

ON THE V -TRANSVERSALITY CONSTRUCTION OF EQUIVARIANT FRAMED MAPS

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Abstract. Let G be a finite group. For a pointed G -map $Y^+ \wedge V^\bullet \rightarrow V^\bullet$, we introduce the notion of V -transversality, where Y is a compact (smooth) G -manifold, V is a real G -module, $Y^+ = Y \amalg \{y_\infty\}$, and $V^\bullet = V \cup \{v_\infty\}$. Every V -transversal G -map $Y^+ \wedge V^\bullet \rightarrow V^\bullet$ gives rise of a G -framed map with the target manifold Y . This yields a one-to-one correspondence from the set of pointed G -homotopy classes to the set of G -framed cobordisms. From this view point, we discuss the Burnside ring of G and the equivariant stable G -cohomotopy group $\omega_G^0(Y)$ which consists of equivalence classes of pointed G -maps $Y^+ \wedge V^\bullet \rightarrow V^\bullet$ for $V = \mathbb{C}[G]^m$ ($m \gg 1$).

1. INTRODUCTION

In this article, we report results obtained from discussions with Takashi Matsunaga and Yasuhiro Hara.

Let G be a finite group throughout this paper. A G -framed map $\mathbf{f}$ is a pair of a G -map $f : (X, \partial X) \rightarrow (Y, \partial Y)$ and a real G -vector bundle isomorphism $b : TX \oplus \varepsilon(\mathbb{R}^m) \rightarrow f^*TY \oplus \varepsilon(\mathbb{R}^m)$, where X and Y are compact (smooth) G -manifolds of same dimension. Let $\mathfrak{N}(G, Y)$ be the set of G -framed cobordism classes of G -framed maps with the target manifold Y . Let $\omega_G^0(Y)$ be the equivariant stable cohomotopy group of Y , of which the definition will be given in Section 2. Let V be a real G -module (of finite dimension). In this paper we introduce the notion that a

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G -map $Y^+ \wedge V^\bullet \rightarrow V^\bullet$ is V -transversal to 0 in $V^\bullet$, essentially due to T. Petrie [8], cf. [5, §5]. The next result indicates importance of the notion in equivariant surgery theory.

Theorem 1.1. *The V -transversality construction gives a one-to-one correspondence*

$$\Phi : \omega_G^0(Y) \rightarrow \mathfrak{N}(G, Y).$$

This is an equivariant version of [4, Theorems B and C].

Let ω_G^n denote the group $\omega_G^n(pt)$. Let $A(G)$ denote the Burnside ring, i.e. the Grothendieck group for the category of finite G -sets. By proofs in equivariant homotopy theory, e.g. [1], [6], we have seen that $A(G)$ is isomorphic to ω_G^0 . However the theorem above gives rise of a geometric proof.

Theorem 1.2. *The V -transversality construction yields an isomorphism*

$$\Psi : \omega_G^0 \rightarrow A(G).$$

Details of the theorem will be given in Section 4.

Let $\mathcal{S}(G)$ denote the set of all subgroups of G . Let $\mathcal{F}$ be a set of subgroups of G . We call $\mathcal{F}$ *lower closed* if $H \in \mathcal{F}$ then $\mathcal{S}(H) \subset \mathcal{F}$. We call $\mathcal{F}$ *conjugation invariant* if $gHg^{-1} \in \mathcal{F}$ holds for all $H \in \mathcal{F}$ and $g \in G$. Hereafter, we suppose that $\mathcal{F}$ is lower closed and conjugacy invariant. Let $\mathfrak{F} = \mathfrak{F}(G, \mathcal{F})$ be the category such that $\text{Obj}(\mathfrak{F})$ is same as $\mathcal{F}$ and $\text{Mor}(\mathfrak{F})$ consists of all (H, g, K) with $H, K \in \mathcal{F}$, $g \in G$ and $gHg^{-1} \subset K$. Let Y be a compact G -manifold. If $H \leq K \leq G$ then we have the restriction map $\text{res}_H^K : \omega_K^n(Y) \rightarrow \omega_H^n(Y)$. If $H \leq G$ and $g \in G$ then we have the conjugation map $c_g : H \rightarrow gHg^{-1}$ and hence the induced map $c_g^* : \omega_{gHg^{-1}}^n(Y) \rightarrow \omega_H^n(Y)$. For $(H, g, K) \in \text{Mor}(\mathfrak{F})$, the homomorphism $(H, g, K)^*$ is defined to be the composition

$$\omega_K^n(Y) \xrightarrow{\text{res}_{gHg^{-1}}^K} \omega_{gHg^{-1}}^n(Y) \xrightarrow{c_g^*} \omega_H^n(Y).$$

