q) 5-branes background

In this subsection, we consider D3-branes embedded in a background with (p, q) 5-branes.*. The brane configuration is summarized as

$$\begin{array}{r|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \text{(probe) D3} & \text{o} & \text{o} & \text{o} & \text{o} & & & & & & \\ (p,q)\,5 & \text{o} & \text{o} & \text{o} & & \text{o} & \text{o} & \text{o} & & & \end{array} \tag{7.1.1}$$

The effective action on the D3-brane world-volume can be written down by using (6.3) and (6.4). As we will soon see, because the (p, q) 5-brane is not extended along the x^3-direction, the gauge coupling and the theta parameter of the D3-brane action depend on the coordinate x^3. This brane configuration is related to the supersymmetric Janus configurations considered in [14]. We will show that the action obtained by using (6.3) and (6.4) is indeed consistent with that obtained in section 2.3, which provides a consistency check of our results in section 6.

In this subsection, the letters for the indices are chosen as $\alpha, \beta = 0, 1, 2$; $a, b, c = 4, 5, 6$ and $p, q, r = 7, 8, 9$. Let us consider n (p, q) 5-branes placed at $x^3 = 0$, $x^p = x_0^p$ $(p = 7, 8, 9)$. The supergravity solution corresponding to the (p, q) 5-branes can be obtained by applying $SL(2, \mathbb{Z})$

*Here, p and q are relativly prime integers and a (p, q) 5-brane is a bound state of p NS5-brane and q D5-brane.

duality to a D5-brane solution. Its explicit form is †

$$ds_E^2 = h(r)^{-\frac{1}{4}}(\eta_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ab}dx^a dx^b) + h(r)^{\frac{3}{4}}((dx^3)^2 + \delta_{pq}dx^p dx^q) \;, \tag{7.1.2}$$

$$e^{-\phi} = \frac{\rho}{p^2 g_s^{-1} h(r)^{1/2} + (q+p\chi_0)^2 g_s h(r)^{-1/2}} \;, \quad C_0 = \frac{pq(1-h(r)) + \rho\chi_0 g_s}{p^2 g_s^{-1} h(r) + (q+p\chi_0)^2 g_s} \;, \tag{7.1.3}$$

$$H_3 = 2np\, l_s^2 \epsilon_3 \;, \quad F_3 = 2nq\, g_s l_s^2 \epsilon_3 \;, \tag{7.1.4}$$

where ds_E^2 denotes the line element in the Einstein frame, χ_0 is a constant,

$$h(r) \equiv 1 + \frac{n\sqrt{\rho g_s}\, l_s^2}{r^2} \;, \quad \rho \equiv p^2 g_s^{-1} + (q+p\chi_0)^2 g_s \;, \quad r^2 \equiv (x^3)^2 + \sum_{p=7,8,9}(x^p - x_0^p)^2 \;, \tag{7.1.5}$$

where ϵ_3 is the volume form of the unit S^3 embedded in the $\mathbb{R}^4$ parametrized by $x^{3,7,8,9}$ with its center at the position of the (p,q) 5-brane. ϵ_3 can be written explicitly as

$$\epsilon_3 = \sin^2\theta \sin\phi_1 d\theta \wedge d\phi_1 \wedge d\phi_2 \;, \tag{7.1.6}$$

where (θ, ϕ_1, ϕ_2) are the coordinates on the unit S^3 with $0 \le \theta \le \pi$, $0 \le \phi_1 \le \pi$ and $0 \le \phi_2 \le 2\pi$, related to $x^{3,7,8,9}$ as

$$\begin{aligned} x^3 &= r\cos\theta \;, \\ x^7 - x_0^7 &= r\sin\theta\cos\phi_1 \;, \\ x^8 - x_0^8 &= r\sin\theta\sin\phi_1\cos\phi_2 \;, \\ x^9 - x_0^9 &= r\sin\theta\sin\phi_1\sin\phi_2 \;. \end{aligned} \tag{7.1.7}$$

Note that H_3 and F_3 in (7.1.4) can be written as

$$H_{p'q'r'} = \frac{p}{\sqrt{\rho g_s}}\, \varepsilon_{p'q'r'}{}^{s'}\partial_{s'}h(r) \;, \quad F_{p'q'r'} = q\sqrt{\frac{g_s}{\rho}}\, \varepsilon_{p'q'r'}{}^{s'}\partial_{s'}h(r) \;, \tag{7.1.8}$$

where $p', q', r', s' = 3, 7, 8, 9$ and $\varepsilon_{p'q'r'}{}^{s'} = \varepsilon_{p'q'r't'}\delta^{t's'}$ is the Levi-Civita symbol for the flat $\mathbb{R}^4$ parametrized by $x^{3,7,8,9}$ with $\varepsilon_{3789} = +1$.‡ In the the expressions (7.1.2), (7.1.3) and (7.1.8), the function $h(r)$ can be replaced with an arbitrary positive harmonic function on $\mathbb{R}^4$, which corresponds to a supergravity solution describing parallel multiple (p,q) 5-branes distributed in $\mathbb{R}^4$.

†See, e.g., [42, 43, 44].

‡ One can easily recovor the expressions of H_3 and F_3 in (7.1.4) from (7.1.8) by using the polar coordinates (7.1.7) with metric of $\mathbb{R}^4$: $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \sin^2\theta\sin^2\phi_1 d\phi_2^2)$ and $\varepsilon_{r\theta\phi_1\phi_2} = r^3\sin^2\theta\sin\phi_1$.

The dilaton and R-R 0-from combined into a complex scalar field $\tau \equiv g_s^{-1}(C_0 + ie^{-\phi})$ can be written as

$$\tau = \tau_0 + 4\pi D\, e^{i2\psi} \ , \tag{7.1.9}$$

where

$$\tau_0 \equiv -\frac{q}{p} + 4\pi D \ , \quad 4\pi D \equiv \frac{\rho}{2p(q + p\chi_0)g_s} \ , \tag{7.1.10}$$

are real constants and $\psi(r)$ is a real function satisfying

$$\tan\psi(r) = \frac{p\, h(r)^{1/2}}{(q + p\chi_0)g_s} \ . \tag{7.1.11}$$

Here, we assume p, q, χ_0 are all positive and $0 < \psi < \frac{\pi}{2}$. The complex filed (7.1.9) evaluated at $x^i = 0$ corresponds to the complex coupling (3.1.12). In fact, the expression in (7.1.9) agrees with the complex coupling obtained in section 2.3. Note here that $\psi|_{x^i=0}$ can be chosen to be a generic real function of x^3, because as mentioned above, $h(r)$ in (7.1.11) can be replaced with an arbitrary positive harmonic function on $\mathbb{R}^4$ transverse to the (p, q) 5-branes.

The metric in the string frame is given by

$$\begin{aligned} ds^2_{\text{string}} &= e^{\frac{1}{2}\phi} ds^2_E \\ &= \frac{p}{\sqrt{\rho g_s}\,\sin\psi}\left[(\eta_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ab}dx^a dx^b) + \left(\frac{\rho}{8\pi D p^2}\tan\psi\right)^2 \left((dx^3)^2 + \delta_{pq}dx^p dx^q\right)\right] . \end{aligned} \tag{7.1.12}$$

It is easy to show that both V_{DBI} and V_{CS} in the potential (6.11) are flat (i.e. Φ^i independent), because $F_5 = 0$, $H_{i\mu\nu} = F_{i\mu\nu} = 0$, $B_{\mu\nu} = 0$ and L_{DBI} defined in (F.1.11) is a constant:

$$L_{\text{DBI}} = e^{-\phi}\sqrt{-\det(g_{\mu\nu})} = 1 \ . \tag{7.1.13}$$

Next, consider N probe D3-branes placed at $x^i = 0$ $(i = 4, \ldots, 9)$ in this background. The metric (7.1.12) evaluated at $x^i = 0$ is written as

$$ds^2_{\text{string}}\big|_{x^i=0} = e^{\xi}\left[\eta_{\mu\nu}dy^\mu dy^\nu + \delta_{ab}dx^a dx^b + e^{2\eta}\delta_{pq}dx^p dx^q\right] \ , \tag{7.1.14}$$

where we have defined

$$e^{\xi} \equiv \left.\frac{p}{\sqrt{\rho g_s}\,\sin\psi(r)}\right|_{x^i=0} \ , \quad e^{\eta} \equiv h(r)^{1/2}\big|_{x^i=0} = \left.\frac{\rho}{8\pi D p^2}\tan\psi(r)\right|_{x^i=0} \ , \tag{7.1.15}$$

and introduced new coordinates y^μ $(\mu = 0,1,2,3)$ satisfying

$$y^0 = x^0 \ , \quad y^1 = x^1 \ , \quad y^2 = x^2 \ , \quad dy^3 = e^\eta dx^3 \ . \tag{7.1.16}$$

Then, using the coordinates y^μ, the bosonic part of the effective action (6.3) becomes

$$\begin{aligned} S_{\rm D3}^{\rm boson} =& \frac{T_3\lambda^2}{2}\int d^4y\, e^{-\phi}\, {\rm tr}\Bigg(-\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} \pm \frac{e^\phi C_0}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &+ e^{2\xi}\eta^{\mu\nu}\delta_{ab}D_\mu\Phi^a D_\nu\Phi^b + e^{2(\xi+\eta)}\eta^{\mu\nu}\delta_{pq}D_\mu\Phi^p D_\nu\Phi^q \\ &+ \frac{1}{2}e^{4\xi}\delta_{ab}\delta_{a'b'}[\Phi^a,\Phi^b][\Phi^{a'},\Phi^{b'}] + \frac{1}{2}e^{4(\xi+\eta)}\delta_{pq}\delta_{p'q'}[\Phi^p,\Phi^q][\Phi^{p'},\Phi^{q'}] \\ &+ e^{4\xi+2\eta}\delta_{ab}\delta_{pq}[\Phi^a,\Phi^b][\Phi^p,\Phi^q] \mp \frac{i}{3}e^{2\xi+\phi}(G^R_\pm)_{ijk}\Phi^i[\Phi^j,\Phi^k]\Bigg) \ . \end{aligned} \tag{7.1.17}$$

In order to compare with the action (3.1.10), it is convenient to rescale the scalar fields as

$$A_a \equiv e^\xi\Phi^a \ , \quad A_p \equiv e^{\xi+\eta}\Phi^p \ . \tag{7.1.18}$$

Then, the kinetic term of the scalar fields can be rewritten as

$$\begin{aligned} & e^{-\phi}{\rm tr}\left(e^{2\xi}\eta^{\mu\nu}\delta_{ab}D_\mu\Phi^a D_\nu\Phi^b + e^{2(\xi+\eta)}\eta^{\mu\nu}\delta_{pq}D_\mu\Phi^p D_\nu\Phi^q\right) \\ =\ & e^{-\phi}{\rm tr}\left(\eta^{\mu\nu}\delta^{AB}D_\mu A_A D_\nu A_B + \frac{1}{2}m^{AB}A_A A_B\right) + (\text{total derivative}) \ , \end{aligned} \tag{7.1.19}$$

where $A, B = 4,\ldots,9$ and the non-zero components of m^{AB} are

$$\begin{aligned} m^{ab} &= 2\left(\xi'^2 + \xi'' - \phi'\xi'\right)\delta^{ab} = 2\left(\psi'^2 - (\psi'\cot\psi)'\right)\delta^{ab} \ , \\ m^{pq} &= 2\left((\xi'+\eta')^2 + \xi'' + \eta'' - \phi'(\xi'+\eta')\right)\delta^{pq} = 2\left(\psi'^2 + (\psi'\tan\psi)'\right)\delta^{pq} \ . \end{aligned} \tag{7.1.20}$$

Here, the prime denotes the derivative with respect to y^3, e.g., $\xi' = \partial_{y^3}\xi$. These expressions precisely agree with (2.3.4)

Note that the non-zero components of $(G^R_\pm)_{ijk}$ are

$$(G^R_\pm)_{456} = -e^{-3\eta}(F_{789} + C_0 H_{789}) = 2\sqrt{\frac{g_s}{\rho}}(q + g_s^{-1}C_0 p)\eta' \ , \tag{7.1.21}$$

$$(G^R_\pm)_{789} = \mp e^{-\phi}H_{789} = \pm e^{-\phi}\frac{2p}{\sqrt{\rho g_s}}e^{3\eta}\eta' \ , \tag{7.1.22}$$

and one can show the following relations:

$$e^{\phi-\xi-3\eta}(G^{\rm R}_\pm)_{456} = \mp\frac{2\psi'}{\cos\psi} \ , \quad e^{\phi-\xi}(G^{\rm R}_\pm)_{789} = \frac{2\psi'}{\sin\psi} \ . \tag{7.1.23}$$

Using these, the last term of (7.1.17) can be written as

$$\mp\frac{i}{3}e^{2\xi+\phi}(G^{\rm R}_\pm)_{ijk}\Phi^i[\Phi^j,\Phi^k] = \mp 2i\left(e^{\phi-\xi}(G^{\rm R}_\pm)_{456}A_4[A_5,A_6] + e^{\phi-\xi-3\eta}(G^{\rm R}_\pm)_{789}A_7[A_8,A_9]\right)$$
$$= 4i\left(\mp\frac{\psi'}{\sin\psi}A_4[A_5,A_6] - \frac{\psi'}{\cos\psi}A_7[A_8,A_9]\right) \ . \tag{7.1.24}$$

These terms agree with (2.3.4) for the upper sign. (The lower sign is obtained, e.g., by a transformation $(x^1,x^4)\to(-x^1,-x^4)$.)

In summary, the bosonic part is written as

$$\begin{aligned} S^{\rm boson}_{\rm D3} =& \frac{T_3\lambda^2}{2}\int d^4y\, e^{-\phi}\,{\rm tr}\Bigg(-\frac{1}{2}\eta^{II'}\eta^{JJ'}F_{IJ}F_{I'J'} \pm \frac{e^\phi C_0}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ & + 4i\left(\mp\frac{\psi'}{\sin\psi}A_4[A_5,A_6] - \frac{\psi'}{\cos\psi}A_7[A_8,A_9]\right) \\ & - (\psi'^2 - (\psi'\cot\psi)'))\delta^{ab}A_aA_b - (\psi'^2 + (\psi'\tan\psi)'))\delta^{pq}A_pA_q\Bigg) \end{aligned} \tag{7.1.25}$$

with ϕ and C_0 given by (7.1.9).

