

STATIONARY SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS AND THE BOUSSINESQ EQUATIONS UNDER GENERAL OUTFLOW CONDITION

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Let D be a bounded domain in $\mathbf{R}^n$ ($n=2$ or 3), ∂D its smooth boundary. We consider the existence of solutions to the stationary Navier-Stokes equations

$$(1) \quad \begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \end{cases}$$

under the boundary condition

$$(2) \quad \mathbf{u} = \boldsymbol{\beta} \quad \text{on } \partial D$$

where $\mathbf{u}$ is the velocity vector, p the pressure, $\mathbf{f}$ the external force, ν kinematic viscosity, $\boldsymbol{\beta}$ the velocity vector given on the boundary.

We consider also the similar problem for the stationary Boussinesq equations

$$(3) \quad \begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p + \alpha g T = \mathbf{f} & \text{in } D, \\ -\chi \Delta T + (\mathbf{u} \cdot \nabla) T = 0 & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \end{cases}$$

under the boundary condition

$$(4) \quad \begin{cases} \mathbf{u} = \boldsymbol{\beta} & \text{on } \partial D \\ T = \theta & \text{on } \partial D \end{cases}$$

where T is the temperature, α coefficient of volume expansion, χ thermal diffusivity, $\boldsymbol{\beta}$ and θ prescribed velocity and temperature on the boundary, respectively.

Firstly, we mention some known results for these problem.

The existence of the stationary solutions to the Navier-Stokes equations (1), (2) and the Boussinesq equations (3), (4) is known in general context if, for any $\varepsilon > 0$, there exists an extension $\mathbf{b}_\varepsilon$ of the boundary value $\boldsymbol{\beta}$ to the domain D such that $\operatorname{div} \mathbf{b}_\varepsilon = 0$ in D and the inequality

$$(L) \quad |((\mathbf{u} \cdot \nabla) \mathbf{b}_\varepsilon, \mathbf{u})| \leq \varepsilon \|\nabla \mathbf{u}\|^2, \quad \forall \mathbf{u} \in C_{0,\sigma}^\infty(D)$$

holds, where

$$(\mathbf{u}, \mathbf{v}) = \sum_i \int_D u_i(x) v_i(x) dx$$

$$\|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{1/2}$$

$$C_{0,\sigma}^\infty(D) = \{\mathbf{u} \in C_0^\infty(D) ; \operatorname{div} \mathbf{u} = 0 \text{ in } D\}.$$

Suppose that the boundary ∂D of D is multiply connected,

$$(5) \quad \partial D = \cup_{i=1}^k \Gamma_i \quad (k \geq 2) \quad (\Gamma_i : \text{connected component of } \partial D)$$

and D is inside of Γ_k .

Since $\operatorname{div} \mathbf{u} = 0$, the integral $\int_{\partial D} \mathbf{u} \cdot \mathbf{n} d\sigma$ must vanish, where $\mathbf{n}$ denotes the outward normal vector to the boundary. Let us call this condition general outflow condition (GOC).

$$(GOC) \quad \int_{\partial D} \beta \cdot \mathbf{n} d\sigma = \sum_{i=1}^k \int_{\Gamma_i} \beta \cdot \mathbf{n} d\sigma = 0$$

Theorem 1. (*Leray[8], Hopf[6], Fujita[3], Ladyzhenskaya[7]*)

Suppose the following condition is satisfied.

$$(OC) \quad \int_{\Gamma_i} \beta \cdot \mathbf{n} d\sigma = 0 \quad (1 \leq i \leq k)$$

Then the inequality (L) holds true.

Remark 1. *If the boundary is multiply connected, the condition (OC) is stronger than the condition (GOC). On the other hand, if ∂D is connected, then (GOC) and (OC) are equivalent and*

$$\int_{\partial D} \beta \cdot \mathbf{n} d\sigma = 0 \implies (L) \text{ holds true.}$$

When (OC) does not hold, we know the following fact due to the work of Takeshita.

Theorem 2. (*Takeshita[14]*) *Let D be a bounded domain in $\mathbf{R}^2$ the boundary of which consists of 2 connected components $\partial D = \Gamma_1 \cup \Gamma_2$. Suppose that we can insert a circle between Γ_1 and Γ_2 . If the boundary integral*

$$\int_{\Gamma_1} \beta \cdot \mathbf{n} d\sigma = - \int_{\Gamma_2} \beta \cdot \mathbf{n} d\sigma \neq 0,$$

then (L) does not hold true.

Therefore we can not use the method of Theorem 1 to show the existence of stationary solutions to the Navier-Stokes equations (1), (2). Nevertheless this does not mean the non-existence of solutions. In fact, Amick showed the existence of solution under the assumption of "symmetry" for 2-D case.

