

New exponents and Betti numbers
of complement of hyperplanes

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§0. Introduction

The aim of this article is to report the results in [8][9][10] and to give the outlines of their proofs. For further details see the original papers.

We define an n-arrangement as a finite family of hyperplanes through the origin O in $\mathbb{C}^{n+1}$. Let X be an n -arrangement. By $|X|$ denote we the union of all hyperplanes belonging to X . Our subject here is the Poincaré polynomial $P_M(t)$ of $M = \mathbb{C}^{n+1} \setminus |X|$. Let $Q \in \mathbb{C}[z_0, \dots, z_n]$ be a defining equation of $|X|$.

(0.1) Definition. We say that X is free if

$$D(X) := \left\{ \text{germ } \theta \text{ at } O \text{ of holomorphic vector field such that } \theta \cdot Q \in Q \cdot \mathcal{O} \right\}$$

is a free $\mathcal{O}$ -module, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1}, O}$.

*) The author gratefully acknowledges support by the Grant in Aid for Scientific Research of the Ministry of Education of Japan No. 574047.

A germ θ of holomorphic vector field at 0 is said to be homogeneous of degree d , denoted by $\deg \theta = d$, if θ has a local expression

$$\theta = \sum_{i=0}^n f_i \frac{\partial}{\partial z_i}$$

at the origin such that all f_i 's are homogeneous polynomials and all non-zero f_i 's have the same degree d . A little observation leads us to the existence of a system of homogeneous free basis $\{\theta_0, \dots, \theta_n\}$ for $D(X)$ if X is a free n -arrangement. It is easy to see that the set $\{\deg \theta_0, \dots, \deg \theta_n\}$ of non-negative integers depends only on X .

(0.2) Definition. We call $(\deg \theta_0, \dots, \deg \theta_n)$ the exponents of a free n -arrangement X .

Let $(d_0, \dots, d_n)$ be the exponents of a free n -arrangement X . Then our main result here is:

Main Theorem. $P_M(t) = \prod_{i=0}^n (1 + d_i t)$.

Let $G \subset GL(n+1; \mathbb{C})$ be a finite unitary reflection groups acting on $\mathbb{C}^{n+1}$. Then the set of the reflecting hyperplanes of the unitary reflections in G makes an n -arrangement X . Such an arrangement is called a unitary reflection arrangement. Then we can prove that X is free. Moreover its exponents coincide with the exponents of G

which were recently introduced by Orlik-Solomon ([3]).

In this special case our Main Theorem is nothing other than the main result in [3]. For details see [10].

Especially when G is real, our Main Theorem was first proved by Brieskorn ([1] Theorem 6(ii)).

Remark. The class of the free arrangements is far wider than that of the unitary reflection arrangements. In fact many examples suggest that the freeness of arrangement is a combinatorial property ([6]).

In Sect. 1, we study an n -arrangement by a combinatorial method. Our main tool for it is the Möbius function on the lattice associated with the n -arrangement. We shall give a characterization of the Möbius function (1.5). For this purpose we need a notion called i -cumulativity which plays a main role in the proof of Main Theorem. At the end of Sect. 1, we state Proposition A concerning the cumulativity of product of Möbius functions.

In Sect. 2, we try to compute the Hilbert polynomial $H(\mathcal{O}/J(X); \nu)$, where $J(X)$ stands for the Jacobian ideal of the defining equation Q of $|X|$. Assume that X is a free n -arrangement. Then we have an explicit formula (2.9) for $H(\mathcal{O}/J(X); \nu)$ by using the exponents of

X. This formula and Proposition B in Sect. 2, which asserts the cumulateness of the coefficients of $H(\theta/J(X); \nu)$, lead us to the proof of Main Theorem which is in Sect. 3.

Our key results for the proof are a characterization of the Möbius function (1.5), Proposition A, B and the explicit formula (2.9) for $H(\theta/J(X); \nu)$.

Let X be a finite family of hyperplanes in $\mathbb{C}^{n+1}$ or $\mathbb{P}^{n+1}(\mathbb{C})$. The intersection of all hyperplanes belonging to X may be void. We can define the notion of the freeness for X also in this case. Moreover we can define the exponents of X if X is free and prove that

$$P_M(t) = \prod_{i=0}^n (1+d_i t).$$

($M = \mathbb{C}^{n+1} \setminus \bigcup_{H \in X} H$ or $\mathbb{P}^{n+1}(\mathbb{C}) \setminus \bigcup_{H \in X} H$ and $(d_0, \dots, d_n)$ are the exponents of X.) This gives a generalization of Main Theorem. For the full explanation on this generalization, see [9].