Let us next consider to the fermionic part. The action (6.4) in the background (7.1.2)–(7.1.4) is

$$S^{\rm fermi}_{\rm D3} = \frac{T_3\lambda^2}{2}\int d^4y\, e^{2\xi}\, e^{-\phi}\,{\rm tr}\left\{i(\overline{\Psi}\Gamma^\mu D_\mu\Psi + \overline{\Psi}\Gamma_k i[\Phi^k,\Psi]) - i\overline{\Psi}M_\pm\Psi\right\} \tag{7.1.26}$$

with

$$M_\pm \equiv \mp\frac{e^\phi}{4}\left((*_4F_1)_{012}\Gamma^{012} + (G^R_\pm)_{456}\Gamma^{456} + (G^R_\pm)_{789}\Gamma^{789}\right) \ . \tag{7.1.27}$$

Rescaling Ψ, $M_\pm$ and the gamma matrices as

$$\begin{aligned} &\widehat{\Psi}\equiv e^{\frac{3}{4}\xi}\Psi \ , \quad \widehat{M_\pm}\equiv e^{\xi/2}M_\pm \ , \\ &\widehat{\Gamma}^\mu \equiv e^{\xi/2}\Gamma^\mu \ , \quad \widehat{\Gamma}_a\equiv\widehat{\Gamma}^a\equiv e^{\xi/2+\eta}\Gamma^a = e^{-(\xi/2+\eta)}\Gamma_a \ , \quad \widehat{\Gamma}_p\equiv\widehat{\Gamma}^p\equiv e^{\xi/2}\Gamma^p = e^{-\xi/2}\Gamma_p \end{aligned} \tag{7.1.28}$$

we obtain

$$S^{\rm fermi}_{\rm D3} = \frac{T_3\lambda^2}{2}\int d^4y\, e^{-\phi}\,{\rm tr}\left\{i(\overline{\widehat{\Psi}}\widehat{\Gamma}^\mu D_\mu\widehat{\Psi} + \overline{\widehat{\Psi}}\widehat{\Gamma}^A i[A_A,\Psi]) - i\overline{\widehat{\Psi}}\widehat{M_\pm}\widehat{\Psi}\right\} \ . \tag{7.1.29}$$

Here, the rescaled gamma matrices $\widehat{\Gamma}^I$ $(I=0,1,\ldots,9)$ satisfy the anti-commutation relations with the flat metric $\{\widehat{\Gamma}^I,\widehat{\Gamma}^J\}=\eta^{IJ}$, and we have used $\overline{\Psi}\Gamma^\mu(\partial_\mu\xi)\Psi=0$, which follows because $\Gamma^0\Gamma^\mu$ is a symmetric matrix.

Again, using (7.1.23), we obtain

$$\widehat{M}_{\pm} = \mp\frac{1}{4}\left(e^{\phi}\partial_3 C_0 \widehat{\Gamma}^{012} + e^{\phi-\xi}(G^R_{\pm})_{456}\widehat{\Gamma}^{456} + e^{\phi-\xi-3\eta}(G^R_{\pm})_{789}\widehat{\Gamma}^{789}\right)$$
$$= \frac{1}{2}\left(\pm\psi'\,\widehat{\Gamma}^{012} \mp \frac{\psi'}{\sin\psi}\,\widehat{\Gamma}^{456} - \frac{\psi'}{\cos\psi}\,\widehat{\Gamma}^{789}\right)\ . \tag{7.1.30}$$

This result also reproduces the results of supersymmetric Janus configuration, (2.3.4), just by changing $\psi \to \frac{\pi}{2} + \psi$.

7.2 D3-branes background

Let us next consider probe D3-branes extended along $x^{0,1,2,3}$ directions in a background corresponding to n D3-branes extended along $x^{0,1,4,5}$ directions:

	0	1	2	3	4	5	6	7	8	9
(probe) D3	o	o	o	o						
D3	o	o			o	o				

(7.2.1)

In this subsection, we use the letters for the indices as $\alpha, \beta = 0, 1$; $m, n = 2, 3$; $a, b = 4, 5$ and $p, q = 6, 7, 8, 9$.

The supergravity solution corresponding to n D3-branes placed at $x^m = 0$ and $x^p = x^p_0$, in the string frame, is

$$e^{\phi} = 1\ , \quad C_0 = \text{constant}\ , \tag{7.2.2}$$
$$ds^2_{\text{string}} = h(r)^{-\frac{1}{2}}(\eta_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ab}dx^a dx^b) + h(r)^{\frac{1}{2}}(\delta_{mn}dx^m dx^n + \delta_{pq}dx^p dx^q)\ , \tag{7.2.3}$$
$$F_5 = f_5 + *f_5\ , \quad f_5 \equiv dh(r)^{-1}\wedge dx^0 \wedge dx^1 \wedge dx^4 \wedge dx^5, \tag{7.2.4}$$

where g_s is a constant and $h(r)$ is given as

$$h(r) \equiv 1 + \frac{Q_3}{r^4}\ , \quad Q_3 \equiv 4\pi g_s n l_s^4\ , \quad r^2 \equiv \sum_{m=2,3}(x^m)^2 + \sum_{p=6}^{9}(x^p - x^p_0)^2\ . \tag{7.2.5}$$

As in the previous subsection, the function $h(r)$ can be replaced with an arbitrary positive harmonic function on the $\mathbb{R}^6$ parametrized by $x^{2,3,6,7,8,9}$.

The metric evaluated at the position of the probe D3-branes, *i.e.* $x^i = 0$ ($i = 4, \ldots, 9$), is

$$ds^2_{\text{string}}\Big|_{x^i=0} = e^{-\frac{1}{2}\varphi}(\bar{g}_{\mu\nu}dx^\mu dx^\nu + \delta_{ab}dx^a dx^b + e^{\varphi}\delta_{pq}dx^p dx^q)\ , \tag{7.2.6}$$

where we have defined

$$e^{\varphi(x^m)} \equiv h(r)|_{x^i=0} \ , \tag{7.2.7}$$

and

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu \equiv \eta_{\alpha\beta}dx^\alpha dx^\beta + e^\varphi \delta_{mn}dx^m dx^n \ . \tag{7.2.8}$$

This metric has $ISO(1,1) \times SO(2) \times SO(4)$ isometry, where $ISO(1,1)$ is the Poincaré symmetry acting on $x^{0,1}$, $SO(2)$ and $SO(4)$ are rotational symmetry acting on $x^{4,5}$ and $x^{6,7,8,9}$, respectively.* We have already obtained the action of deformed $\mathcal{N}=4$ SYM (3.1.10) with this symmetry in section 6. Since our brane configuration (7.2.1) preserves part of the supersymmetry, the action (6.3) and (6.4) for this background should reproduce one of the general solutions obtained there.

It is again easy to see that both $V_{\rm DBI}$ and $V_{\rm CS}$ in the potential (6.11) are flat, because $F_3 = H_3 = 0$, $F_{\mu\nu\rho\sigma i} = 0$ and $L_{\rm DBI}$ defined in (F.1.11) is a constant. Then, the bosonic part (6.3) becomes

$$\begin{aligned} S^{\rm boson}_{\rm D3} =& \frac{T_3\lambda^2}{2}\int d^4x\sqrt{-\bar{g}}\,{\rm tr}\bigg\{ -\frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} \pm \frac{C_0}{4}\bar{\epsilon}^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ & - \bar{g}^{\mu\nu}(e^{-\varphi}\delta_{ab}D_\mu\Phi^a D_\nu\Phi^b + \delta_{pq}D_\mu\Phi^p D_\nu\Phi^q) \\ & + \frac{1}{2}e^{-2\varphi}\delta_{ab}\delta_{a'b'}[\Phi^a,\Phi^b][\Phi^{a'},\Phi^{b'}] + \frac{1}{2}\delta_{pq}\delta_{p'q'}[\Phi^p,\Phi^q][\Phi^{p'},\Phi^{q'}] \\ & + e^{-\varphi}\delta_{ab}\delta_{pq}[\Phi^a,\Phi^b][\Phi^p,\Phi^q] \mp e^{-\varphi}(*_4\widetilde{F}_5)^\mu{}_{ij}\Phi^i D_\mu\Phi^j \bigg\} \ , \end{aligned} \tag{7.2.9}$$

where $\bar{\epsilon}^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol with $\bar{\epsilon}^{0123} \equiv 1/\sqrt{-\bar{g}}$. In order to compare with the results in section 6, we redefine the scalar fields as

$$A_a \equiv e^{-\varphi/2}\Phi^a \ , \quad A_p \equiv \Phi^p \ . \tag{7.2.10}$$

Then, the kinetic terms for the scalar fields become

$$\begin{aligned} & \sqrt{-\bar{g}}\,{\rm tr}\left(e^{-\varphi}\bar{g}^{\mu\nu}\delta_{ab}D_\mu\Phi^a D_\nu\Phi^b + \bar{g}^{\mu\nu}\delta_{pq}D_\mu\Phi^p D_\nu\Phi^q\right) \\ = \ & \sqrt{-\bar{g}}\,{\rm tr}\left(\bar{g}^{\mu\nu}\delta^{AB}D_\mu A_A D_\nu A_B + \frac{1}{2}m^{AB}A_A A_B\right) + (\text{total derivative}) \ , \end{aligned} \tag{7.2.11}$$

where

$$m^{ab} = \bar{g}^{mn}\left(\frac{1}{2}\partial_m\varphi\partial_n\varphi - \partial_m\partial_n\varphi\right)\delta^{ab} \ , \quad m^{pq} = 0 \ . \tag{7.2.12}$$

* If we use $h(r)$ in (7.2.5), the metric also has a rotational symmetry on the $x^{2,3}$ plane. However, as mentioned above, $h(r)$ can be replace with an arbitrary positive harmonic function on $\mathbb{R}^6$, in which case the rotational symmetry on $x^{2,3}$ is broken in general.

The non-zero components of $(*_4\widetilde{F}_5)^{\mu}{}_{ij}$ are

$$(*_4\widetilde{F}_5)^{n}{}_{45} = -\bar{\epsilon}^{mn}\partial_m\varphi \ , \tag{7.2.13}$$

where $\bar{\epsilon}^{23} = -\bar{\epsilon}^{32} = \sqrt{\bar{g}^{22}\bar{g}^{33}} = e^{-\varphi}$. The last term in (7.2.9) becomes

$$e^{-\varphi}(*_4\widetilde{F}_5)^{\mu}{}_{ij}\Phi^i D_\mu\Phi^j = -\bar{\epsilon}^{mn}\partial_m\varphi\,\varepsilon^{ab}A_a D_n A_b \ , \tag{7.2.14}$$

where $\varepsilon^{45} = -\varepsilon^{54} = 1$. This gives

$$d^{nab} = \pm\frac{1}{2}\bar{\epsilon}^{mn}\partial_m\varphi\,\varepsilon^{ab} \ , \tag{7.2.15}$$

in (3.1.10).

Collecting all these results, (7.2.9) becomes

$$\begin{aligned} S_{\rm D3}^{\rm boson} =& \frac{T_3\lambda^2}{2}\int d^4x\sqrt{-\bar{g}}\,\mathrm{tr}\Bigg\{ -\frac{1}{2}\bar{g}^{II'}\bar{g}^{JJ'}F_{IJ}F_{I'J'} \pm \frac{C_0}{4}\bar{\epsilon}^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &+\frac{1}{2}\bar{g}^{mn}\left(\frac{1}{2}\partial_m\varphi\partial_n\varphi - \partial_m\partial_n\varphi\right)\delta^{ab}A_aA_b \pm \bar{\epsilon}^{mn}\partial_m\varphi\,\varepsilon^{ab}A_aD_nA_b\Bigg\} \ . \end{aligned} \tag{7.2.16}$$

The fermionic part (6.4) is

$$S_{\rm D3}^{\rm fermi} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-\bar{g}}\,e^{-\varphi}\,\mathrm{tr}\left\{i(\overline{\Psi}\Gamma^\mu D_\mu\Psi + \overline{\Psi}\Gamma_k i[\Phi^k,\Psi]) - i\overline{\Psi}M_\pm\Psi\right\} \tag{7.2.17}$$

with

$$M_\pm = \pm\frac{1}{4}(*_4\widetilde{F}_5)^{n}{}_{45}\Gamma^{m45}g_{mn} \ . \tag{7.2.18}$$

As in the previous subsection, we rescale Ψ, $M_\pm$ and the gamma matrices by

$$\begin{aligned} &\widehat{\Psi} \equiv e^{-\frac{3}{8}\varphi}\Psi \ , \quad \widehat{M_\pm} \equiv M_\pm e^{-\frac{1}{4}\varphi} \ , \\ &\widehat{\Gamma}^\mu \equiv e^{-\frac{1}{4}\varphi}\Gamma^\mu \ , \quad \widehat{\Gamma}_a \equiv \widehat{\Gamma}^a \equiv e^{-\frac{1}{4}\varphi}\Gamma^a = e^{\frac{1}{4}\varphi}\Gamma_a \ , \quad \widehat{\Gamma}_p \equiv \widehat{\Gamma}^p \equiv e^{\frac{1}{4}\varphi}\Gamma^p = e^{-\frac{1}{4}\varphi}\Gamma_p \ . \end{aligned} \tag{7.2.19}$$

The rescaled gamma matrices satisfy

$$\{\widehat{\Gamma}^\mu,\widehat{\Gamma}^\nu\} = 2\bar{g}^{\mu\nu} \ , \quad \{\widehat{\Gamma}^a,\widehat{\Gamma}^b\} = 2\delta^{ab} \ , \quad \{\widehat{\Gamma}^p,\widehat{\Gamma}^q\} = 2\delta^{pq} \ . \tag{7.2.20}$$

Then, we obtain

$$S_{\rm D3}^{\rm fermi} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-\bar{g}}\ \mathrm{tr}\left\{i\left(\overline{\widehat{\Psi}}\widehat{\Gamma}^\mu D_\mu\widehat{\Psi} + \overline{\widehat{\Psi}}\widehat{\Gamma}^A i[A_A,\widehat{\Psi}]\right) - i\overline{\widehat{\Psi}}\widehat{M_\pm}\widehat{\Psi}\right\} \tag{7.2.21}$$

with

$$\widehat{M}_{\pm} = \pm\frac{1}{4}(*\widetilde{F}_5)^n{}_{45}\widehat{\Gamma}^{m45}\bar{g}_{nm} = \mp\frac{1}{4}\bar{\epsilon}^n{}_m\partial_n\varphi\widehat{\Gamma}^{m45} \ , \tag{7.2.22}$$

where $\bar{\epsilon}^n{}_m \equiv \bar{\epsilon}^{nn'}\bar{g}_{n'm}$. This gives

$$\beta_m = \mp\frac{1}{4}\bar{\epsilon}^n{}_m\partial_n\varphi \ . \tag{7.2.23}$$

The results (7.2.12), (7.2.15) and (7.2.23) agree (5.1.21), (5.1.9) and (5.1.25 respectively, for the case (C1) with $\varphi_1 = \text{constant}$, $s = \mp$.