Let $\omega_{G, \mathcal{F}}^n(Y)$ denote the inverse limit

$$\text{inv-lim}_{\mathfrak{F}} \omega_H^n(Y) \quad (H \in \mathcal{F}).$$

We set $\omega_{G, \mathcal{F}}^n = \omega_{G, \mathcal{F}}^n(pt)$. It is interesting to study the image and the kernel of the canonical map

$$\text{res}_{\mathcal{F}} : \omega_G^0(Y) \rightarrow \omega_{G, \mathcal{F}}^0(Y).$$

Theorem 1.3 (Y. Hara–M. M.). *Let G be a nontrivial nilpotent group and $\mathcal{F} = \mathcal{S}(G) \setminus \{G\}$. The canonical map $\text{res}_{\mathcal{F}} : \omega_G^0 \rightarrow \omega_{G,\mathcal{F}}^0$ is surjective if and only if G is a cyclic group of order a prime or a product of distinct primes.*

This implies

Corollary 1.4. *Let G and $\mathcal{F}$ be as above and let Y be a compact G -manifold such that $Y^G \neq \emptyset$. If the canonical map $\text{res}_{\mathcal{F}} : \omega_G^0(Y) \rightarrow \omega_{G,\mathcal{F}}^0(Y)$ is surjective then G is a cyclic group of order a prime or a product of distinct primes.*

Let k_G be the product of the primes p such that G contains a normal subgroup N with index p . Here k_G is understood to be 1 if G is a perfect group. The next lemma is essentially due to R. Oliver [7, Lemma 8].

Lemma 1.5. *There exists an element $\gamma_G = [X_1] - [X_2]$ of $A(G)$ such that $|X_1^G| - |X_2^G| = k_G$ and $\text{res}_H^G \gamma_G = 0$ for all $H < G$, where X_1 and X_2 are finite G -sets.*

Proposition 1.6. *Let G be a nontrivial group and $\mathcal{F} = \mathcal{S}(G) \setminus \{G\}$. Then the kernel of $\text{res}_{\mathcal{F}} : \omega_G^0 \rightarrow \omega_{G,\mathcal{F}}^0$ is generated by γ_G .*

Corollary 1.7. *Let G and $\mathcal{F}$ be as above. Then $\gamma_G \omega_G^0(Y)$ is contained in the kernel of $\omega_G^0(Y) \rightarrow \omega_{G,\mathcal{F}}^0(Y)$.*

2. THE EQUIVARIANT STABLE COHOMOTOPY GROUP $\omega_G^n(Y)$

Let Y be a compact G -manifold. We denote by Y^+ the disjoint union $Y \amalg \{y_\infty\}$ and the point y_∞ is regarded as the base point of Y^+ . Let V be a real G -module. We denote by $V^\bullet$ the one-point compactification $V \cup \{v_\infty\}$ and v_∞ is regarded as the base point of $V^\bullet$. The smash product

$$Y^+ \wedge V^\bullet = \frac{Y^+ \times V^\bullet}{(Y^+ \times v_\infty) \cup (y_\infty \times V^\bullet)}$$

can be regarded as the Thom space of real G -vector bundle $\pi : Y \times V \rightarrow Y$ with fiber V .