Theorem 3. (*Amick[1]*) *Let D be a bounded domain in $\mathbf{R}^2$. If D , $\mathbf{f}$, β are symmetric with respect to a line ℓ , and every Γ_i intersects with ℓ , then a solution exists.*

Motivated the work of Takeshita, we found the following exact solution for 2-D annular domain

$$D = \{\mathbf{x} \in \mathbf{R}^2; R_1 < |\mathbf{x}| < R_2\}, \partial D = \Gamma_1 \cup \Gamma_2, \Gamma_i = \{|\mathbf{x}| = R_i\} (i = 1, 2).$$

Example 1. (Morimoto[9], see also [10]) Suppose $\mathbf{f} = \mathbf{0}$ and the boundary value:

$$\beta = \frac{\mu}{R_i} \mathbf{e}_r + \omega_i R_i \mathbf{e}_\theta \text{ on } \Gamma_i (i = 1, 2),$$

where μ, ω_1, ω_2 are given constants. Then the boundary value problem (1) (2) has the following solution. The velocity $\mathbf{u}_0$ is given by

$$\mathbf{u}_0 = \mathbf{u}_0(\mu) = \frac{\mu}{r} \mathbf{e}_r + b(\mu, r) \mathbf{e}_\theta.$$

$$(i) \text{ If } \mu \neq -2\nu, b(\mu, r) = \frac{1}{r} (c_1 + c_2 r^{2+\frac{\mu}{\nu}}),$$

$$(ii) \text{ If } \mu = -2\nu, b(\mu, r) = \frac{1}{r} (c_1 + c_2 \log r),$$

where c_1, c_2 are appropriate constants. The pressure $p_0 = p_0(\mu)$ can be obtained from the equation.

As for the perturbation of the above solution, we have

Theorem 4. (Morimoto-Ukai[13])

Let $D = \{\mathbf{x} \in \mathbf{R}^2; R_1 < |\mathbf{x}| < R_2\}$, $\mathbf{f} = \mathbf{0}$ and the boundary value:

$$\beta = \left\{ \frac{\mu}{R_i} + \varphi_i(\theta) \right\} \mathbf{e}_r + \left\{ \omega_i R_i + \psi_i(\theta) \right\} \mathbf{e}_\theta \text{ on } \Gamma_i (i = 1, 2),$$

where μ, ω_1, ω_2 are given constants and $\varphi_i(\theta), \psi_i(\theta)$ are 2π -periodic functions, the integral of which over the interval $[0, 2\pi]$ vanishes. Suppose the inequality

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1} \right)^2 < 2\nu$$

hold. Then there exists at most discrete countable set $\mathcal{M}$ such that for each $\mu \in \mathbf{R} \setminus \mathcal{M}$ the boundary value problem (1), (2) has a solution for sufficiently small $\varphi_i(\theta), \psi_i(\theta)$ ($i = 1, 2$).

Remark 2. ω_i ($i = 1, 2$) can be large but the difference $|\omega_1 - \omega_2|$ should be small.

For the general domain D in $\mathbf{R}^2$ or $\mathbf{R}^3$, the boundary of which is multiply connected, we have

Theorem 5. (Fujita-Morimoto[4])

Suppose that $\mathbf{f} \in V'$ is a potential force, that $\beta = \mu \beta_0 + \beta_1$, where μ is a constant, β_0 is the boundary value of gradient of a harmonic function $\varphi \in H^2(D)$, and that β_1 is in $H^{1/2}(\partial D)$ with

$$\int_{\partial D} \beta_1 \cdot \mathbf{n} d\sigma = 0.$$

Then, there exists a discrete countable set $\mathcal{M} \subset \mathbf{R}$ such that for each $\mu \in \mathbf{R} \setminus \mathcal{M}$, there exists a weak solution to (1), (2) if β_1 satisfies the inequality $\|\beta_1\|_{H^{1/2}(\partial D)} < C^*$ for some positive constant $C^* = C^*(\nu, \mu, D, \beta_0)$.

Remark 3. The boundary value β_0 may not satisfy the vanishing outflow condition. A non-trivial example of such β_0 in 3-dimensinal case is

$$\sum_{i=1}^{k-1} \nabla \left(\frac{q_i}{4\pi|x - a_i|} \right)$$

where q_i 's are constants and a_i 's are points outside D , each a_i being enclosed by Γ_i .

In the following case, the set $\mathcal{M}$ is void, that is, for every μ , solutions exist for sufficiently small β_1 .

Theorem 6. (Fujita-Morimoto-Okamoto[5], Morimoto[12])

In case of 2-D annular domain and

$$\beta_0 = \nabla \log r|_{\partial D}$$

the set of exceptional values $\mathcal{M}$ in Theorem 5 is void.

As for the Boussinesq equations, we obtain the following results.

Theorem 7. (Morimoto[11]) Suppose that $\mathbf{f} \in V'$ is a potential force, that $\beta = \mu\beta_0 + \beta_1$, where μ is a constant, β_0 is the boundary value of gradient of a harmonic function $\varphi \in H^2(D)$, and that β_1 is in $H^{1/2}(\partial D)$ with

$$\int_{\partial D} \beta_1 \cdot \mathbf{n} d\sigma = 0.$$

Suppose that θ_0 is in $H^{1/2}(\partial D)$. Then, there exists a discrete countable set $\mathcal{M} \subset \mathbf{R}$ such that for each $\mu \in \mathbf{R} \setminus \mathcal{M}$, there exists a solution to (3) (4), if α , $\|\beta_1\|_{H^{1/2}(\partial D)}$, $\|\theta_0\|_{H^{1/2}(\partial D)} < C^*$ holds for some positive constant $C^* = C^*(\nu, \chi, \mu, D, \beta_0)$.

Remark 4. The set of exceptional value $\mathcal{M}$ is the same as in the Navier-Stokes equations case.

Theorem 8. (Morimoto[11]) In case of 2-D annular domain and

$$\beta_0 = \nabla \log r|_{\partial D}$$

the set of exceptional values $\mathcal{M}$ in Theorem 7 is void.

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