Chapter 8

Conclusion and outlook

In summary, we have studied space-time dependent deformation of $\mathcal{N} = 4$ SYM as preserving SUSY and the relation with the world-volume theory of D3-branes in supergravity background. As a result, we got general form of conditions for preserving SUSY while breaking Poincare symmetry to give couplings space-time dependence. Also we have studied the interpretation of this type of SUSY gauge theory in type IIB string theory; the non-Abelian world-volume action of D3-branes in BPS supergravity backgrounds. And we compared the coupling of background fields with D3-branes and the parameters in the field theory.

We have investigated some of examples imposing specific symmetries. For $ISO(1,1)\times SO(3)\times SO(3)$ case, the solution of this case is not the most general one, but we found wide range of class of solutions. And we found the special solutions for $ISO(1,2) \times SO(3) \times SO(3)$ case and $ISO(1,1) \times SO(6)$ case by limiting the parameters to recover these symmetries. This means this solution is generalization of SUSY Janus configuration solution and F-theory solution. We compare the solution of $ISO(1,2) \times SO(3) \times SO(3)$ case and $ISO(1,1) \times SO(6)$ case with the non-Abelian world-volume action of D3-branes in (p, q) 5-branes background and $[p, q]$ 7-branes background respectively. And they agreed with each other.

For $ISO(1,1) \times SO(2) \times SO(4)$ case, we found the general solution. We got four types of solutions corresponding to the D3-branes embedding in trivial, D3, D7 and D3-D7 background in supergravity. Again, we compared this solution and the non-Abelian world-volume action of D3-branes in D3 branes background, and we found they agree with each.

We have also studied the cases that couplings are time dependent, $ISO(3)$ and $ISO(2)\times SO(6)$ cases. For $ISO(3)$ case, there was no non-trivial solutions, and for $ISO(2) \times SO(6)$ case, we got general solution that τ should be left-moving or right-moving one.

It could be a lot of future extension of our work. Since our analysis was completely classical, so it would be important to take into account the quantum effects. Localization, which allows to evaluate physical quantities including all quantum effect exactly, is one of the example. This implies that we can also calculate something corresponding in supergravity side exactly. It would also be interesting to consider how the BPS solitonic objects, such as dyons or instantons and their dual objects in supergravity.

As mentioned in Section 3.1, our ansatz for the action in (3.1.10) is not completely general, even if we restrict the deformation to be of dimension less than 4. As an obvious extension of the analysis, one may try to include all the terms that are compatible with renormalizability. It would also be interesting to consider the case with $U(N)$ gauge group and include the terms like $\mathrm{tr}\, F_{IJ}$, $\mathrm{tr}\, A_A$. The $U(1)$ part will be important to consider the configurations of D3-branes, when the system is embedded in string theory. Since our strategy should work for arbitrary supersymmetric theory, further generalization would also be possible.

The form of non-Abelian world-volume action, (6.3) and (6.4), is not manifest of $SL(2,\mathbb{Z})$ symmetry of type IIB supergravity. So, it will be highly non-trivial to formulate world-volume action of D-branes in $SL(2,\mathbb{Z})$ invariant way. Also, since we made mapping of parameters in the gauge theory to the supergravity fields, we expect the theory, (3.1.10), has S-duality as $SL(2,\mathbb{Z})$ transformation of type IIB supergravity fields. But it also not clear at all.

Even though the map between the parameters in field theory and supergravity fields is established, it's not still clear whether the conditions to preserve SUSY are equivalent or not. In other words, we can study the relation between Killing spinor equations or equations of motion in supergravity and the equations to preserve SUSY presented in section 3.2.

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Appendix A

Complex coordinates

We use complex coordinates to parametrize the 2-3 plane. Our convention used in Sections 4 and 5.1 is as follows:

$$z = \frac{1}{\sqrt{2}}(x^2 + ix^3) \ , \quad \overline{z} = \frac{1}{\sqrt{2}}(x^2 - ix^3) \ , \tag{A.1}$$

$$\partial_z = \frac{1}{\sqrt{2}}(\partial_2 - i\partial_3) \ , \quad \partial_{\overline{z}} = \frac{1}{\sqrt{2}}(\partial_2 + i\partial_3) \ , \tag{A.2}$$

$$g_{z\overline{z}} = g_{\overline{z}z} = e^{\Phi} \ , \quad g^{z\overline{z}} = g^{\overline{z}z} = e^{-\Phi} \ , \tag{A.3}$$

$$\epsilon^{z\overline{z}} = -\epsilon^{\overline{z}z} = -ie^{-\Phi} \ , \quad \epsilon_{z\overline{z}} = -\epsilon_{\overline{z}z} = ie^{\Phi} \ , \quad \epsilon^{z}{}_{z} = \epsilon_{\overline{z}}{}^{\overline{z}} = -i \ , \quad \epsilon^{\overline{z}}{}_{\overline{z}} = \epsilon_{z}{}^{z} = i \ . \tag{A.4}$$

$$\begin{aligned} \partial_z \tau &= 4\pi a\, i(4\alpha_z + q_z) \ , \quad \partial_{\overline{z}}\tau = 4\pi a\, i(-4\alpha_{\overline{z}} + q_{\overline{z}}) \ , \\ \partial_z \overline{\tau} &= 4\pi a\, i(4\alpha_z - q_z) \ , \quad \partial_{\overline{z}}\overline{\tau} = 4\pi a\, i(-4\alpha_{\overline{z}} - q_{\overline{z}}) \ . \end{aligned} \tag{A.5}$$

$$4\alpha_z = -ia^{-1}\partial_z(ac) \ , \quad 4\alpha_{\overline{z}} = ia^{-1}\partial_{\overline{z}}(ac) \ . \tag{A.6}$$

Appendix B

Gamma matrices

In this paper, we have chosen the $SO(1,9)$ gamma matrices Γ^I to be real 32×32 matrices.* They can be written as

$$\Gamma^0 = i\sigma_2 \otimes 1_{16} \ , \quad \Gamma^i = \sigma_1 \otimes \gamma^i_{SO(8)} \ , \quad (i = 1, 2, \cdots, 9) \tag{B.1}$$

where $\gamma^{1\sim 8}_{SO(8)}$ are $SO(8)$ gamma matrices and $\gamma^9_{SO(8)}$ is the chirality operator.† Γ^0 is an anti-symmetric matrix and $\Gamma^{1\sim 9}$ are symmetric matrices. $\Gamma^0\Gamma^{I_1 I_2 \cdots I_n}$ are symmetric for $n = 1, 2$ (mod 4) and anti-symmetric for $n = 0, 3$ (mod 4).

The following formulas are useful.

$$\begin{aligned}
\Gamma^I\Gamma^J &= \Gamma^{IJ} + \eta^{IJ} \ , && \text{(B.2)}\\
\Gamma^{IJ}\Gamma^K &= \Gamma^{IJK} + \eta^{JK}\Gamma^I - \eta^{IK}\Gamma^J \ , && \text{(B.3)}\\
\Gamma^{IJK}\Gamma^L &= \Gamma^{IJKL} + \eta^{KL}\Gamma^{IJ} - \eta^{JL}\Gamma^{IK} + \eta^{IL}\Gamma^{JK} \ , && \text{(B.4)}\\
\Gamma^{IJ}\Gamma_{KL} &= \Gamma^{IJ}{}_{KL} - 4\delta^{[I}_{[K}\Gamma^{J]}{}_{L]} - 2\delta^{[I}_{[K}\delta^{J]}_{L]} \ , && \text{(B.5)}\\
\Gamma^{IJK}\Gamma_{LM} &= \Gamma^{IJK}{}_{LM} + 6\delta^{[I}_{[L}\Gamma^{JK]}{}_{M]} - 6\delta^{[I}_{[L}\delta^J_{M]}\Gamma^{K]} \ , && \text{(B.6)}\\
\Gamma_{LM}\Gamma^{IJK} &= \Gamma_{LM}{}^{IJK} - 6\delta^{[I}_{[L}\Gamma^{JK]}{}_{M]} - 6\delta^{[I}_{[L}\delta^J_{M]}\Gamma^{K]} \ , && \text{(B.7)}\\
\Gamma^{IJK}\Gamma_{LMN} &= \Gamma^{IJK}{}_{LMN} + 9\,\delta^{[I}_{[L}\Gamma^{JK]}{}_{MN]} - 18\,\delta^{[I}_{[L}\delta^J_M\Gamma^{K]}{}_{N]} - 6\,\delta^{[I}_L\delta^J_M\delta^{K]}_N \ . && \text{(B.8)}
\end{aligned}$$

* In this appendix, we assume the metric is flat.

†See, e.g., Appendix 5.B of [36] for an explicit realization.

$$
\begin{aligned}
\Gamma^{I_1 I_2 \cdots I_n}\Gamma_{J_1 J_2 \cdots J_m} &= \Gamma^{I_1 I_2 \cdots I_n}{}_{J_1 J_2 \cdots J_m} + (-1)^{n-1} nm\, \delta^{[I_1}_{[J_1}\Gamma^{I_2 \cdots I_n]}{}_{J_2 \cdots J_m]} \\
&\quad + (-1)^{(n-1)+(n-2)} \frac{n(n-1)m(m-1)}{2}\, \delta^{[I_1}_{[J_1}\delta^{I_2}_{J_2}\Gamma^{I_3 \cdots I_n]}{}_{J_3 \cdots J_m]} + \cdots \\
&= \sum_{k=0}^{\min\{m,n\}} (-1)^{(2n-k-1)k/2} k!\, {}_nC_k\, {}_mC_k\, \delta^{[I_1}_{[J_1} \cdots \delta^{I_k}_{J_k}\Gamma^{I_{k+1} \cdots I_n]}{}_{J_{k+1} \cdots J_m]} \ . \qquad \text{(B.9)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{I'}\Gamma^{I_1 \cdots I_n}\Gamma_{I'} &= (-1)^n (D-2n)\Gamma^{I_1 \cdots I_n} \ , \qquad \text{(B.10)} \\
\Gamma^{I'J'}\Gamma^{I_1 \cdots I_n}\Gamma_{J'I'} &= ((D-2n)^2 - D)\Gamma^{I_1 \cdots I_n} \ . \qquad \text{(B.11)}
\end{aligned}
$$

where D is the number of dimension. These formulas work for any D, though we are mainly interested in the case $D = 10$.

Appendix C

Useful local transformations

C.1 Weyl transformation

Let us consider the transformations

$$\begin{aligned} g_{\mu\nu} &\to e^{-2\omega} g_{\mu\nu} \ , \\ e^{\hat{\mu}}_{\mu} &\to e^{-\omega} e^{\hat{\mu}}_{\mu} \ , \\ \Gamma^{\mu} &\to e^{\omega}\Gamma^{\mu} \ , \\ A_{\mu} &\to A_{\mu} \ , \\ A_{A} &\to e^{\omega} A_{A} \ , \\ \Psi &\to e^{\frac{3}{2}\omega}\Psi \ , \end{aligned} \tag{C.1.1}$$

where $\omega = \omega(x^{\mu})$ is a real function of x^{μ}.

Then the following quantities entering the action transform as:

$$D_{\mu}A^{A} \to e^{\omega}(D_{\mu}A^{A} + \partial_{\mu}\omega\, A^{A}) \ , \tag{C.1.2}$$

$$\begin{aligned} g^{\mu\mu'}\,\mathrm{tr}(D_{\mu}A^{A}D_{\mu'}A_{A}) &\to e^{4\omega}g^{\mu\mu'}\,\mathrm{tr}(D_{\mu}A^{A}D_{\mu'}A_{A} + 2\partial_{\mu}\omega\,(D_{\mu'}A^{A})A_{A} + \partial_{\mu}\omega\partial_{\mu'}\omega\, A^{A}A_{A}) \\ &= e^{4\omega}g^{\mu\mu'}\,\mathrm{tr}(D_{\mu}A^{A}D_{\mu'}A_{A} + \partial_{\mu}\omega\, D_{\mu'}(A^{A}A_{A}) + \partial_{\mu}\omega\partial_{\mu'}\omega\, A^{A}A_{A}) \ . \end{aligned} \tag{C.1.3}$$

According to the definition

$$D_{\mu}\Psi = \partial_{\mu}\Psi + i[A_{\mu},\Psi] + \frac{1}{4}\omega_{\mu\hat{\nu}\hat{\rho}}\Gamma^{\hat{\nu}\hat{\rho}}\Psi \ , \tag{C.1.4}$$

where the spin connection is

$$\omega_{\mu\hat{\nu}\hat{\rho}} = \frac{1}{2}e^{\nu'}_{\hat{\nu}}(\partial_{\mu}e_{\nu'\hat{\rho}} - (\partial_{\nu'}g_{\mu\mu'})e^{\mu'}_{\hat{\rho}}) - (\hat{\nu} \leftrightarrow \hat{\rho}) \ , \tag{C.1.5}$$

these objects transform as follows:

$$\omega_{\mu\hat{\nu}\hat{\rho}} \rightarrow \omega_{\mu\hat{\nu}\hat{\rho}} + (e_{\mu\hat{\rho}}e^{\nu'}_{\hat{\nu}} - e_{\mu\hat{\nu}}e^{\nu'}_{\hat{\rho}})\partial_{\nu'}\omega \ , \tag{C.1.6}$$

$$\begin{aligned} D_\mu\Psi &\rightarrow e^{\frac{3}{2}\omega}\left(D_\mu + \frac{3}{2}\partial_\mu\omega + \frac{1}{4}\partial_{\nu'}\omega(e_{\mu\hat{\rho}}e^{\nu'}_{\hat{\nu}} - e_{\mu\hat{\nu}}e^{\nu'}_{\hat{\rho}})\Gamma^{\hat{\nu}\hat{\rho}}\right)\Psi \\ &= e^{\frac{3}{2}\omega}\left(D_\mu + \frac{1}{2}\partial_{\nu'}\omega\,(\Gamma^{\nu'}{}_{\mu} + 3\,\delta^{\nu'}_{\mu})\right)\Psi \ . \end{aligned} \tag{C.1.7}$$

