

A Rich Hierarchy on the Time Complexity of Uniform PRAMs

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1. Introduction.

Motivation of introducing the unbounded fan-in circuit model is not unique: In [1] the authors found a connection between the complexity theory over this model and a famous open question (at that time) on the polynomial hierarchy. [4] proves that the model can simulate the parallel random access machines with simultaneous writes (CRCW-PRAMs) of almost the same depth and size. Recently the present author claims in [2] that the model can be much more useful than the bounded fan-in model to study the depth hierarchy of Boolean functions. To this goal, however, it turns out that the essential nonuniformity circuit models generally possess is a major obstacle and that the best possible model we have currently seems to be (uniform) CRCW-PRAMs. The main result of [2] is the following.

Let $A(i, j)$ be the Ackermann function and let $\bar{A}_k(n)$ be its inverse function defined by $\bar{A}_k(n) = \text{least } j \text{ such that } A(k, j) \geq n$. CRCW-PRAMs here denote parallel RAMs with simultaneous writes and with operations $+$, $-$ and $\mid$ (bitwise OR). It should be noted again that CRCW-PRAMs in this paper are *uniform*, namely, each RAM has the same program not depending on the size of inputs. Then it follows:

Theorem 1. For any constant $c \geq 4$, there is a nondegenerate Boolean function G_c of n variables such that it takes $\Theta(\bar{A}_c(n))$ steps to compute G_c by CRCW-PRAMs with polynomial number of processors.

In this paper we extend this result, which suggests the existence of a rich hierarchy on the time complexity of Boolean functions. We will also show a result on a relation between the star-free regular expressions and the unbounded fan-in circuits which might help removing the obstacle above mentioned (the nonuniformity of the circuit models).

2. Time Hierarchy.

It should be noted that Theorem 1 is still true if $\bar{A}_c(n)$ is replaced by its composition like $\bar{A}_{c_1}(\bar{A}_{c_2}(\cdots(\bar{A}_{c_i}(n))\cdots))$ for constants $c_1, c_2, \cdots, c_i \geq 4$ (e.g., $\bar{A}_4(\bar{A}_4(\bar{A}_4(n))) = \log^* \log^* \log^* n$). In this section we demonstrate that the hierarchy is much more dense. Let M be a Turing machine computing an integer function $f(n)$ by producing $1^{f(n)}$ on its output tape from 1^n on its input tape. Then f is called a

polynomial-time function if M halts within $T(n)$ steps for some polynomial T . For simplicity, we assume that f is monotone, i.e., $f(n+1) \geq f(n)$ for all n .

Theorem 2. Theorem 1 is also true if $\bar{A}_c(n)$ is replaced by any polynomial-time function f .

Proof. A straightforward modification of the proof of Theorem 1 [2]. Use a sequence of configurations of the Turing machine that computes the polynomial-time function f instead of the sequence of binary numbers. Details are omitted. $\square$

Corollary 1. Theorem 1 is still true for $\bar{A}_n(n)$.

Proof. We will show that Ackermann function $A(i, j)$ can be computed in a polynomial number of steps on its answer (not on its input). Then it immediately follows that $\bar{A}_n(n)$ is a polynomial-time function.

By definition $A(i, j)$ can be obtained by applying the following operations as far as possible:

$$A(0, y) = y + 1 \quad (1)$$

$$A(x+1, 0) = A(x, 1) \quad (2)$$

$$A(x+1, y+1) = A(x, A(x+1, y)) \quad (3)$$

Thus, during the computation we always handle the form like

$$A(a_0, A(a_1, \dots, A(a_{n-3}, A(a_{n-2}, a_{n-1}))) \dots),$$

which we denote by string

$$\sigma = a_0 a_1 \dots a_{n-1}.$$

Let $L(\sigma)$ be the length of σ ($=n$) and $S(\sigma)$ be the sum of the integers ($=a_0 + \dots + a_{n-1}$). Then one can see that:

(i) $L(\sigma) + S(\sigma)$ does not change before and after applying operation (1) or (2).

(ii) Applying operation (3) increases $L(\sigma) + S(\sigma)$ by x . Namely if $x=0$ then $L(\sigma) + S(\sigma)$ does not change either.

Since the final answer clearly does not exceed $L(\sigma) + S(\sigma)$, the number of times operation (3) for positive x (that makes $L(\sigma) + S(\sigma)$ larger at least one) is applied must be less than the value of the answer. Therefore the total number of necessary applications of operations (1)-(3) depends on the number of applications of those that do not change $L(\sigma) + S(\sigma)$ or operations (1), (2) and (3) for $x=0$. It should be observed that the application of the operations is deterministic, i.e., the next operation is determined uniquely by the current σ . For example, when $\sigma = a_0 \dots a_{n-2} a_{n-1}$, operation (3) is applied if and only if both a_{n-2} and a_{n-1} are positive. Now one can see that a lot of consecutive applications of operations (1), (2) and (3) for $x=0$ occur when σ looks like

$$\sigma = \sigma_1 a 11 \dots 1b \quad (m \text{ 1's}, a \geq 2 \text{ and } b \geq 1).$$

For σ of this form, operation (3) has to be applied b times, operation (2) once and then operation (1) $b+1$ times, which leaves

$$\sigma = \sigma_1 a 11 \dots 1(b+1) \quad (m-1 \text{ 1's}).$$

This sequence of operations is repeated until we run out all the 1's. Thus those operations that do not change $L(\sigma)+S(\sigma)$ can only continue $O(y \cdot m)$ times. Since both y and m are less than the final answer, the total number of the whole operations do not exceed the cube of the answer.