§1. Combinatorial study of an n-arrangement

Let X be an n -arrangement in this section.

(1.1) Definition. Let

$$L(X) := \left\{ \bigcap_{H \in A} H; A \subset X \right\},$$

where we interpret that

$$\mathbb{C}^{n+1} = \bigcap_{H \in \emptyset} H.$$

Define the join and meet operations in $L(X)$ by

$$s \vee t = s \cup t,$$

$$\text{and } s \wedge t = \bigcap_{H \text{ runs over a set } \{L \in X; L \supset s \cup t\}} H \text{ for } s, t \in L(X).$$

Then $L(X)$ becomes a lattice which is called the lattice associated with an n -arrangement X .

Write $s \leq t$ if $s \vee t = t$ ($s, t \in L(X)$).

(1.2) Definition. Define the Möbius function μ on $L(X)$ inductively defined by

$$\mu(\mathbb{C}^{n+1}) = 1$$

$$\mu(s) = - \sum_{\substack{t \leq s \\ t \neq s}} \mu(t).$$

(1.3) Definition. The rank of $s \in L(X)$, denoted by $r(s)$, is the length of the longest chain in $L(X)$ below s . Thus

$$r(s) = \text{codim}_{\mathbb{C}^{n+1}} s.$$

For any integer $i \geq 0$, put

$$\mu_i(L(X)) := \sum_{\substack{s \in L(X) \\ r(s)=i}} |\mu(s)|.$$

For any $s \in L(X)$, define a new n -arrangement

$$X_s := \{H \in X; s \subset H\}.$$

Put $\mathcal{A}(X) := \{X_s; s \in L(X)\}$. Consider the mappings

$$\mu_i \circ L : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (i \geq 0)$$

corresponding $Y \in \mathcal{A}(X)$ to $\mu_i(L(Y))$.

We will give a characterization of these mappings $\mu_i \circ L$ ($i \geq 0$). For this purpose we need

(1.4) Definition. For a mapping

$$q : \mathcal{A}(X) \longrightarrow \mathbb{Z},$$

define a new mapping

$$r_i q : \mathcal{A}(X) \longrightarrow \mathbb{Z}$$

by
$$(r_i q)(Y) = q(Y) - \sum_{\substack{s \in L(Y) \\ r(s)=i}} q(Y_s)$$

for any $Y \in \mathcal{A}(X)$ and any integer $i \geq 0$. Denote $r_i r_{i-1} \dots r_0 q$ by $R_i q$.

We say that q is i -cumulative ($i \geq 0$) on X if

$$(R_i q)(X) = 0.$$

(1.5) Theorem. (A characterization of $\mu_i \cdot L$ ($i \geq 0$)).

Assume that the mappings

$$q_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j = 0, 1, 2, \dots)$$

satisfy the following conditions:

- I. $q_0(\phi) = 1$.
- II. $q_j(X_s) = 0$ if $s \in L(X)$ and $r(s) < j$ ($j \geq 0$).
- III. The alternating sum of $q_j(Y)$ ($j = 0, 1, 2, \dots$) is zero if $Y \in \mathcal{A}(X) \setminus \{\phi\}$.
- IV. q_j is j -cumulative on any $Y \in \mathcal{A}(X)$ ($j = 0, \dots, i$).

Then $q_j = \mu_j \cdot L$ ($j = 0, \dots, i$) on $\mathcal{A}(X)$.

Proof. see [8].

Define the mappings

$$q_j : \mathcal{A}(X) \longrightarrow \mathbb{Z} \quad (j \geq 0)$$

by $q_j(Y) = b_j(\mathbb{C}^{n+1} \setminus |Y|) \quad (Y \in \mathcal{A}(X)),$

where the right handside stands for the j -th Betti number of $\mathbb{C}^{n+1} \setminus |Y|$. Then it is not too difficult to show that the conditions I-IV in (1.5) hold true for any $i \geq 0$ (cf. [1] Lemma 3). Thus we have

(1.6) Theorem. For any n -arrangement, we have

$$b_j(\mathbb{C}^{n+1} \setminus |X|) = \mu_j \cdot L(X) \quad (j = 0, 1, 2, \dots).$$

This theorem was first proved by Orlik-Solomon [2].