Therefore

$$\Gamma^\mu D_\mu\Psi \rightarrow e^{\frac{5}{2}\omega}\,\Gamma^\mu D_\mu\Psi \ , \tag{C.1.8}$$

and

$$\overline{\Psi}\Gamma^\mu D_\mu\Psi \rightarrow e^{4\omega}\,\overline{\Psi}\Gamma^\mu D_\mu\Psi \ . \tag{C.1.9}$$

The action (3.1.10) is invariant under the Weyl transformation (C.1.1), if we also transform the couplings as

$$\begin{aligned} d^{\mu\nu A} &\rightarrow e^{3\omega}d^{\mu\nu A} \ , \\ d^{\mu BA} &\rightarrow e^{2\omega}d^{\mu BA} \ , \\ d^{BCA} &\rightarrow e^{\omega}d^{BCA} \ , \\ m^{AB} &\rightarrow e^{2\omega}m^{AB} + 2e^{-2\omega}g^{\mu\mu'}(-\partial_\mu\omega\partial_{\mu'}\omega + a^{-1}D_\mu(a\partial_{\mu'}\omega))\delta^{AB} \ , \\ M &\rightarrow e^{\omega}M \ . \end{aligned} \tag{C.1.10}$$

C.2 $SO(6)_R$ transformation

Let us consider a local $SO(6)_R$ transformation

$$A_A \rightarrow O_A{}^B A_B \ , \tag{C.2.1}$$

$$\Psi \rightarrow \mathcal{O}\Psi \ , \tag{C.2.2}$$

where $O = (O_A{}^B) \in SO(6)$ and $\mathcal{O}$ is the corresponding $SO(6)$ element in the spinor representation of $SO(1,9)$ acting on the fermion. This implies

$$D_\mu A_A \rightarrow O_A{}^B(D_\mu A_C + (O^{-1}\partial_\mu O)_B{}^C A_C) \ , \tag{C.2.3}$$

$$\begin{aligned} g^{\mu\mu'}\,\mathrm{tr}(D_\mu A^A D_{\mu'}A_A) &\rightarrow g^{\mu\mu'}\,\mathrm{tr}(D_\mu A^A D_{\mu'}A_A + 2(O^{-1}\partial_{\mu'}O)^{AC}(D_\mu A_A)A_C \\ &\quad -(O^{-1}\partial_{\mu'}OO^{-1}\partial_{\mu'}O)^{AB}A_A A_B) \ , \end{aligned} \tag{C.2.4}$$

$$D_\mu\Psi \rightarrow \mathcal{O}(D_\mu\Psi + \mathcal{O}^{-1}\partial_\mu\mathcal{O}\Psi) \ . \tag{C.2.5}$$

Then, the action (3.1.10) is invariant if we also transform the couplings as

$$
\begin{aligned}
d^{\mu\nu A} &\to d^{\mu\nu B}(O^{-1})_B{}^A\ , && \text{(C.2.6)}\\
d_{\mu B}{}^A &\to O_B{}^{B'} d_{\mu B'}{}^{A'}(O^{-1})_{A'}{}^A + (O\partial_\mu O^{-1})_B{}^A\ , && \text{(C.2.7)}\\
d^{ABC} &\to d^{A'B'C'}(O^{-1})_{A'}{}^A(O^{-1})_{B'}{}^B(O^{-1})_{C'}{}^C\ , && \text{(C.2.8)}\\
m_A{}^B &\to O_A{}^{A'} m_{A'}{}^{B'}(O^{-1})_{B'}{}^B - 2g^{\mu\nu}(O\partial_\mu O^{-1}O\partial_\nu O^{-1})_A{}^B \\
&\quad -2g^{\mu\nu}(O_A{}^{A'} d_{\mu A'}{}^{B'}\partial_\nu(O^{-1})_{B'}{}^B - \partial_\mu O_A{}^{A'} d_{\nu A'}{}^{B'}(O^{-1})_{B'}{}^B)\ , && \text{(C.2.9)}\\
M &\to \mathcal{O}M\mathcal{O}^{-1} - \Gamma^\mu\mathcal{O}\partial_\mu\mathcal{O}^{-1}\ . && \text{(C.2.10)}
\end{aligned}
$$

Since $d_{\mu A}{}^B$ behaves as the gauge field of the $SO(6)$ symmetry, it is useful to define the covariant derivative including the $SO(6)$ gauge field:

$$
\begin{aligned}
\widehat{D}_\mu A_A &= D_\mu A_A + d_{\mu A}{}^B A_B\ , && \text{(C.2.11)}\\
\widehat{D}_\mu \Psi &= D_\mu\Psi + \frac{1}{4} d_\mu{}^{AB}\Gamma_{AB}\ , && \text{(C.2.12)}
\end{aligned}
$$

which transform as

$$
\begin{aligned}
\widehat{D}_\mu A_A &\to O_A{}^B \widehat{D}_\mu A_B\ , && \text{(C.2.13)}\\
\widehat{D}_\mu \Psi &\to \mathcal{O}\widehat{D}_\mu\Psi\ . && \text{(C.2.14)}
\end{aligned}
$$

The action (3.1.10) can be written as

$$
\begin{aligned}
S = \int d^4x\sqrt{-g_4}\, a\, \mathrm{tr}\Big\{ &-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - (\widehat{D}_\mu A_A)^2 + \frac{1}{2}[A_A, A_B]^2 + i(\overline{\Psi}\Gamma^\mu\widehat{D}_\mu\Psi + \overline{\Psi}\Gamma^A\, i[A_A,\Psi]) \\
&+\frac{c}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - d^{\mu\nu A}F_{\mu\nu}A_A - d^{BCA}i[A_B, A_C]A_A - \frac{\widehat{m}^{AB}}{2}A_A A_B - i\overline{\Psi}\widehat{M}\Psi\Big\}\ , \quad \text{(C.2.15)}
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{m}^{AB} &= m^{AB} + 2g^{\mu\nu} d_\mu{}^{AA'} d_{\nu A'}{}^B\ , && \text{(C.2.16)}\\
\widehat{M} &= M + \frac{1}{4}\Gamma^{\mu AB} d_{\mu AB}\ . && \text{(C.2.17)}
\end{aligned}
$$

$\widehat{m}^{AB}$ and $\widehat{M}$ transform as

$$
\begin{aligned}
\widehat{m}_A{}^B &\to (O\,\widehat{m}\,O^{-1})_A{}^B\ , && \text{(C.2.18)}\\
\widehat{M} &\to \mathcal{O}\widehat{M}\mathcal{O}^{-1}\ . && \text{(C.2.19)}
\end{aligned}
$$

Appendix D

Derivation of the SUSY conditions

D.1 Derivation of (3.2.7) and (3.2.8)

First, the variation of the action (3.1.1) is given by

$$\begin{aligned}\delta S = \int d^4x\sqrt{-g}\,a\,\mathrm{tr}\Bigg\{&\delta A_I\Big[-2g^{II'}g^{JJ'}D_JF_{I'J'} + \left(-2a^{-1}D_\mu(a\,d^{I\mu A}) - m^{IA}\right)A_A \\ &+ \left(-2g^{IJ}g^{\mu K}a^{-1}\partial_\mu a - a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} - 3d^{[IJK]}\right)F_{JK}\Big] \\ &+ 2i\,\delta\overline{\Psi}\left(\Gamma^I D_I\Psi - \widetilde{M}\Psi\right) - \overline{\Psi}\Gamma^I[\delta A_I,\Psi]\Bigg\} + (\text{total derivative terms})\ ,\end{aligned} \tag{D.1.1}$$

with the understanding that d^{JKI} and m^{IA} can be non-zero only if $I = 4\sim 9$, and ϵ^{IJKL} can be non-zero only for $I, J, K, L = 0 \sim 3$. Here, we have used

$$\delta F_{IJ} = D_I\delta A_J - D_J\delta A_I\ , \quad \epsilon^{\mu\nu\rho\sigma}D_\nu F_{\rho\sigma} = 0\ , \quad \overline{\Psi}\Gamma^I\delta\Psi = -\delta\overline{\Psi}\Gamma^I\Psi\ , \tag{D.1.2}$$

and defined

$$d^{[IJK]} \equiv \frac{1}{3}(d^{IJK} + d^{JKI} + d^{KIJ})\ , \tag{D.1.3}$$

$$\widetilde{M} \equiv M - \frac{1}{2}\Gamma^\mu\partial_\mu\log a\ . \tag{D.1.4}$$

The SUSY variation of the action with respect to the transformations (3.1.16) is given by

$$\delta_\epsilon S = \int d^4x\sqrt{-g}\,a\ \mathrm{tr}\Bigg\{(i\bar\epsilon\,\Gamma_I\Psi)\Big[-2g^{II'}g^{JJ'}D_JF_{I'J'} + \left(-2a^{-1}D_\mu(a\,d^{I\mu A}) - m^{IA}\right)A_A$$
$$+\left(-2g^{IJ}g^{\mu K}a^{-1}\partial_\mu a - a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} - 3d^{[IJK]}\right)F_{JK}\Big]$$
$$+\,i\,\bar\epsilon\,(-F_{JK}\Gamma^{JK} + A_A\overline{B}^A)\left(\Gamma^I D_I\Psi - \widetilde{M}\Psi\right)\Bigg\} + (\text{total deriv. terms})\ , \tag{D.1.5}$$

where we have used the identity *

$$\mathrm{tr}(\overline{\Psi}\Gamma^I[\delta_\epsilon A_I,\Psi]) = 0\ . \tag{D.1.6}$$

We would like to find a condition under which the integrand of (D.1.5) is a total derivative. Our ansatz for the total derivative is

$$S_{\text{der}} \ = \ \int d^4x\sqrt{-g}\,\mathrm{tr}\left\{D_I(i\bar\epsilon A^{IJK}\Psi F_{JK}) + D_I(i\bar\epsilon B^{IA}\Psi A_A)\right\}\ , \tag{D.1.7}$$

where A^{IJK} and B^{IA} are 32×32 matrices with $A^{IJK} = -A^{IKJ}$ to be determined. This total derivative term can be expanded as

$$S_{\text{der}} \ = \ \int d^4x\sqrt{-g}\,\mathrm{tr}\Bigg\{i\bar\epsilon A^{IJK}(\Psi(D_IF_{JK}) + (D_I\Psi)F_{JK}) + iD_I(\bar\epsilon B^{IA})\Psi A_A$$
$$+i\left(\bar\epsilon B^{[JK]} + D_I(\bar\epsilon A^{IJK})\right)\Psi F_{JK} + i\bar\epsilon B^{IA}(D_I\Psi)A_A\Bigg\}\ . \tag{D.1.8}$$

Comparing (D.1.5) and (D.1.8), we obtain the following conditions:

$$\bar\epsilon\Gamma_I(-g^{II'}g^{JJ'} + g^{IJ'}g^{JI'})a = \bar\epsilon(A^{JI'J'} + E^{JI'J'})\ , \tag{D.1.9}$$

$$a\bar\epsilon(-\Gamma^{JK}\Gamma^I) = \bar\epsilon A^{IJK}\ , \tag{D.1.10}$$

$$\bar\epsilon\Gamma_I\left((-g^{IJ}g^{\mu K} + g^{IK}g^{\mu J})\partial_\mu a + \partial_\nu(ac)\,\epsilon^{I\nu JK} - 3a\,d^{[JKI]}\right) + a\,\bar\epsilon\Gamma^{JK}\widetilde{M} = \bar\epsilon B^{[JK]} + D_I(\bar\epsilon A^{IJK})\ , \tag{D.1.11}$$

$$a\bar\epsilon\overline{B}^A\Gamma^I = \bar\epsilon B^{IA}\ , \tag{D.1.12}$$

*See appendix 4.A of [36] for a proof of this identity.

$$\bar{\epsilon}\Gamma_I \left(-2D_J(a\, d^{IJA}) - a\, m^{IA}\right) - a\bar{\epsilon}\overline{B}^A\widetilde{M} = D_I(\bar{\epsilon}B^{IA}) \ . \tag{D.1.13}$$

In (D.1.9), E^{IJK} is a 32×32 matrix which is totally anti-symmetric with respect to I, J, K, which can be added because of the Bianchi identity:

$$D_I F_{JK} + D_J F_{KI} + D_K F_{IJ} = 0 \ . \tag{D.1.14}$$

(D.1.10) and (D.1.12) determine $\bar{\epsilon}A^{IJK}$ and $\bar{\epsilon}B^{IA}$, respectively. It is easy to see that (D.1.9) is satisfied with $E^{IJK} = a\Gamma^{IJK}$, using the identity

$$\Gamma^{JK}\Gamma^I = \Gamma^{IJK} + g^{IK}\Gamma^J - g^{IJ}\Gamma^K \ . \tag{D.1.15}$$

Then, using (D.1.10) and (D.1.12), (D.1.11) becomes

$$\begin{aligned} &\bar{\epsilon}\left(\left(-\Gamma^J g^{\mu K} + \Gamma^K g^{\mu J}\right)\partial_\mu a + \partial_\nu(ac)\,\Gamma_I\epsilon^{I\nu JK} - 3a\,\Gamma_I d^{[JKI]}\right) + a\,\bar{\epsilon}\Gamma^{JK}\widetilde{M} \\ = \ & a\bar{\epsilon}\overline{B}^{[K}\Gamma^{J]} - D_I(a\bar{\epsilon}(\Gamma^{JK}\Gamma^I)) \ , \end{aligned} \tag{D.1.16}$$

which is equivalent to

$$(D_K\bar{\epsilon})\Gamma^{IJ}\Gamma^K = \bar{\epsilon}\left(\overline{B}^{[J}\Gamma^{I]} - \Gamma^{IJ\mu}a^{-1}\partial_\mu a + (a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} + 3\, d^{[IJK]})\Gamma_K - \Gamma^{IJ}\widetilde{M}\right) . \tag{D.1.17}$$

This is the condition (3.2.7). Similarly, (D.1.13) becomes

$$\bar{\epsilon}\Gamma_I \left(-2D_J(a\, d^{IJA}) - a\, m^{IA}\right) - a\bar{\epsilon}\overline{B}^A\widetilde{M} = D_I(a\bar{\epsilon}\overline{B}^A)\Gamma^I \ , \tag{D.1.18}$$

which is equivalent to

$$a^{-1}D_I(a\bar{\epsilon}\overline{B}^A)\Gamma^I = \bar{\epsilon}\left(-2a^{-1}\Gamma_I D_J(a\, d^{IJA}) - m^{AB}\Gamma_B - \overline{B}^A\widetilde{M}\right) \ . \tag{D.1.19}$$

This gives (3.2.8).