We denote by $\mathbb{R}$ the 1-dimensional real vector space with trivial G -action. For a finite G -CW complex X and an integer n , we define $\omega_G^n(X)$ by

$$\omega_G^n(X) = \lim_{m \rightarrow \infty} [A, B]_0^G$$

where

$$\begin{aligned} A &= \begin{cases} X^+ \wedge M_m^\bullet & (n \geq 0) \\ X^+ \wedge (\mathbb{R}^{|n|} \oplus M_m)^\bullet & (n < 0), \end{cases} \\ B &= \begin{cases} (\mathbb{R}^n \oplus M_m)^\bullet & (n \geq 0) \\ M_m^\bullet & (n < 0), \end{cases} \\ M_m &= \mathbb{C}[G]^{\oplus m}, \end{aligned}$$

and $[-, -]_0^G$ stands for the set of all homotopy classes of maps in the category of pointed G -spaces. We set $\omega_G^n = \omega_G^n(pt)$. Define the map $\deg_H : \omega_G^0 \rightarrow \mathbb{Z}$ by

$$\deg_H([f : M_m^\bullet \rightarrow M_m^\bullet]) = \deg(f^H : M_m^{H^\bullet} \rightarrow M_m^{H^\bullet}).$$

It is known that the map

$$\prod_{H \in \mathcal{S}(G)} \deg_H : \omega_G^n \rightarrow \prod_{H \in \mathcal{S}(G)} \mathbb{Z}$$

is injective and

$$[V^\bullet, V^\bullet]_0^G \cong \omega_G^0$$

via the canonical map whenever $V \supset \mathbb{C}[G]$.

3. V -TRANSVERSALITY OF G -MAPS $Y^+ \wedge V^\bullet \rightarrow V^\bullet$

In this section we introduce the notion of V -transversality due to T. Petrie [8]. Let V be a real G -module with a G -invariant inner product. For $H \leq G$, V is decomposed into $V^H \oplus V_H$ as real $N_G(H)$ -modules, where V^H is the H -fixed point set of V . A base point preserving G -map $\alpha : Y^+ \wedge V^\bullet \rightarrow V^\bullet$ is called V -transversal to 0 in the target space $V^\bullet$ if the following conditions are fulfilled.

- (1) α is smooth on a neighborhood of X .
- (2) α is transversal to 0 in V , i.e. $d_x \alpha : T_x(Y \times V) \rightarrow T_0 V$ is surjective at every $x \in X$.
- (3) the normal derivative $\nu_x(\alpha) : V_H \rightarrow V_H$ at x , where $H = G_x$, coincides with the identity map $V_H \rightarrow V_H$ for every $x \in X$,

where $X = \alpha^{-1}(0)$ and $\nu_x(\alpha)$ is defined to be the composition

$$V_H \xrightarrow{\text{incl}} V = T_{\pi_V(x)} V \xrightarrow{\text{incl}} T_x(Y \times V) \xrightarrow{d_x \alpha} T_0(V) \xrightarrow{\text{proj}} V_H.$$

Lemma 3.1. *Let Y be a compact G -manifold with G -invariant Riemannian metric, A a closed G -subset of Y , V a real G -module, and $\varepsilon : Y^+ \wedge V^\bullet \rightarrow \mathbb{R}$ a G -invariant positive function. Let $\alpha : Y^+ \wedge V^\bullet \rightarrow V^\bullet$ be a base-point preserving G -map such that $\alpha : A^+ \wedge V^\bullet \rightarrow V^\bullet$ is V -transversal to 0 in $V^\bullet$. Then there exists a base-point preserving G -homotopy $H : I \times (Y^+ \wedge V^\bullet) \rightarrow V^\bullet$, where $I = [0, 1]$, from α to $\beta : Y^+ \wedge V^\bullet \rightarrow V^\bullet$ satisfying the following conditions.*

- (1) $H(t, (x, v)) = \alpha(x, v)$ for all $t \in I$, $x \in A$, and $v \in V$.
- (2) $d(\alpha(y, v), H(t, (y, v))) < \varepsilon(y, v)$ for all $(y, v) \in Y \times V$.
- (3) β is V -transversal to 0 in $V^\bullet$.

A G -framed map $\mathbf{f} = (f, b)$ consists of a G -map $f : X \rightarrow Y$, where X and Y are compact (smooth) G -manifolds, and a real G -vector bundle isomorphism $b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \rightarrow f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^m)$ for some integer $m \geq 0$.