Let X be a finite family of hyperplanes in $\mathbb{C}^{n+1}$ or $\mathbb{P}^{n+1}(\mathbb{C})$. The intersection of all hyperplanes belonging to X may be void. Put

$$M = \mathbb{C}^{n+1} \setminus \bigcup_{H \in X} H \quad \text{or} \quad \mathbb{P}^{n+1}(\mathbb{C}) \setminus \bigcup_{H \in X} H.$$

We have a formula for $P_M(t)$ by using the Möbius functions also in this case. For further details of this generalization, see [9].

Assume that $Q \in \mathbb{R}[z_0, \dots, z_n]$, a product of real linear forms, is a defining equation of a free n -arrangement X . By combining Main Theorem with (1.6) and the Zaslavsky's result ([11] p. 16 Theorem A), we have

$$\begin{aligned} & \# \{ \text{connected component of } \mathbb{R}^{n+1} \setminus \{Q = 0\} \} \\ &= \sum_{i=0}^{n+1} b_i(\mathbb{R}^{n+1} \setminus |X|) = \prod_{i=0}^n (1+d_i). \end{aligned}$$

This equality was proved when $n = 2$ in [7]. K. Saito proved

$$\# \{ \text{connected component of } \mathbb{R}^{n+1} \setminus \{Q = 0\} \} \leq \prod_{i=0}^n (1+d_i)$$

in [4].

For an arbitrary multi-index $I = (I(1), \dots, I(k))$ composing of k non-negative integers, define

$$\mu_{I \circ L} : \mathcal{A}(X) \longrightarrow \mathbb{Z}$$

$$\text{by } \mu_{I \circ L}(Y) = \prod_{j=1}^k \mu_{I(j) \circ L}(Y). \text{ Define } |I| = \sum_{j=1}^k I(j).$$

One reason why the notion of i -cumulateness plays an important role in our theory is the following

Proposition A. $\mu_{I \circ L}$ is $|I|$ -cumulative.

The proof, which is omitted here, is purely combinatorial (see [8]).

§2. The Hilbert polynomial of $\mathcal{O}/J(X)$

From now on we denote $\mathcal{O}_{\mathbb{A}^{n+1},0}$ simply by $\mathcal{O}$.

Let Q be a defining equation of $|X|$. By ∂Q denote we the Jacobian ideal of Q in $\mathcal{O}$ (i.e., $\partial Q = (\partial Q/\partial z_0, \dots, \partial Q/\partial z_n)\mathcal{O}$). Then ∂Q depends only on X . Define the Jacobian ideal $J(X)$ of X by

$$J(X) = \begin{cases} \partial Q & \text{if } X \neq \emptyset \\ \mathcal{O} & \text{if } X = \emptyset. \end{cases}$$

(2.1) Definition. Introduce a decreasing filtration

$$(\mathcal{O}^k)_m = \underbrace{\mathcal{M}^m \oplus \dots \oplus \mathcal{M}^m}_k \quad (m \geq 0)$$

on an $\mathcal{O}$ -module $\mathcal{O}^k$ ($k > 0$). Then this filtration $((\mathcal{O}^k)_m)_{m \geq 0}$ makes $\mathcal{O}^k$ to be an $\mathcal{M}$ -bonne filtered $\mathcal{O}$ -module (see [5]).

By the natural projection $\mathcal{O} \rightarrow \mathcal{O}/J(X)$, we can introduce an $\mathcal{M}$ -bonne filtration on $\mathcal{O}/J(X)$.

On the other hand, $D(X)$ can be embedded in $\mathcal{O}^{n+1}$ by the correspondence

$$\sum_{i=0}^n f_i (\partial/\partial z_i) \mapsto (f_0, \dots, f_n) \quad (f_i \in \mathcal{O} \ (i = 0, \dots, n)).$$

Denote this mapping by $\alpha: D(X) \rightarrow \mathcal{O}^{n+1}$. So one can induce an $\mathcal{M}$ -bonne filtration on $D(X)$.

From now on we regard $\mathcal{O}^{n+1}$, $\mathcal{O}$, $\mathcal{O}/J(X)$ and $D(X)$ as

$\mathcal{M}$ -bonne filtered $\mathcal{O}$ -modules in the above manners.

(2.2) Definition. Let $M = (M_n)_{n \geq 0}$ be an $\mathcal{M}$ -bonne (decreasingly) filtered $\mathcal{O}$ -module. A polynomial $H(M; \nu)$ is characterized by the property that:

$H(M; \nu) \in \mathbb{Q}[\nu]$ equals the dimension of $\mathcal{O}/\mathcal{M} \simeq \mathbb{C}$ -vector space $M_\nu / M_{\nu+1}$ for sufficiently large ν .