D.2 Derivation of (3.2.9)∼(3.2.12)

Using the identity

$$\Gamma^{IJ}\Gamma^K\left(\frac{1}{16}\Gamma_{[I}g_{J]L} - \frac{7}{16\times 54}\Gamma_{IJ}\Gamma_L\right) = \delta^K_L \ , \tag{D.2.1}$$

we rewrite (3.2.7) as

$$D_L\bar{\epsilon} = \bar{\epsilon}\, C^{IJ}\left(\frac{1}{16}\Gamma_{[I}g_{J]L} - \frac{7}{16\times 54}\Gamma_{IJ}\Gamma_L\right) \ , \tag{D.2.2}$$

where

$$C^{IJ} \equiv \overline{B}^{[J}\Gamma^{I]} - \Gamma^{IJ\mu}a^{-1}\partial_\mu a + (a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} + 3\,d^{[IJK]})\Gamma_K - \Gamma^{IJ}\widetilde{M}\ . \tag{D.2.3}$$

Inserting (D.2.2) back to (3.2.7), we obtain

$$\bar\epsilon\, C^{IJ} = \bar\epsilon\, C^{I'J'}P_{I'J'}{}^{IJ}\ , \tag{D.2.4}$$

where we have defined

$$\begin{aligned} P_{I'J'}{}^{IJ} &\equiv \left(\frac{1}{16}\Gamma_{[I'}g_{J']K} - \frac{7}{16\times 54}\Gamma_{I'J'}\Gamma_K\right)\Gamma^{IJ}\Gamma^K \\ &= \frac{1}{72}\Gamma_{I'J'}\Gamma^{IJ} + \frac{1}{4}\Gamma_{[I'}\Gamma^{[I}\delta^{J]}_{J']}\ . \end{aligned} \tag{D.2.5}$$

One can check that $P_{I'J'}{}^{IJ}$ is a projection operator satisfying

$$P_{IJ}{}^{I'J'}P_{I'J'}{}^{KL} = P_{IJ}{}^{KL}\ , \tag{D.2.6}$$

and

$$G^{[I'}\Gamma^{J']}P_{I'J'}{}^{IJ} = G^{[I}\Gamma^{J]}\ , \quad P_{I'J'}{}^{IJ}G_{[I}\Gamma_{J]} = G_{[I'}\Gamma_{J']}\ , \tag{D.2.7}$$

with arbitrary G^J. Therefore, if $\bar\epsilon\, C^{IJ}$ can be written as

$$\bar\epsilon\, C^{IJ} = \bar\epsilon\, G^{[I}\Gamma^{J]}\ , \tag{D.2.8}$$

with some G^I, (D.2.4) is satisfied. Conversely, if (D.2.4) is satisfied,

$$G^I \equiv \frac{1}{72}(C^{I'J'}\Gamma_{I'J'})\Gamma^I + \frac{1}{4}C^{II'}\Gamma_{I'} \tag{D.2.9}$$

satisfies (D.2.8). It can also be shown, using (D.2.7), that $v^{I'J'}P_{I'J'}{}^{IJ} = 0$ is equivalent to $v^{[IJ]}\Gamma_J = 0$.

Then, using the property (D.2.7), one can easily show

$$C^{I'J'}P_{I'J'}{}^{IJ} - C^{IJ} = \left((a^{-1}\partial_\nu(ac)\,\epsilon^{\nu I'J'K'} + 3\,d^{[I'J'K']})\Gamma_{K'} - \Gamma^{I'J'}\widetilde{M}\right)\left(P_{I'J'}{}^{IJ} - \delta^I_{I'}\delta^J_{J'}\right)\ . \tag{D.2.10}$$

Therefore, (D.2.4) can be written as

$$0 = \bar\epsilon\left((a^{-1}\partial_\nu(ac)\,\epsilon^{\nu I'J'K'} + 3\,d^{[I'J'K']})\Gamma_{K'} - \Gamma^{I'J'}\widetilde{M}\right)\left(P_{I'J'}{}^{IJ} - \delta^I_{I'}\delta^J_{J'}\right)\ . \tag{D.2.11}$$

In this equation, $\widetilde{M}$ can be replaced with M, because of the relation (D.2.7).

When M is expanded as

$$M = m_{IJK}\Gamma^{IJK} \ , \tag{D.2.12}$$

it satisfies

$$\Gamma^{I'J'}M = M\Gamma^{I'J'} - 6m_{IJK}(g^{I'K}\Gamma^{IJ}\Gamma^{J'} - g^{J'K}\Gamma^{IJ}\Gamma^{I'} + 4g^{I'I}g^{JJ'}\Gamma^{K}) \ . \tag{D.2.13}$$

Therefore, (D.2.11) is equivalent to

$$0 = \overline{\epsilon}\, e^{I'J'K'}\Gamma_{K'}\left(P_{I'J'}{}^{IJ} - \delta^{I}_{I'}\delta^{J}_{J'}\right) \ , \tag{D.2.14}$$

where

$$e^{IJK} \equiv a^{-1}\partial_\nu(ac)\,\epsilon^{\nu IJK} + 3\,d^{[IJK]} + 24\,m^{IJK} \ . \tag{D.2.15}$$

This is (3.2.9).

Next, we write (D.2.2) as

$$D_J\overline{\epsilon} = \overline{\epsilon}\,E_J \ , \tag{D.2.16}$$

where

$$\begin{aligned} E_J \ \equiv \ & \overline{B}^A\left(\frac{1}{4}\,g_{JA} - \frac{1}{24}\Gamma_A\Gamma_J\right) - \frac{1}{12}\Gamma^\mu\Gamma_J\partial_\mu \log a \\ & + e^{I'J'K'}\left(\frac{1}{16}g_{JJ'}\Gamma_{K'I'} - \frac{7}{16\times 54}\Gamma_{I'J'K'}\Gamma_J\right) - m^{I'J'K'}\left(3g_{JJ'}\Gamma_{K'I'} - \frac{1}{3}\Gamma_{I'J'K'}\Gamma_J\right) \ . \end{aligned} \tag{D.2.17}$$

(D.2.16) with $J = A$, implies

$$\begin{aligned} 0 \ = \ & \overline{\epsilon}\Bigg(\overline{B}^{A'}\left(\frac{1}{4}\,\delta^A_{A'} - \frac{1}{24}\Gamma_{A'}\Gamma^A\right) - \frac{1}{12}\Gamma^{\mu A}\partial_\mu \log a \\ & + e^{I'J'K'}\left(\frac{1}{16}\delta^A_{J'}\Gamma_{K'I'} - \frac{7}{16\times 54}\Gamma_{I'J'K'}\Gamma^A\right) - m^{I'J'K'}\left(3\delta^A_{J'}\Gamma_{K'I'} - \frac{1}{3}\Gamma_{I'J'K'}\Gamma^A\right)\Bigg) \ . \end{aligned} \tag{D.2.18}$$

This equation can be written as

$$\overline{\epsilon}\left(\overline{B}^A + \left(\frac{1}{4}e^{AK'I'} - 12m^{AK'I'}\right)\Gamma_{K'I'}\right) = \overline{\epsilon}\,F\Gamma^A \ , \tag{D.2.19}$$

where

$$F \equiv \frac{1}{6}\overline{B}^{A'}\Gamma_{A'} + \frac{7}{4\times 54}e^{I'J'K'}\Gamma_{I'J'K'} - \frac{4}{3}m^{I'J'K'}\Gamma_{I'J'K'} + \frac{1}{3}\Gamma^{\mu}\partial_{\mu}\log a \ . \tag{D.2.20}$$

This is (3.2.11). Multiplying (D.2.18) by Γ_A, we obtain

$$0 = \overline{\epsilon}\left(\frac{1}{72}e^{IJK}\Gamma_{IJK} - \frac{1}{2}\Gamma^{\mu}\partial_{\mu}\log a - \left(\frac{1}{16}e_{\mu JK} - 3m_{\mu JK}\right)\Gamma^{\mu JK} - M\right) , \tag{D.2.21}$$

which is (3.2.10).

With (D.2.21), the explicit form of F in (D.2.20) is not needed anymore, because (D.2.21) implies that (D.2.20) can be replaced with

$$F = \frac{1}{6}\left(\overline{B}^{A} + \left(\frac{1}{4}e^{AK'I'} - 12m^{AK'I'}\right)\Gamma_{K'I'}\right)\Gamma_A \ , \tag{D.2.22}$$

which can be obtained by contracting (D.2.19) with Γ_A.

(D.2.16) with $J = \mu$ is written as

$$D_{\mu}\overline{\epsilon} = \overline{\epsilon}\left(-\frac{1}{4}F\Gamma_{\mu} + \left(\frac{1}{16}e_{\mu IJ} - 3m_{\mu IJ}\right)\Gamma^{IJ}\right) \ . \tag{D.2.23}$$

This equation is equivalent to (3.2.12).

Appendix E

On e^{IJK}

E.1 $e^{I\alpha a} = 0 \Rightarrow e^{IJK} = 0$

In this section, we split the 10-dimensional gamma matrices $\{\Gamma^I\}$ into two sectors $\{\Gamma^\alpha\}$ and $\{\Gamma^a\}$, where $I, J, K = 0, 1, \cdots, 9$; $\alpha, \beta, \gamma = 0, 1, \cdots, d-1$ and $a, b, c = d, d+1, \cdots, 9$, with $0 < d < 10$. For simplicity, the 10 dimensional metric is assumed to be the flat Minkowski metric, though the generalization to a curved metric is straightforward.

Here, we prove the following statement: Suppose e^{IJK} with mixed indices such as $e^{\alpha\beta a}$ and $e^{\alpha ab}$ are all zero. Then, the condition (3.2.9) implies that all the components of e^{IJK} vanish when $d \neq 3, 7$. It also holds for the case with $d = 7$, which will be shown separately in Appendix E.2.

The condition (3.2.9) is equivalent to

$$0 = \bar{\epsilon}\left(e^{I'J'K'}\Gamma_{I'J'K'}\Gamma^{IJ} + 9e^{K'I'J}\Gamma_{K'I'}\Gamma^{I} - 9e^{K'I'I}\Gamma_{K'I'}\Gamma^{J} - 72e^{IJK'}\Gamma_{K'}\right) . \tag{E.1.1}$$

When $e^{\alpha\beta a} = 0$ and $e^{\alpha ab} = 0$, this condition for $(I, J) = (\alpha, \beta)$, $(I, J) = (a, b)$ and $(I, J) = (\alpha, a)$ becomes, respectively,

$$\begin{aligned}
0 &= \bar{\epsilon}\left(e^{I'J'K'}\Gamma_{I'J'K'}\Gamma^{\alpha\beta} + 9e^{\gamma\delta\beta}\Gamma_{\gamma\delta}\Gamma^{\alpha} - 9e^{\beta'\gamma'\alpha}\Gamma_{\beta'\gamma'}\Gamma^{\beta} - 72e^{\alpha\beta\gamma'}\Gamma_{\gamma'}\right) , &\text{(E.1.2)}\\
0 &= \bar{\epsilon}\left(e^{I'J'K'}\Gamma_{I'J'K'}\Gamma^{ab} + 9e^{b'c'b}\Gamma_{b'c'}\Gamma^{a} - 9e^{b'c'a}\Gamma_{b'c'}\Gamma^{b} - 72e^{abc'}\Gamma_{c'}\right) , &\text{(E.1.3)}\\
0 &= \bar{\epsilon}\left(e^{I'J'K'}\Gamma_{I'J'K'}\Gamma^{\alpha a} + 9e^{b'c'a}\Gamma_{b'c'}\Gamma^{\alpha} - 9e^{\beta'\gamma'\alpha}\Gamma_{\beta'\gamma'}\Gamma^{a}\right) , &\text{(E.1.4)}
\end{aligned}$$

respectively. Multiplying (E.1.2) by $\Gamma_{\alpha\beta}$ we obtain

$$0 = \bar{\epsilon}\left(d(d-1)e^{abc}\Gamma_{abc} + (9-d)(10-d)e^{\alpha\beta\gamma}\Gamma_{\alpha\beta\gamma}\right) , \tag{E.1.5}$$

and we have

$$\bar\epsilon\, e^{IJK}\Gamma_{IJK} = \frac{18(d-5)}{d(d-1)}\,\bar\epsilon\, e^{\alpha\beta\gamma}\Gamma_{\alpha\beta\gamma} = \frac{18(5-d)}{(9-d)(10-d)}\,\bar\epsilon\, e^{abc}\Gamma_{abc}\ . \tag{E.1.6}$$

Inserting this equation into (E.1.2) and (E.1.3), we obtain

$$0 = \bar\epsilon\left(\frac{2(5-d)}{(9-d)(10-d)} e^{a'b'c'}\Gamma_{a'b'c'}\Gamma^{ab} + e^{b'c'b}\Gamma_{b'c'}\Gamma^{a} - e^{b'c'a}\Gamma_{b'c'}\Gamma^{b} - 8e^{abc'}\Gamma_{c'}\right)\ . \tag{E.1.7}$$

Multiplying (E.1.7) by Γ_b, we obtain

$$0 = \bar\epsilon\left(e^{a'b'c'}\Gamma_{a'b'c'}\Gamma^{a} - (10-d)e^{ab'c'}\Gamma_{b'c'}\right)\ , \tag{E.1.8}$$

which implies

$$0 = \bar\epsilon\left(2e^{a'b'c'}\Gamma_{a'b'c'}\Gamma^{ab} + (10-d)e^{bb'c'}\Gamma_{b'c'}\Gamma^{a} - (10-d)e^{ab'c'}\Gamma_{b'c'}\Gamma^{b}\right)\ . \tag{E.1.9}$$

Inserting this back to (E.1.7), we get

$$\begin{aligned}
0 &= \bar\epsilon\left(e^{bb'c'}\Gamma_{b'c'}\Gamma^{a} - e^{ab'c'}\Gamma_{b'c'}\Gamma^{b} - 2(9-d)e^{abc'}\Gamma_{c'}\right)\ , && \text{(E.1.10)}\\
0 &= \bar\epsilon\left(e^{a'b'c'}\Gamma_{a'b'c'}\Gamma^{ab} + (9-d)(10-d)e^{abc'}\Gamma_{c'}\right)\ . && \text{(E.1.11)}
\end{aligned}$$

From these equations, it is easy to show

$$\begin{aligned}
0 &= \bar\epsilon\left(e^{abc}e_{ab'c'}\Gamma_{bc}\Gamma^{b'c'} + (9-d)e^{abc}e_{abc}\right)\ , && \text{(E.1.12)}\\
0 &= \bar\epsilon\left((e^{a'b'c'}\Gamma_{a'b'c'})^2 + (9-d)(10-d)e^{abc}e_{abc}\right)\ . && \text{(E.1.13)}
\end{aligned}$$

Using the identities

$$\begin{aligned}
\{\Gamma^{ab},\Gamma_{a'b'}\} &= 2\left(\Gamma^{ab}{}_{a'b'} - 2\delta^{[a}_{a'}\delta^{b]}_{b'}\right)\ , && \text{(E.1.14)}\\
\{\Gamma^{abc},\Gamma_{a'b'c'}\} &= 2\left(9\,\Gamma^{[ab}{}_{[a'b'}\delta^{c]}_{c']} - 6\,\delta^{[a}_{a'}\delta^{b}_{b'}\delta^{c]}_{c'}\right)\ , && \text{(E.1.15)}
\end{aligned}$$

(E.1.12) and (E.1.13) can be written as

$$\begin{aligned}
0 &= \bar\epsilon\left(e^{abc}e_{ab'c'}\Gamma_{bc}{}^{b'c'} + (7-d)e^{abc}e_{abc}\right)\ , && \text{(E.1.16)}\\
0 &= \bar\epsilon\left(9\,e^{abc}e_{ab'c'}\Gamma_{bc}{}^{b'c'} + (12-d)(7-d)e^{abc}e_{abc}\right)\ . && \text{(E.1.17)}
\end{aligned}$$

These equations imply

$$0 = (3-d)(7-d)\,\bar\epsilon\, e^{abc}e_{abc}\ , \tag{E.1.18}$$

and hence we have $e^{abc} = 0$ for $d \neq 3, 7$.

