

ON AUGMENTED SCHOTTKY SPACES AND INTERCHANGE OPERATORS

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50. Introduction.

Problem 1. Let S be a compact Riemann surface with nodes.

Does there exist a point in an augmented Schottky space representing the surface S ?

Problem 2. We give a point τ in an augmented Schottky space

$\widehat{\mathcal{G}}_g(\widetilde{\Sigma}_0)$ associated with a basic system of Jordan curves $\widetilde{\Sigma}_0$,

which represents a compact Riemann surface S with nodes. Then

for any sequence of points $\{\tau_n\}$ in the Schottky space $\mathcal{G}_g(\widetilde{\Sigma}_0)$

tending to the point τ , does the Riemann surface $S(\tau_n)$ repre-

sented by τ_n converge to S as marked surfaces as $n \rightarrow \infty$?

The answer to Problem 1 is affirmative:

THEOREM 1. There exists a point in an augmented Schottky

space which represents a given Riemann surface with nodes.

The answer to Problem 2 is negative in the general case,

namely in the case where $\widetilde{\Sigma}_0$ is a basic system of Jordan curves.

However the answer is affirmative in a special case, namely in

the case where $\widetilde{\Sigma}_0$ is a standard system of Jordan curves. Now

the following question arises: To what Riemann surface does the

sequence of Riemann surfaces $\{S(\tau_n)\}$ converge as marked surface as $n \rightarrow \infty$ in the general case ?

THEOREM 2. Given a point $\tau \in \widehat{\mathcal{G}}_g^*(\tilde{\Sigma}_0)$. Then there exists a sequence of points $\{\tau_n\} \subset \mathcal{G}_g(\tilde{\Sigma}_0)$ tending to τ such that $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces.

THEOREM 3. Let $\langle G_0 \rangle$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed basic system of Jordan curves for $\langle G_0 \rangle$, respectively. Given a point $\tau \in \delta^{I,J} \mathcal{G}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Let $\tilde{\Sigma}_0^*$, I^* , and J^* be a basic system of loops, a subset of I , and a subset of J , respectively, obtained from $\tilde{\Sigma}_0$, I and J by applying certain interchange operators. Let $\tau^* \in \delta^{I^*,J^*} \mathcal{G}_g(\tilde{\Sigma}_0^*)$ be a point representing a compact Riemann surface with $|I^*| + |J^*|$ nodes. Then there exists the following sequence of points $\{\tau_n\} \subset \mathcal{G}_g(\tilde{\Sigma}_0)$:

$$\tau_n \rightarrow \tau \quad \text{and} \quad S(\tau_n) \rightarrow S(\tau^*) \quad \text{as } n \rightarrow \infty,$$

as marked surfaces.

§ 1. Definitions.

DEFINITION 1. Let $C_1, C_{g+1}; C_2, C_{g+2}; \dots; C_g, C_{2g}$ be a set of $2g$ mutually disjoint Jordan curves on the Riemann sphere $\hat{\mathbb{C}}$ which comprize the boundary of a $2g$ -ply connected region ω . Suppose there are g Möbius transformations $A_1, \dots, A_g$ which have the property that A_j maps C_j onto C_{g+j} and $A_j(\omega) \cap \omega = \emptyset$ ($1 \leq j \leq g$). Then A_j ($j=1, 2, \dots, g$) generates a marked Schottky

group $\langle G \rangle = \langle A_1, A_2, \dots, A_g \rangle$. $C_1, \dots, C_{2g}$ are called defining curves of $\langle G \rangle$.

We say two marked Schottky groups $\langle G \rangle = \langle A_1, \dots, A_g \rangle$ and $\langle \hat{G} \rangle = \langle \hat{A}_1, \dots, \hat{A}_g \rangle$ being equivalent if there exists a Möbius transformation T such that $\hat{A}_j = TA_jT^{-1}$ ($j=1, 2, \dots, g$), and we denote it by $\langle G \rangle \sim \langle \hat{G} \rangle$.

DEFINITION 2. The Schottky space of genus g , denoted by $\mathcal{G}_g$, is the set of all equivalent classes of Schottky groups of genus $g \geq 1$.