We call $H(M; \nu)$ the Hilbert polynomial of
 $M = (M_n)_{n \geq 0}$.

(2.3) Definition. Let $M = (M_n)_{n \geq 0}$ be a filtered $\mathcal{O}$ -module. Then $M(k) = (M(k)_n)_{n \geq 0}$ is another $\mathcal{O}$ -module defined by $M(k)_n = M_{k+n}$ for $k \in \mathbb{Z}$, $k \geq 0$. Then it is easy to see that

$$H(M(k); \nu) = H(M; k + \nu)$$

for $k \in \mathbb{Z}$, $k \geq 0$.

Let $m = \#X = \deg Q$. Then we have an exact sequence

$$(2.4) \quad 0 \longrightarrow D(X) \xrightarrow{\alpha} \mathcal{O}^{n+1} \xrightarrow{\beta} (\mathcal{O}/Q \cdot \mathcal{O})^{(m-1)} \\ \xrightarrow{\gamma} (\mathcal{O}/J(X))^{(m-1)} \longrightarrow 0,$$

where

$$\beta(f_0, \dots, f_n) = \sum_{i=0}^n f_i (\partial Q / \partial z_i) \quad (f_i \in \mathcal{O} \ (i = 0, \dots, n))$$

and γ is the natural projection. Each mapping above is strictly compatible with each filtration. Thus we have

$$\begin{aligned} & H(\mathcal{O}/J(X); \nu_{+m-1}) \\ &= H(\mathcal{O}/Q \cdot \mathcal{O}; \nu_{+m-1}) - H(\mathcal{O}^{n+1}; \nu) + H(D(X); \nu). \end{aligned}$$

For our convenience, put

$$f^{(m)} = \frac{(f+1) \cdots (f+m)}{m} \text{ and } f^{(0)} = 1$$

for any polynomial f and $m > 0$. Then

$$H(\mathcal{O}; \nu) = \nu^{(n)},$$

and thus

$$H(\mathcal{O}^{n+1};) = (n+1)\nu^{(n)}.$$

It is easy to see that

$$\begin{aligned} & H(\mathcal{O}/Q \cdot \mathcal{O}; \nu_{+m-1}) \\ &= (\nu_{+m-1})^{(n)} - (\nu_{-1})^{(n)} \\ &= m \cdot \nu^{(n-1)} + \sum_{i=2}^n \binom{m+i-2}{i} \nu^{(n-i)}. \end{aligned}$$

Let X be free with its exponents $(d_0, \dots, d_n)$ throughout this section. Then we have

$$H(D(X); \nu) = \sum_{i=0}^n (\nu - d_i)^{(n)},$$

and thus

$$\begin{aligned} (2.5) \quad & H(\mathcal{O}/J(X); \nu + m - 1) \\ &= m \cdot \nu^{(n-1)} + \sum_{i=2}^n \binom{m+i-2}{i} \nu^{(n-i)} - (n+1) \cdot \nu^{(n)} + \sum_{i=0}^n (\nu - d_i)^{(n)} \\ &= \left(m - \sum_{i=0}^n d_i \right) \cdot \nu^{(n-1)} + \sum_{i=2}^n \left\{ \binom{m+i-2}{i} + (-1)^i \sum_{j=0}^n \binom{d_j}{j} \right\} \nu^{(n-i)}. \end{aligned}$$

On the other hand we know that

$$\deg H(\mathcal{O}/J(X); \nu) = \deg(\mathcal{O}/\partial Q; \nu) = \dim \operatorname{Spec}(\mathcal{O}/\partial Q) - 1 \leq n-2$$

if $X \neq \emptyset$. If $X = \emptyset$, then

$$H(\mathcal{O}/J(X); \nu) = 0.$$

Thus we have proved

$$(2.6) \quad \underline{\text{Proposition.}} \quad m = \sum_{i=0}^n d_i.$$

Define $P_i(X)$ ($i = 2, \dots, n$) $\in \mathbb{Z}$ by

$$H(\mathcal{O}/J(X); \nu) = \sum_{i=2}^n P_i(X) \cdot \nu^{(n-i)}.$$

Then we can explicitly compute

$$(2.7) \quad P_i(X)$$

$$= \sum_{j=0}^{i-2} \left\{ (-1)^j \binom{d_0 + \dots + d_n + i - j - 2}{i-j} + (-1)^i \sum_{k=0}^n \binom{d_k}{i-j} \right\} \cdot \binom{d_0 + \dots + d_n - 1}{j}$$

because of (2.5) and (2.6).