Then, (E.1.2)–(E.1.4) with $e^{abc} = 0$ imply

$$\bar{\epsilon}\, e^{\alpha\beta\gamma}\Gamma_{\alpha\beta\gamma} = 0 \ , \quad \bar{\epsilon}\, e^{\alpha\beta\gamma}\Gamma_{\beta\gamma} = 0 \ , \quad \bar{\epsilon}\, e^{\alpha\beta\gamma}\Gamma_{\gamma} = 0 \ . \tag{E.1.19}$$

Multiplying the last equation by $e^{\alpha\beta\gamma'}\Gamma_{\gamma'}$ without summing over α and β, we obtain

$$-(e^{\alpha\beta 0})^2 + \sum_{\gamma=1}^{d-1}(e^{\alpha\beta\gamma})^2 = 0 \tag{E.1.20}$$

for all $\alpha, \beta = 0, 1, \ldots, d-1$ to have non-zero $\bar{\epsilon}$. Choosing $\alpha = 0$, this equation implies $e^{0\beta\gamma} = 0$ and then again using this equation, we see that all the components $e^{\alpha\beta\gamma}$ must vanish.

E.2 Proof of $e^{IJK} = 0$ for $ISO(3)$ symmetric case

Here, we show that $e^{IJK} = 0$ is the only solution of (3.2.9) with $\bar{\epsilon} \neq 0$ for the cases with $ISO(3)$ symmetry considered in Section 5.2.1. Because of the $ISO(3)$ symmetry, the allowed components of e^{IJK} are e^{ABC}, e^{0AB} and e^{123}, where $A, B, C = 4, \ldots, 9$. This is the case with $d = 7$ considered in appendix E.1 by assigning the indices as $\alpha, \beta, \gamma = 0, 4, \ldots, 9$ and $a, b, c = 1, 2, 3$

The condition (3.2.9) with $(I, J) = (1, 2)$ implies

$$0 = \bar{\epsilon}\left(\frac{1}{72}e^{I'J'K'}\Gamma_{I'J'K'} + \frac{1}{2}e^{123}\Gamma_{123}\right) \ . \tag{E.2.1}$$

Using this equation, the condition (3.2.9) with $(I, J) = (0, 1)$ implies

$$0 = \bar{\epsilon}\left(\frac{3}{4}e^{123}\Gamma_{123} + \frac{1}{8}e^{0AB}\Gamma_{0AB}\right) \ . \tag{E.2.2}$$

Multiplying the condition (3.2.9) with $(I, J) = (1, C)$ by Γ_{1C} and summing over $C = 4, \ldots, 9$, we obtain

$$0 = \bar{\epsilon}\left(3e^{123}\Gamma_{123} + \frac{1}{8}e^{ABC}\Gamma_{ABC}\right) \ . \tag{E.2.3}$$

Because $\Gamma_{ABC}\Gamma_{123}$ is an anti-symmetric matrix, this equation implies that e^{123} is an eigenvalue of a real anti-symmetric matrix. This is possible only when

$$e^{123} = 0 \ , \quad \bar{\epsilon}\, e^{ABC}\Gamma_{ABC} = 0 \ . \tag{E.2.4}$$

These equations, together with (E.2.1) and (E.2.2), imply

$$\bar{\epsilon}\, e^{IJK}\Gamma_{IJK} = 0 \ , \quad \bar{\epsilon}\, e^{0AB}\Gamma_{AB} = 0 \ . \tag{E.2.5}$$

Using these results, the condition (3.2.9) with $(I,J) = (0,C)$ becomes

$$0 = \bar{\epsilon}\left(-\frac{3}{4} e^{0BC}\Gamma_{0B} + \frac{1}{8} e^{ABC}\Gamma_{AB} \right) \ . \tag{E.2.6}$$

Multiplying the condition (3.2.9) with $(I,J) = (B,C)$ by Γ_B and summing over B, we obtain

$$\bar{\epsilon}\, e^{ABC}\Gamma_{AB} = 0 \ . \tag{E.2.7}$$

This equation and (E.2.6) imply

$$\bar{\epsilon}\, e^{0AB}\Gamma_{B} = 0 \ . \tag{E.2.8}$$

Multiplying this equation by $e^{0AC}\Gamma_C$, without summing over the index A, we obtain

$$0 = \bar{\epsilon}\left(\sum_{B=4}^{6} (e^{0AB})^2 \right) \ . \tag{E.2.9}$$

Therefore, we get $e^{0AB} = 0$ for all $A, B = 4, \ldots, 9$. Using this result and (E.2.7), the condition (3.2.9) with $(I,J) = (B,C)$ implies

$$\bar{\epsilon}\, e^{ABC}\Gamma_{A} = 0 \ . \tag{E.2.10}$$

Again, multiplying this equation by $e^{A'BC}\Gamma_{A'}$, we obtain $e^{ABC} = 0$. In conclusion, all the components of e^{IJK} are zero for this case.

E.3 Proof of $e^{IJK} = 0$ for $ISO(1,2)$ case

In this subsection, we show that the statement discussed in appendix E.1 also holds for the $d = 3$ case. For this Lorentz symmetry, only non trivial components are e^{012}, e^{abc} with $a, b, c = 3, \cdots, 9$. Basically we'll follow the same logic of E.2.

The condition (3.2.9) with $(I,J) = (0,1)$ implies

$$0 = \bar{\epsilon}\left(\frac{1}{72} e^{I'J'K'}\Gamma_{I'J'K'} + \frac{1}{2} e^{012}\Gamma_{012} \right). \tag{E.3.1}$$

The condition (3.2.9) with $(I,J)=(0,\bar{a})$ by using (E.3.1),

$$0=\bar{\epsilon}\bigg(-\frac{3}{4}e^{012}\Gamma_{012}-\frac{1}{8}e^{bc\bar{a}}\Gamma_{bc\bar{a}}\bigg), \tag{E.3.2}$$

where $\bar{a}$ means this equation doesn't sum over $\bar{a}$ but it holds any $a=3\sim 9$.

Multiplying the condition (3.2.9) with $(I,J)=(\bar{a},\bar{b})$ and using (E.3.1) and (E.3.2), we obtain

$$0=\bar{\epsilon}\bigg(e^{012}\Gamma_{012}+e^{\bar{a}\bar{b}c}\Gamma_{\bar{a}\bar{b}c}\bigg). \tag{E.3.3}$$

This is the last equation we are given. This equation means that for any combination of $(\bar{a},\bar{b})$ doesn't make $\bar{\epsilon}(e^{\bar{a}\bar{b}c}\Gamma_{\bar{a}\bar{b}c})\neq 0$, because if so, e^{012} will become zero, so we can show all e^{abc} also zero. So, assuming this let deform more,

$$\bar{\epsilon}(e^{012})=\bar{\epsilon}\bigg(e^{\bar{a}\bar{b}c}\Gamma_{012\bar{a}\bar{b}c}\bigg), \tag{E.3.4}$$

and at least one of $e^{\bar{a}\bar{b}c}$ should have non zero component. All combination of $(\bar{a},\bar{b})$ makes the eigenvalue equations for $\bar{\epsilon}$, so matrices of right hand side should commute each other for any $\bar{a}$, $\bar{b}$. Therefore, for any $\bar{a},\bar{b},\bar{d},\bar{e}$,

$$0=\bar{\epsilon}\bigg(e^{\bar{a}\bar{b}c}e_{\bar{d}\bar{e}f}(\Gamma_{\bar{a}\bar{b}c}{}^{\bar{d}\bar{e}f}+36\,\delta^{[\bar{d}}_{[c}\delta^{\bar{e}}_{\bar{b}}\Gamma_{\bar{a}]}{}^{f]})\bigg). \tag{E.3.5}$$

Choose the case of $\bar{b}=\bar{e}\neq\bar{d}\neq\bar{a}$, then above equation means,

$$0=\bar{\epsilon}\bigg(e^{\bar{a}\bar{b}\bar{d}}e_{\bar{d}\bar{b}f}\Gamma_{\bar{a}}{}^{f}-e^{\bar{a}\bar{b}f}e_{\bar{d}\bar{b}f}\Gamma_{\bar{a}}{}^{\bar{d}}\bigg). \tag{E.3.6}$$

Because contraction index f doesn't contain $\bar{d}$, multiplying $\Gamma_{\bar{d}}{}^{\bar{a}}$ we can show that

$$e^{\bar{a}\bar{b}\bar{d}}e_{\bar{d}\bar{b}\bar{f}}=0\ \Rightarrow e^{\bar{a}\bar{b}\bar{d}}=0\text{ or }e_{\bar{d}\bar{b}\bar{f}}=0, \tag{E.3.7}$$

$$e^{\bar{a}\bar{b}f}e_{\bar{d}\bar{b}f}=0, \tag{E.3.8}$$

where $\bar{f}\neq\bar{a}$ is arbitrary. Because we assumed $e^{012}\neq 0$, it means there are no zero combination of $e^{\bar{b}\bar{d}c}\Gamma_{\bar{b}\bar{d}c}$ for any $\bar{b},\bar{d}$. So, this means there exist some $\bar{a}$ which makes zero component $e^{\bar{a}\bar{b}\bar{d}}$. And the number of non zero component $e^{\bar{b}\bar{d}f}$ should be one by (E.3.7).

We can find this non zero e^{IJK} combination, for example, as

$$e^{345},\ e^{367},\ e^{389},\ e^{468},\ e^{479},\ e^{569},\ e^{578}. \tag{E.3.9}$$

And other cases can be obtained by changing the coordinate each other. In this case, we can get the eigenvalue equation by (E.3.7) as

$$\bar{\epsilon}\Gamma_{4567} = \pm\bar{\epsilon}, \quad \bar{\epsilon}\Gamma_{4589} = \pm\bar{\epsilon}, \quad \bar{\epsilon}\Gamma_{3478} = \pm\bar{\epsilon}. \tag{E.3.10}$$

Also from , the additional eigenvalue equation is given as,

$$\bar{\epsilon}\Gamma_{012345} = \pm\bar{\epsilon}. \tag{E.3.11}$$

After all those conditions show the number of preserving supersymmetry charge is $16 \times \left(\frac{1}{2}\right)^4 = 1$.

Because we impose spacetime symmetry as $ISO(1,2)$, the number of supersymmetry charge should be at least two, for $\mathcal{N} = 1$ in 3 dimension. If not, the super-Poincare algebra won't be closed. So, this means that even if we got the case for e^{IJK} satisfying (3.2.9), other SUSY conditions will make this $e^{IJK} = 0$.

Appendix F

Derivation of the D3-brane effective action

F.1 DBI action

In this section, we present the extended calculations of the expansion of the DBI term (2.4.14)

$$S_{\mathrm{D}p}^{\mathrm{DBI}} = -\,T_p \int d^{p+1}x\,\mathrm{Str}\left\{e^{-\phi}\sqrt{-\det(M_{\mu\nu})\det(Q^i{}_j)}\right\}\ , \tag{F.1.1}$$

$M_{\mu\nu}$ is defined as

$$M_{\mu\nu} \equiv \mathrm{P}\left[\widehat{E}_{\mu\nu} + \widehat{E}_{\mu i}(Q^{-1}-\delta)^{ij}\widehat{E}_{j\nu}\right] + \lambda F_{\mu\nu}\ . \tag{F.1.2}$$

The pull-back of the first term is given in (2.4.12) and it is expanded as

$$\begin{aligned}\mathrm{P}[\widehat{E}_{\mu\nu}] &= \widehat{E}_{\mu\nu} + \lambda\widehat{E}_{\mu i}D_\nu\Phi^i + \lambda\widehat{E}_{i\nu}D_\mu\Phi^i + \lambda^2\widehat{E}_{ij}D_\mu\Phi^i D_\nu\Phi^j \\ &= \widehat{E}_{\mu\nu} + \lambda^2(\partial_j B_{\mu i}\Phi^j D_\nu\Phi^i + \partial_j B_{i\nu}\Phi^j D_\mu\Phi^i + g_{ij}D_\mu\Phi^i D_\nu\Phi^j) + \mathcal{O}(\lambda^3)\ ,\end{aligned} \tag{F.1.3}$$

where we have used the assumptions in (6.2). The expansion of $Q^i{}_j$ is

$$\begin{aligned}Q^i{}_j &= \delta^i{}_j + i\lambda[\Phi^i,\Phi^k]\widehat{E}_{kj} \\ &= \delta^i{}_j + i\lambda[\Phi^i,\Phi^k]g_{kj} + i\lambda^2[\Phi^i,\Phi^k]\Phi^l\partial_l E_{kj} + \mathcal{O}(\lambda^3)\ .\end{aligned} \tag{F.1.4}$$

Because $(Q^{-1}-\delta)^i_j = \mathcal{O}(\lambda)$, it is easy to see that

$$\mathrm{P}\left[E_{\mu i}(Q^{-1}-\delta)^{ij}E_{j\nu}\right] = \mathcal{O}(\lambda^3)\ , \tag{F.1.5}$$

and we can discard such higher-order contributions.