Suppose $\alpha : Y^+ \wedge V^\bullet \rightarrow V^\bullet$ is V -transversal to 0 in $V^\bullet$. Then set $X = \alpha^{-1}(0)$ and let $f : X \rightarrow Y$ be the composition of the inclusion $j_X : X \rightarrow Y \times V$ and the projection $\pi_Y : Y \times V \rightarrow Y$. There is a canonical isomorphism

$$T(Y \times V)|_X = (\pi_Y^*TY \oplus \pi_V^*TV)|_X = f^*TY \oplus \varepsilon(V).$$

We also have an isomorphism

$$T(Y \times V)|_X = TX \oplus \nu(X, Y \times V) \cong TX \oplus (\alpha|_X)^*\nu(0, V) = TX \oplus \varepsilon(V).$$

Thus we get a G -vector bundle isomorphism

$$\beta : TX \oplus \varepsilon(V) \rightarrow f^*TY \oplus \varepsilon(V)$$

such that $(\beta|_x)(x, v) = (x, v) \in \varepsilon(V)$ for all $x \in X$ and $v \in V_H$, where $H = G_x$. By Lück-Madsen [3, Appendix, Proposition (A2)] and [5, §6], in the case $m > \dim Y$, we obtain a G -vector bundle isomorphism

$$b : TX \oplus \varepsilon(\mathbb{R}^m) \rightarrow f^*TY \oplus \varepsilon(\mathbb{R}^m)$$

such that β and b are stably regularly G -homotopic. This procedure obtaining $\mathbf{f} = (f, b)$ from α is called the *V -transversality construction* of G -framed maps. The construction may also be called *Pontryagin-Petrie construction* of G -framed maps, cf. [4, §7].

4. THE ISOMORPHISM $\Psi : \omega_G^0 \rightarrow A(G)$

Let $\alpha : V^\bullet \rightarrow V^\bullet$ be a base-point preserving G -map such that $V \supset \mathbb{C}[G]$. Then we have $\dim V^G \geq 2$. Suppose α is V -transversal to 0 in $V^\bullet$. The V -transversality construction yields $X = \alpha^{-1}(0)$, $f = \alpha|_X : X \rightarrow \{pt\}$, $\beta : \varepsilon(V) \rightarrow \varepsilon(V)$ and $b : \varepsilon(V^G) \rightarrow \varepsilon(V^G)$ such that $\beta|_x : V \rightarrow V$ coincides with $d\alpha_x : V \rightarrow V$ for $x \in X$, and β is regularly G -homotopic to $b \oplus id_{\varepsilon(V^G)}$. Decompose the G -set $X = \alpha^{-1}(0)$ into the disjoint union of G -orbits $X_i = Gx_i$, where $i = 1, \dots, k$; i.e. $X = \coprod_{i=1}^k X_i$. We define

$$\varepsilon(d\alpha_{x_i}) = \begin{cases} 1 & \text{(if } d\alpha_{x_i} \text{ is orientation preserving),} \\ -1 & \text{(if } d\alpha_{x_i} \text{ is orientation reversing).} \end{cases}$$

Set

$$X_+ = \coprod_i Gx_i : \varepsilon(d\alpha_{x_i}) = 1,$$

$$X_- = \coprod_i Gx_i : \varepsilon(d\alpha_{x_i}) = -1.$$

Then we obtain an element $[X_+] - [X_-]$ of $A(G)$. The correspondence $[f] \mapsto [X_+] - [X_-]$ gives the map $\Psi : \omega_G^0 \rightarrow A(G)$. We have a canonical one-to-one correspondence $\Xi : \mathfrak{N}(G, pt) \rightarrow A(G)$. Moreover the diagram

$$\begin{array}{ccc} \omega_G^0 & \xrightarrow{\Phi} & \mathfrak{N}(G, pt); \\ \Psi \searrow & \cong \swarrow \Xi & \\ & A(G) & \end{array} \quad \begin{array}{ccc} [\alpha] & \xrightarrow{\Phi} & [f, b] \\ \Psi \searrow & \swarrow \Xi & \\ & [X_+] - [X_-] & \end{array}$$

commutes. Once it was admitted that Φ is a one-to-one correspondence, the map Ψ is bijective and hence an isomorphism.

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