To get $\bar{A}_n(n)$, we compute $A(i,i)$ for $i=1,2,\dots$ successively until the value exceeds the input n . On the tape, each a_i on the string σ may be represented by a binary number or a unary number. Clearly it does not take so many steps to carry out the operations (1)-(3). Also it should be noted that if $L(\sigma)$ becomes larger than n for the first time when computing $A(i,i)$ then $\bar{A}_n(n)$ is $i-1$. $\square$

Corollary 2. There do not exist nonconstant lower bounds for the computation time of CRCW-PRAMs.

Proof. It is enough to show that for any recursive function $g(n)$ there exists a polynomial-time function $f(n)$ such that:

$$(i) f(n) \leq \max_{0 \leq i \leq n} g(i) \text{ and}$$

(ii) $f(n)$ is not bounded by any constant if g is not.

Let M be a Turing machine which computes $g(n)$. We construct the Turing machine T that computes $f(n)$ as follows: By simulating M , T computes $f(1)$, $f(2)$ and so on successively and at the same time it counts the number N of its moving steps (one step for the simulation of M 's one step). Suppose that when N becomes n (the input to T) T is computing $f(i)$. Then T halts with leaving on its output tape the maximum value in $f(1)$, $f(2)$, $\dots$, and $f(i-1)$. It is not hard to see this $f(n)$ meets the above conditions (i) and (ii). $\square$

3. Star-Free Regular Expressions vs. Unbounded Fan-In Circuits.

In this section we show a sufficient condition analogous to the Unger's well-known one [5] that says if a language L over $\{0,1\}$ is a regular set then L can be recognized by *bounded fan-in* circuits of depth $O(\log n)$ and size $O(n)$. Our present one is:

Theorem 3. Let Σ be an alphabet, h be a homomorphism from Σ into $\{0,1\}^*$ and R be a star-free regular expression over Σ . Then if the on-set of a Boolean function f can be given by $h(L(R))$ ($L(R)$ is the language generated by R), f is computed by an unbounded fan-in circuit C of a constant depth and a polynomial size on n .

By definition, a *star-free regular expression* over alphabet Σ is a regular expression that can use ϕ , ϵ , σ for each σ in Σ and, as operations, complement $\bar{}$, union $\cup$, intersection $\cap$ and concatenation $\cdot$. Theorem 3 could help to introduce a desirable uniformity to the unbounded fan-in circuits. (The common uniformity for the bounded fan-in circuits [3] can of course be applied to the unbounded fan-in model but it is too weak to discuss relatively low depth complexity. See [2].)

Proof of Theorem 3. Let $\Sigma=\{a_1, a_2, \dots, a_m\}$ and suppose that the regular expression R consists of k subexpressions $R_1, R_2, \dots, R_k=R$. Then we construct Boolean expressions $f_{i,j}^1, f_{i,j}^2, \dots, f_{i,j}^k$ for each i and j such that $0 \leq i \leq j \leq n$ where n is the number of variables $x_1, x_2, \dots, x_n$ of the target Boolean expression (or

equivalently the circuit C). f is obtained as $f = f_{0,n}^k$. Now the expressions $f_{i,j}^l$ are of the following form:

- (i) $R_l = \phi$. Then $f_{i,j}^l = 0$ for all i and j .
- (ii) $R_l = \epsilon$. Then $f_{i,i}^l = 1$ for all i and $f_{i,j}^l = 0$ for all i and j such that $i \neq j$.
- (iii) $R_l = a_t$ ($\in \Sigma$). Suppose that $h(a_t) = c_1 c_2 \cdots c_p$ ($c_1, \dots, c_p \in \{0,1\}$). Then $f_{i,j}^l = 0$ if $j \neq i+p$. Otherwise $f_{i,i+p}^l = x'_{i+1} x'_{i+2} \cdots x'_{i+p}$ where x'_{i+s} is x_{i+s} if $c_s = 1$ and $\overline{x_{i+s}}$ if $c_s = 0$.
- (iv) $R_l = R_p \cup R_q$. Then $f_{i,j}^l = (f_{i,j}^p + f_{i,j}^q)$ for all i and j .
- (v) $R_l = R_p \cap R_q$. Then $f_{i,j}^l = (f_{i,j}^p \cdot f_{i,j}^q)$ for all i and j .
- (vi) $R_l = \overline{R_p}$. Then $f_{i,j}^l = \overline{(f_{i,j}^p)}$ for all i and j .
- (vii) $R_l = R_p \cdot R_q$. Then $f_{i,j}^l = (\sum_{i \leq s \leq j} f_{i,s}^p \cdot f_{s,j}^q)$ for all i and j .

To show the correctness of the construction, we prove the validity of the following sentence by the mathematical induction on the number of operations involved in the expression R :

$$f_{i,j}^l(x_1, x_2, \dots, x_n) = 1 \text{ if and only if } v_{i+1} v_{i+2} \cdots v_j \in h(L(R))$$

where v_{i+s} is the value (0 or 1) of the variable x_{i+s} . Details may be omitted since it is a standard application of the induction method.

As for the number of (unbounded fan-in) gates to realize the expression f , the following observation will be enough: (a) The number of Boolean expressions $f_{i,j}^l$ is $O(n^2)$. (Note that the length of R or the number k of its subexpressions is a constant.) (b) To realize $f_{i,j}^l$ by circuit, we need only $O(1)$ gates for all the construction rules (i)-(vi). (c) For the rule (vii) also, one can see that $O(n)$ gates are enough. Thus the total number of gates necessary for the above construction is $O(n^3)$. $\square$

References

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