On the other hand, to obtain the expansion of $\sqrt{\det(Q^i{}_j)}$, we use the following formula. For general matrices X and δX, the following expansion is obtained:

$$\sqrt{\det(X+\delta X)} = \sqrt{\det X}\left(1+\frac{1}{2}\operatorname{tr}(X^{-1}\delta X)+\frac{1}{8}(\operatorname{tr}(X^{-1}\delta X))^2 \right.$$
$$\left. -\frac{1}{4}\operatorname{tr}(X^{-1}\delta X X^{-1}\delta X)+\mathcal{O}(\delta X^3)\right) \ . \quad \text{(F.1.6)}$$

Using this formula, we get

$$\sqrt{\det(Q^i{}_j)} = 1+\frac{i\lambda^2}{2}[\Phi^i,\Phi^k]\Phi^l\partial_l B_{ki} - \frac{\lambda^2}{4} g_{ii'}g_{jj'}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}]+\mathcal{O}(\lambda^3) \ . \quad \text{(F.1.7)}$$

Let us consider now the term $\sqrt{-\det(M_{\mu\nu})}$. By following the same procedure, we work out its Taylor expansion and find:

$$\sqrt{-\det(M_{\mu\nu})} = \sqrt{-\det(\widehat{E}_{\mu\nu})}$$
$$+\sqrt{-g}\ \frac{\lambda^2}{2}\left(g^{\mu\nu}g_{ij}D_\mu\Phi^i D_\nu\Phi^j+\frac{1}{2}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}+(\partial_i B^{\mu\nu})\Phi^i F_{\mu\nu}\right)+\mathcal{O}(\lambda^3) \ . \quad \text{(F.1.8)}$$

Now, making use of the partial results of the expansions (F.1.7) and (F.1.8), we calculate the full integrand:

$$\begin{aligned}
&\operatorname{Str}\left\{e^{-\widehat{\phi}}\sqrt{-\det(M_{\mu\nu})\det(Q^i{}_j)}\right\} \\
&\quad = \operatorname{Str}\left\{e^{-\widehat{\phi}}\sqrt{-\det(\widehat{E}_{\mu\nu})}\right\} \\
&\qquad + \frac{\lambda^2}{2}e^{-\phi}\sqrt{-g}\operatorname{tr}\left(\frac{1}{2}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}+g^{\mu\nu}g_{ij}D_\mu\Phi^iD_\nu\Phi^j-\frac{1}{2}g_{ii'}g_{jj'}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}]\right. \\
&\qquad \left. +(\partial_i B^{\mu\nu})\Phi^i F_{\mu\nu}-i(\partial_i B_{jk})\Phi^i[\Phi^j,\Phi^k]\right)+\mathcal{O}(\lambda^3) \\
&\quad = \operatorname{Str}\left\{e^{-\widehat{\phi}}\sqrt{-\det(\widehat{E}_{\mu\nu})}\right\} \\
&\qquad + \frac{\lambda^2}{2}e^{-\phi}\sqrt{-g}\operatorname{tr}\left(\frac{1}{2}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}+g^{\mu\nu}g_{ij}D_\mu\Phi^iD_\nu\Phi^j-\frac{1}{2}g_{ii'}g_{jj'}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}]\right) \\
&\qquad \left. + H_i{}^{\mu\nu}\Phi^iF_{\mu\nu}-\frac{i}{3}H_{ijk}\Phi^i[\Phi^j,\Phi^k]\right)+\mathcal{O}(\lambda^3) \ .
\end{aligned} \quad \text{(F.1.9)}$$

The first term in (F.1.9) gives the DBI part of the scalar potential. Let us define

$$V_{\mathrm{DBI}}(\Phi) \equiv \frac{1}{\sqrt{-g}}e^{-\widehat{\phi}}\sqrt{-\det(\widehat{E}_{\mu\nu})} = \frac{1}{\sqrt{-g}}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\Phi^{i_1}\cdots\Phi^{i_n}\partial_{i_1}\cdots\partial_{i_n}L_{\mathrm{DBI}}|_{x^i=0} \quad \text{(F.1.10)}$$

with

$$L_{\rm DBI} \equiv e^{-\phi}\sqrt{-\det(E_{\mu\nu})}\ . \tag{F.1.11}$$

The derivatives are

$$\partial_i L_{\rm DBI} = e^{-\phi}\sqrt{-\det(E_{\mu\nu})}\left(-\partial_i\phi + \frac{1}{2}(E^{-1})^{\mu\nu}\partial_i E_{\nu\mu}\right)\ , \tag{F.1.12}$$

$$\begin{aligned}\partial_i\partial_j L_{\rm DBI} = e^{-\phi}\sqrt{-\det(E_{\mu\nu})}\Bigg[&\left(-\partial_i\phi + \frac{1}{2}(E^{-1})^{\mu\nu}\partial_i E_{\nu\mu}\right)\left(-\partial_j\phi + \frac{1}{2}(E^{-1})^{\mu'\nu'}\partial_j E_{\nu'\mu'}\right)\\ &-\partial_i\partial_j\phi + \frac{1}{2}\left(-(E^{-1})^{\mu\mu'}\partial_i E_{\mu'\nu'}(E^{-1})^{\nu'\nu}\partial_j E_{\nu\mu} + (E^{-1})^{\mu\nu}\partial_i\partial_j E_{\nu\mu}\right)\Bigg]\ .\end{aligned} \tag{F.1.13}$$

Evaluating these quantities at $x^i = 0$, we obtain

$$V_{\rm DBI}(\Phi) = e^{-\phi}\left(1 + \lambda v_i^{\rm DBI}\Phi^i + \frac{\lambda^2}{2}m_{ij}^{\rm DBI}\Phi^i\Phi^j + \mathcal{O}(\lambda^3)\right)\ , \tag{F.1.14}$$

where the coefficients are

$$v_i^{\rm DBI} \equiv\ -\partial_i\phi + \frac{1}{2}g^{\mu\nu}\partial_i g_{\nu\mu} = \partial_i\log(\sqrt{-g}\,e^{-\phi})\ , \tag{F.1.15}$$

$$\begin{aligned}m_{ij}^{\rm DBI} &\equiv v_i^{\rm DBI}v_j^{\rm DBI} - \partial_i\partial_j\phi + \frac{1}{2}\left(g^{\mu\nu}\partial_i\partial_j g_{\nu\mu} - g^{\mu\mu'}\partial_i g_{\mu'\nu'}g^{\nu'\nu}\partial_j g_{\nu\mu} - g^{\mu\mu'}H_{i\mu'\nu'}g^{\nu'\nu}H_{j\nu\mu}\right)\\ &= \frac{1}{\sqrt{-g}\,e^{-\phi}}\partial_i\partial_j\left(\sqrt{-g}\,e^{-\phi}\right) + \frac{1}{2}H_i{}^{\mu\nu}H_{j\mu\nu}\ .\end{aligned} \tag{F.1.16}$$

Then, the final expression for the DBI action is

$$\begin{aligned}S_{\rm D3}^{\rm DBI} = T_3\lambda^2\int d^4x\sqrt{-g}\,e^{-\phi}\,{\rm tr}\Bigg(&-\frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{2}g^{\mu\nu}g_{ij}D_\mu\Phi^i D_\nu\Phi^j\\ &+\frac{1}{4}g_{ii'}g_{jj'}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}] - \frac{1}{2}H_i{}^{\mu\nu}\Phi^i F_{\mu\nu} + \frac{i}{3!}H_{ijk}\Phi^i[\Phi^j,\Phi^k] - e^{\phi}\lambda^{-2}V_{\rm DBI}(\Phi)\Bigg)\ .\end{aligned} \tag{F.1.17}$$

F.2 CS term

Let us study now the expansion of the CS-term of the D3-brane action, (2.4.15).

First, we define

$$K \equiv \sum_{n:\text{even}} C_n\wedge e^{B_2} \equiv K_0 + K_2 + K_4 + K_6 + \cdots\ , \tag{F.2.1}$$

where*

$$K_0 \equiv C_0 \ , \quad K_2 \equiv C_2 + C_0 B_2 \ , \quad K_4 \equiv C_4 + C_2 B_2 + \frac{1}{2} C_0 B_2^2 \ ,$$
$$K_6 \equiv C_6 + C_4 B_2 + \frac{1}{2} C_2 B_2^2 + \frac{1}{3!} C_0 B_2^3 \ . \tag{F.2.2}$$

Note that it satisfies

$$dK = \sum_{n:\text{odd}} \widetilde{F}_n \wedge e^{B_2} \ . \tag{F.2.3}$$

For an n-form ω_n, we define an m-form with $(n-m)$ indices in the transverse directions $(\omega_n)_{m,i_1\cdots i_{n-m}}$ as

$$(\omega_n)_{m,i_1\cdots i_{n-m}} \equiv \frac{1}{m!} (\omega_n)_{\mu_1\cdots\mu_m i_1\cdots i_{n-m}} dx^{\mu_1} \cdots dx^{\mu_m} \ . \tag{F.2.4}$$

For example,

$$(\widetilde{F}_5)_{3,ij} \equiv \frac{1}{3!} (\widetilde{F}_5)_{\mu\nu\rho ij} dx^\mu dx^\nu dx^\rho \ , \quad (\widetilde{F}_5)_{4,j} \equiv \frac{1}{4!} \widetilde{F}_{\mu\nu\rho\sigma j} dx^\mu dx^\nu dx^\rho dx^\sigma \ , \quad \text{etc.} \tag{F.2.5}$$

Under the assumptions (6.2), the CS-term (2.4.15) is expanded as

$$S^{\text{CS}}_{\text{D3}} = \mu_3 \int \text{Str} \left\{ \text{P} \left[e^{i\lambda \imath_\Phi \imath_\Phi} \widehat{K} \right] \wedge e^{\lambda F} \right\}$$
$$= \mu_3 \int \text{Str} \left\{ \text{P}[\widehat{K}_4] + \lambda^2 \left(i\text{P}[\imath_\Phi \imath_\Phi \partial_i K_6] \Phi^i + \text{P}[\partial_i K_2] \Phi^i F + \frac{1}{2} C_0 F^2 \right) + \mathcal{O}(\lambda^3) \right\} . \tag{F.2.6}$$

Expanding the first term, we get

$$\text{P}[\widehat{K}_4] = \left[\lambda \partial_i K_4 \Phi^i + \lambda^2 \partial_i (K_4)_{3,j} \Phi^i D\Phi^j + \frac{\lambda^2}{2} \partial_i \partial_j K_4 \Phi^i \Phi^j \right]_0 + \mathcal{O}(\lambda^3) \ , \tag{F.2.7}$$

where $[\cdots]_0$ denotes the pull-back on the world-volume at $x^i = 0$ (obtained by setting $x^i = 0$ and $dx^i = 0$), $D\Phi^j$ is a 1-form defined as

$$D\Phi^j \equiv D_\mu \Phi^j dx^\mu = d\Phi^j + i[A, \Phi^j] \ , \tag{F.2.8}$$

and we have used the notation (F.2.4).

The trace of the second term in (F.2.7) can be rewritten as

$$\left[\lambda^2 \partial_i (K_4)_{3,j} \, \text{tr}(\Phi^i D\Phi^j) \right]_0$$
$$= \frac{\lambda^2}{2} \left[\frac{1}{2} (\partial_i (K_4)_{3,j} + \partial_j (K_4)_{3,i}) d \, \text{tr}(\Phi^i \Phi^j) + (\partial_i (K_4)_{3,j} - \partial_j (K_4)_{3,i}) \, \text{tr}(\Phi^i D\Phi^j) \right]_0$$
$$= \frac{\lambda^2}{2} \left[(\partial_i d (K_4)_{3,j} \, \text{tr}(\Phi^i \Phi^j) - (\widetilde{F}_5)_{3,ij} \, \text{tr}(\Phi^i D\Phi^j) + (\text{total derivative}) \right]_0 \ . \tag{F.2.9}$$

*In this section, we often omit the symbol "$\wedge$" in the products of differential forms.

Using this equation and the identities

$$\begin{aligned}
&[\partial_i K_4]_0 = [(\widetilde{F}_5)_{4,i}]_0 \ , \\
&[\partial_i d(K_4)_{3,j} + \partial_i\partial_j K_4]_0 = [\partial_i (dK_4)_{4,j}]_0 = [\partial_i (\widetilde{F}_5)_{4,j} + (\widetilde{F}_3)_{2,j}(H_3)_{2,i}]_0 \ ,
\end{aligned} \tag{F.2.10}$$

which are valid under the assumptions (6.2), we obtain

$$\int \mathrm{Str}\,\mathrm{P}[\widehat{K}_4] = \int \mathrm{tr}\left\{\lambda(\widetilde{F}_5)_{4,i}\Phi^i + \frac{\lambda^2}{2}\left(\partial_i(\widetilde{F}_5)_{4,j} + (\widetilde{F}_3)_{2,j}(H_3)_{2,i}\right)\Phi^i\Phi^j - \frac{\lambda^2}{2}(\widetilde{F}_5)_{3,ij}\Phi^i D\Phi^j\right\} \ . \tag{F.2.11}$$

The second and third terms in (F.2.6) are rewritten by using

$$\mathrm{tr}\{i\mathrm{P}[\imath_\Phi\imath_\Phi\partial_i K_6]\Phi^i\} = -\frac{i}{2}[\partial_i(K_6)_{4,jk}]_0\,\mathrm{tr}\{[\Phi^j,\Phi^k]\Phi^i\} = -\frac{i}{3!}[(\widetilde{F}_7)_{4,ijk}]_0\,\mathrm{tr}\{[\Phi^j,\Phi^k]\Phi^i\} \ , \tag{F.2.12}$$

$$\mathrm{P}[\partial_i K_2]\Phi^i F_2 = [(\widetilde{F}_3)_{2,i}]_0\,\Phi^i F \ , \tag{F.2.13}$$

respectively.