(2.8) Definition. Let $k \geq 1$. Let $I = (I(1), \dots, I(k))$

be a multi-index composing of k non-negative integers.

Define

$$\sigma_I(X) = \prod_{i=1}^k \sigma_{I(i)}(d_0, \dots, d_n),$$

where $\sigma_j \in \mathbb{C}[t_0, \dots, t_n]$ ($j \geq 0$) is the elementary symmetric polynomial of degree j . When $k = 1$, we write $\sigma_j(X)$ instead of $\sigma_{(j)}(X)$ ($j \geq 0$). Thus (2.6) asserts that $\#X = \sigma_1(X)$.

The following key lemma is not difficult to be verified:

(2.9) Lemma. For each integer i ($2 \leq i \leq n$), there exist real numbers $c(I; i)$ ($I \in I[i]$), which are independent of X , such that

$$P_i(X) + \frac{1}{(i-1)!} \sigma_i(X) = \sum_{I \in I[i]} c(I; i) \sigma_I(X).$$

Here

$$I[i] := \left\{ I = (I(1), \dots, I(k)); 0 \leq I(j) < i \ (j = 1, \dots, k), \right. \\ \left. \sum_{j=1}^k I(j) \leq i \right\}.$$

Since X is free, any element in $A(X)$ is also free (see [8] (5.5)). Thus we can define the mappings

$$\begin{array}{ccc} P_j : A(X) & \longrightarrow & \mathbb{Z} \ (2 \leq j \leq n) \\ \downarrow & & \downarrow \\ Y & \longmapsto & P_j(Y). \end{array}$$

The following is the most important proposition for the proof of Main Theorem:

Proposition B. P_j is j -cumulative ($2 \leq j \leq n$).

Our proof is difficult and long. See [8] (5.10).

§3. Proof of Main Theorem

In this section we shall prove Main Theorem. The crucial results for our proof are (1.5), Proposition A (§1), Proposition B (§2) and (2.9).

The following is stronger than Main Theorem:

(3.1) Theorem. Let $i \geq 0$. Then we have

- 1)_i $\sigma_i(X) = \mu_i \circ L(X)$ for any free n -arrangement X ,
- 2)_i $\sigma_i : A(X) \rightarrow \mathbb{Z}$ is i -cumulative for any free n -arrangement X .

Proof. When $i \leq 1$, we can verify 1)_i and 2)_i because of (2.6).

Let $i \geq 2$. Assume that 1)_j ($j = 0, 1, \dots, i-1$) hold true. Let X be a free n -arrangement. Recall (2.9), then we have

$$P_i(X) + \frac{1}{(i-1)!} \sigma_i(X) = \sum_{I \in I[i]} c(I; i) (\mu_I \circ L)(X).$$

By Proposition A, we know that $\mu_I \circ L$ is $|I|$ -cumulative. Since $|I| \leq i$ for $I \in I[i]$, we can see that $\mu_I \circ L$ is i -cumulative. Thus we have the i -cumulativity of μ_i because the sum of two i -cumulative mappings is also i -cumulative. This is 2)_i.

Next assume 2)_j ($j = 0, 1, \dots, i$). Let X be a free n -arrangement. Then the assumption implies that the

mappings

$$\sigma_j : \mathcal{A}(X) \rightarrow \mathbb{Z} \quad (j \geq 0)$$

satisfy the condition IV in (1.5). Moreover it is not too difficult to see that the mappings σ_j ($j \geq 0$) also satisfy the conditions I, II and III in (1.5). Thus we can apply (1.5) and have

$$\sigma_i = \mu_i \circ L$$

on $\mathcal{A}(X)$. This is $1)_i$.

Q.E.D.

(3.2) The observation so far shows that the following four data concerning a free n -arrangement X are equivalent:

- (1) The set of the exponents $(d_0, \dots, d_n)$ of X , which is equivalent to the polynomial

$$\sum_{i=0}^n \sigma_i(X) t^i = \prod_{i=0}^n (1 + d_i t),$$

- (2) The Hilbert polynomial $H(\mathcal{O}/J(X); \nu)$ together with $\#X$, which is equivalent to the data

$$(\#X, P_2(X), \dots, P_n(X)),$$

- (3) The polynomial $\sum_{i=0}^n (\mu_i \circ L(X)) t^i,$

(4) The Poincaré polynomial of $M = \mathbb{C}^{n+1} \setminus |x|$, which is equivalent to the data

$$(b_0(M), b_1(M), \dots, b_{n+1}(M)).$$

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