Plugging these results in (F.2.6), the CS-term becomes

$$\begin{aligned}
S^{\mathrm{CS}}_{\mathrm{D3}} = \mu_3\lambda^2\int \mathrm{tr}\Big\{&\lambda^{-1}(\widetilde{F}_5)_{4,i}\Phi^i + \frac{1}{2}\left(\partial_i(\widetilde{F}_5)_{4,j} + (\widetilde{F}_3)_{2,j}(H_3)_{2,i}\right)\Phi^i\Phi^j - \frac{1}{2}(\widetilde{F}_5)_{3,ij}\Phi^i D\Phi^j \\
&- \frac{i}{3!}(\widetilde{F}_7)_{4,ijk}\Phi^i[\Phi^j,\Phi^k] + (\widetilde{F}_3)_{2,i}\Phi^i F + \frac{1}{2}C_0 F^2\Big\} \ .
\end{aligned} \tag{F.2.14}$$

It can also be written as

$$\begin{aligned}
S^{\mathrm{CS}}_{\mathrm{D3}} = \mu_3\lambda^2\int d^4x\sqrt{-g}\,\mathrm{tr}\Big\{&\frac{1}{2}(*_4\widetilde{F}_3)_i{}^{\mu\nu}\Phi^i F_{\mu\nu} - \frac{i}{3!}(*_6\widetilde{F}_3)_{ijk}\Phi^i[\Phi^j,\Phi^k] \\
&- \frac{1}{2}(*_4\widetilde{F}_5)^\mu{}_{ij}\Phi^i D_\mu\Phi^j + \frac{1}{8}C_0 F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - \lambda^{-2}V_{\mathrm{CS}}(\Phi)\Big\} \ ,
\end{aligned} \tag{F.2.15}$$

where the potential $V_{\mathrm{CS}}(\Phi)$ is

$$V_{\mathrm{CS}}(\Phi) \equiv -\lambda(*_4\widetilde{F}_5)_i\Phi^i - \frac{\lambda^2}{2}\left(\partial_i(*_4\widetilde{F}_5)_j + \frac{1}{2}(*_4\widetilde{F}_3)_j{}^{\mu\nu}(H_3)_{\mu\nu i}\right)\Phi^i\Phi^j \tag{F.2.16}$$

and we have defined

$$\begin{aligned}
&(*_4\widetilde{F}_3)_i{}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\rho\sigma\mu\nu}\widetilde{F}_{i\rho\sigma} \ , \quad (*_6\widetilde{F}_3)_{ijk} \equiv \frac{1}{3!}\epsilon_{lmnijk}\widetilde{F}^{lmn} \ , \\
&(*_4\widetilde{F}_5)^\mu{}_{ij} \equiv \frac{1}{3!}\epsilon^{\nu\rho\sigma\mu}\widetilde{F}_{\nu\rho\sigma ij} \ , \quad (*_4\widetilde{F}_5)_i \equiv \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\widetilde{F}_{\mu\nu\rho\sigma i} \ ,
\end{aligned} \tag{F.2.17}$$

and used the relation

$$\widetilde{F}_7 = -*\widetilde{F}_3 \ . \tag{F.2.18}$$

F.3 Bosonic part

Summing (F.1.17) and (F.2.15), we obtain

$$S_{\rm D3}^{\rm boson} = \frac{T_3\lambda^2}{2}\int d^4x\sqrt{-g}\,{\rm tr}\bigg(-\frac{e^{-\phi}}{2}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} \pm \frac{C_0}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$
$$- e^{-\phi}g^{\mu\nu}g_{ij}D_\mu\Phi^i D_\nu\Phi^j + \frac{e^{-\phi}}{2}g_{ii'}g_{jj'}[\Phi^i,\Phi^j][\Phi^{i'},\Phi^{j'}] - 2V(\Phi)$$
$$\pm (G^R_\pm)_i{}^{\mu\nu}\Phi^i F_{\mu\nu} \mp \frac{i}{3}(G^R_\pm)_{ijk}\Phi^i[\Phi^j,\Phi^k] \mp (*_4\widetilde{F}_5)^\mu{}_{ij}\Phi^i D_\mu\Phi^j\bigg) \ , \quad \text{(F.3.1)}$$

where we have set

$$\mu_3 = \pm T_3 \quad \text{(F.3.2)}$$

for D3-branes and $\overline{\rm D3}$-branes, respectively, and defined

$$G_3 \equiv F_3 + (C_0 + ie^{-\phi})H_3 = \widetilde{F}_3 + ie^{-\phi}H_3 \ , \quad \text{(F.3.3)}$$
$$(G^R_\pm)_i{}^{\mu\nu} \equiv {\rm Re}((*_4G_3)_i{}^{\mu\nu} \pm iG_i{}^{\mu\nu}) = (*_4\widetilde{F}_3)_i{}^{\mu\nu} \mp e^{-\phi}H_i{}^{\mu\nu} \ , \quad \text{(F.3.4)}$$
$$(G^R_\pm)_{ijk} \equiv {\rm Re}((*_6G_3)_{ijk} \pm iG_{ijk}) = (*_6\widetilde{F}_3)_{ijk} \mp e^{-\phi}H_{ijk} \ , \quad \text{(F.3.5)}$$

and

$$V(\Phi) \equiv \lambda^{-2}(V_{\rm DBI}(\Phi) \pm V_{\rm CS}(\Phi)) \ . \quad \text{(F.3.6)}$$

F.4 Fermionic part

Let us consider the fermionic action (2.4.23) for a D3-brane. To make the kinetic term of the fermions $\mathcal{O}(\lambda^0)$, we rescale the fermion as

$$\psi = \lambda\Psi \ . \quad \text{(F.4.1)}$$

Since we are interested in the terms that survive in the $l_s \to 0$ limit, we can set $M_{\mu\nu} = g_{\mu\nu}$ and $\breve{\Gamma}^{-1}_{\rm D3} = -\Gamma^{(0)}_{\rm D3} = \Gamma^{(4)}$ where

$$\Gamma^{(4)} \equiv \Gamma^{\hat{0}}\Gamma^{\hat{1}}\Gamma^{\hat{2}}\Gamma^{\hat{3}} \ . \quad \text{(F.4.2)}$$

Inserting (2.4.25)-(2.4.28) into the action (2.4.23) we obtain

$$
\begin{aligned}
S_{\mathrm{D3}}^{\mathrm{fermi}} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-g}\,e^{-\phi}\,i\overline{\Psi}\Big[\Gamma^\mu\nabla_\mu &+ \frac{1}{4\cdot 2!}H_{\mu IJ}\Gamma^{\mu IJ} - \frac{1}{4\cdot 3!}H_{IJK}\Gamma^{IJK} \\
&\mp e^{\phi}\Gamma^{(4)}\left(\frac{1}{8}\Gamma^\mu\left(-F_I\Gamma^I + \frac{1}{3!}\widetilde{F}_{IJK}\Gamma^{IJK} - \frac{1}{2\cdot 5!}\widetilde{F}_{IJKL}\Gamma^{IJKL}\right)\Gamma_\mu \right. \\
&\left. - \frac{1}{2}F_I\Gamma^I + \frac{1}{4\cdot 3!}\widetilde{F}_{IJK}\Gamma^{IJK}\right)\Big]\Psi\ , \qquad \text{(F.4.3)}
\end{aligned}
$$

where ∇_μ is

$$
\nabla_\mu\Psi \equiv \partial_\mu\Psi + \frac{1}{4}\omega_\mu{}^{\hat{I}\hat{J}}\Gamma_{\hat{I}\hat{J}}\Psi\ . \qquad \text{(F.4.4)}
$$

Again, the upper (lower) signs correspond to the case of D3- ($\overline{\text{D3}}$-) branes. Note that the first term in (2.4.27) does not contribute, because $\Gamma^0\Gamma^I$ is a symmetric matrix. In general, one can show

$$
\overline{\Psi}\,\Gamma^{I_1\cdots I_n}\Psi = 0 \quad \text{for} \quad n \neq 3\ (\mathrm{mod}\,4)\ . \qquad \text{(F.4.5)}
$$

Using this fact and the identities:

$$
\begin{aligned}
\Gamma^\mu(F_I\Gamma^I)\Gamma_\mu &= -2F_\mu\Gamma^\mu - 4F_i\Gamma^i\ , && \text{(F.4.6)} \\
\Gamma^\mu(\widetilde{F}_{IJK}\Gamma^{IJK})\Gamma_\mu &= 2\widetilde{F}_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - 6\widetilde{F}_{\mu jk}\Gamma^{\mu jk} - 4\widetilde{F}_{ijk}\Gamma^{ijk}\ , && \text{(F.4.7)} \\
\Gamma^\mu(\widetilde{F}_{IJKLM}\Gamma^{IJKLM})\Gamma_\mu &= 20\widetilde{F}_{\mu\nu\rho\sigma m}\Gamma^{\mu\nu\rho\sigma m} + 20\widetilde{F}_{\mu\nu\rho lm}\Gamma^{\mu\nu\rho lm} - 10\widetilde{F}_{\mu jklm}\Gamma^{\mu jklm} - 4\widetilde{F}_{ijklm}\Gamma^{ijklm}\ , && \text{(F.4.8)}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
S_{\mathrm{D3}}^{\mathrm{fermi}} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-g}\,e^{-\phi}\,i\overline{\Psi}\Big[\Gamma^\mu\nabla_\mu &+ \frac{1}{4\cdot 3!}(3H_{i\mu\nu}\Gamma^{i\mu\nu} - H_{ijk}\Gamma^{ijk}) \\
&\mp e^{\phi}\Gamma^{(4)}\Big(- \frac{1}{4}F_\mu\Gamma^\mu + \frac{1}{4\cdot 3!}(3\widetilde{F}_{i\mu\nu}\Gamma^{i\mu\nu} - \widetilde{F}_{ijk}\Gamma^{ijk}) \\
&- \frac{1}{3!\cdot 2^5}(2\widetilde{F}_{\mu\nu\rho ij}\Gamma^{\mu\nu\rho ij} - \widetilde{F}_{\mu ijkl}\Gamma^{\mu ijkl})\Big)\Big]\Psi\ . \qquad \text{(F.4.9)}
\end{aligned}
$$

Futhermore, using the following identities

$$
\begin{aligned}
&\Gamma^{(4)}\Gamma^\mu = \frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}\Gamma_{\nu\rho\sigma}\ , \quad \Gamma^{(4)}\Gamma^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Gamma_{\rho\sigma}\ , \quad \Gamma^{(4)}\Gamma^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho\sigma}\Gamma_\sigma\ , \\
&\Gamma^{(4)}\Gamma^{ijk} = \frac{1}{3!}\epsilon^{ijklmn}\Gamma_{lmn}\Gamma^{(10)}\ , \quad \Gamma^{(4)}\Gamma^{ijkl} = -\frac{1}{2}\epsilon^{ijklmn}\Gamma_{mn}\Gamma^{(10)}\ , \qquad \text{(F.4.10)}
\end{aligned}
$$

together with the chirality condition (6.1) and the relation

$$(*_4\widetilde{F}_5)_{\mu ij} = (*_6\widetilde{F}_5)_{\mu ij} \ , \tag{F.4.11}$$

that follows from the self-duality condition (2.1.5), the action can be rewritten as

$$\begin{aligned} S^{\text{fermi}}_{\text{D3}} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-g}\,e^{-\phi}\,i\overline{\Psi}\Big[\Gamma^\mu\nabla_\mu + \frac{1}{4\cdot 3!}(3H_{i\mu\nu}\Gamma^{i\mu\nu} - H_{ijk}\Gamma^{ijk}) \\ \mp e^\phi\Big(-\frac{1}{4\cdot 3!}(*_4F_1)_{\mu\nu\rho}\Gamma^{\mu\nu\rho} + \frac{1}{4\cdot 3!}\Big(3(*_4\widetilde{F}_3)_{i\mu\nu}\Gamma^{i\mu\nu} - (*_6\widetilde{F}_3)_{ijk}\Gamma^{ijk}\Big) + \frac{1}{8}(*_4\widetilde{F}_5)_{\mu ij}\Gamma^{\mu ij}\Big)\Big]\Psi \ , \end{aligned} \tag{F.4.12}$$

where we have used the notation (F.2.17) and defined

$$(*_4F_1)_{\nu\rho\sigma} \equiv \epsilon_{\mu\nu\rho\sigma}F^\mu \ . \tag{F.4.13}$$

For the non-Abelian case, the covariant derivative ∇_μ should be replaced with

$$\Gamma^\mu\nabla_\mu\Psi \quad \rightarrow \quad \Gamma^\mu D_\mu\Psi + i\Gamma_k[\Phi^k,\Psi] + \frac{1}{4}\omega_{\mu\hat{i}\hat{j}}\Gamma^{\mu\hat{i}\hat{j}} \ , \tag{F.4.14}$$

where $D_\mu\Psi$ is defined in (3.1.5) and we have assumed $g_{\mu i} = 0$ and $\omega_{\mu\hat{\nu}\hat{i}} = -\omega_{\mu\hat{i}\hat{\nu}} = 0$.

Then, our final expression for the fermionic part of the action is

$$S^{\text{fermi}}_{\text{D3}} = \frac{T_3\lambda^2}{2}\int d^4x\,\sqrt{-g}\,e^{-\phi}\,\mathrm{tr}\left\{i(\overline{\Psi}\Gamma^\mu D_\mu\Psi + \overline{\Psi}\Gamma_k i[\Phi^k,\Psi]) - i\overline{\Psi}\left(M_\pm - \frac{1}{4}\omega_{\mu\hat{i}\hat{j}}\Gamma^{\mu\hat{i}\hat{j}}\right)\Psi\right\} \tag{F.4.15}$$

with

$$M_\pm \equiv \mp\frac{e^\phi}{8}\left(\frac{1}{3}(*_4F_1)_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - (G^R_\pm)_{i\mu\nu}\Gamma^{i\mu\nu} + \frac{1}{3}(G^R_\pm)_{ijk}\Gamma^{ijk} - (*_4\widetilde{F}_5)_{\mu ij}\Gamma^{\mu ij}\right) \ , \tag{F.4.16}$$

where $(G^R_\pm)_{i\mu\nu}$ and $(G^R_\pm)_{ijk}$ are defined in (F.3.4) and (F.3.5), respectively.

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