DEFINITION 3. Let $C_1, \dots, C_{2g}$ be defining curves of $\langle G \rangle = \langle A_1, \dots, A_g \rangle$. If mutually disjoint Jordan curves $C_1, \dots, C_{2g}; C_{2g+1}, \dots, C_{4g-3}$ on $\hat{\mathbb{C}}$ have the following properties (i) and (ii), then we call $\tilde{\Sigma} = \{C_1, \dots, C_{2g}; C_{2g+1}, \dots, C_{4g-3}\}$ a basic system of Jordan curves (B.S.J.C.) for $\langle G \rangle$: (i) C_{2g+j} ($j=1, \dots, 2g-3$) lie in ω . (ii) Each component of $\hat{\mathbb{C}} \setminus \bigcup_{j=1}^{2g-3} C_{2g+j}$ is a triply connected domain. In particular, if a B.S.J.C. $\tilde{\Sigma}$ has the following property (iii), we call $\tilde{\Sigma}$ a standard system of Jordan curves (S.S.J.C.) for $\langle G \rangle$: (iii) For each $i=1, 2, \dots, g$ and $j=1, 2, \dots, 2g-3$, C_i and C_{g+i} lie on the same side of C_{2g+j} . See Examples 1 and 2 on p.13.

DEFINITION 4. Let S be a compact Riemann surface. We call the set $\Sigma = \{\alpha_1, \dots, \alpha_g; \gamma_1, \dots, \gamma_{2g-3}\}$ of loops on S having the following property a basic system of loops (B.S.L.): Each component of $S \setminus \bigcup_{i=1}^g \alpha_i \setminus \bigcup_{j=1}^{2g-3} \gamma_j$ is a planar and triply connected region. If, in particular, the number of nondividing loops is equal

to g , we call a B.S.L. Σ a standard system of loops (S.S.L.).

Let $\Omega(G)$ be the region of discontinuity of $\langle G \rangle$. Let $\Pi: \Omega(G) \rightarrow \Omega(G)/\langle G \rangle = S$ be the natural projection. If $\tilde{\Sigma} = \{C_1, \dots, C_{2g}; C_{2g+1}, \dots, C_{4g-3}\}$ is a B.S.J.C. (resp. S.S.J.C.), then the projection $\Sigma = \Pi(\tilde{\Sigma}) = \{\alpha_1, \dots, \alpha_g; \gamma_1, \dots, \gamma_{2g-3}\}$, $\alpha_i = \Pi(C_i)$ and $\gamma_j = \Pi(C_{2g+j})$, is a B.S.L. (resp. S.S.L.). We call Σ the projection of $\tilde{\Sigma}$. See Examples 1 and 2 on p.13.

§2. Introduction of new coordinates to $\mathcal{G}_g$.

We fix a marked Schottky group $\langle G_0 \rangle = \langle A_{1,0}, \dots, A_{g,0} \rangle$. Let $\tilde{\Sigma}_0 = \{C_{1,0}, \dots, C_{2g,0}; C_{2g+1,0}, \dots, C_{4g-3,0}\}$ be a fixed B.S.J.C. for $\langle G_0 \rangle$. Let $\langle G \rangle = \langle A_1, \dots, A_g \rangle$ be a marked Schottky group. Let λ_j ($|\lambda_j| > 1$), p_j and p_{g+j} be the multiplier, the repelling and the attracting fixed points of A_j , respectively. We normalize $\langle G \rangle$ by setting $p_1 = 0$, $p_{g+1} = \infty$ and $p_2 = 1$. Then a point in the Schottky space $\mathcal{G}_g$ is identified with

$$\tilde{\tau} = (\lambda_1, \dots, \lambda_g, p_{g+2}, p_3, p_{g+3}, \dots, p_g, p_{2g}) \in \mathbb{C}^{3g-3}.$$

Now we will introduce new coordinates with respect to $\tilde{\Sigma}_0$:

$$\tau = (t_1, t_2, \dots, t_g, \rho_1, \dots, \rho_{2g-3}) \in \mathbb{C}^{3g-3}.$$

First define t_i by setting $t_i = 1/\lambda_i$ ($i=1, \dots, g$). Thus $t_i \in D^* = \{z | 0 < |z| < 1\}$. Next in order to define ρ_j associated with $C_{2g+j} = C(i_0, i_1, \dots, i_\mu) \in \tilde{\Sigma}_0$ ($j=1, 2, \dots, 2g-3$), we determine integers $k(j)$, $\ell(j)$, $m(j)$ and $n(j)$ which are ≥ 1 and $\leq 2g$ as

follows, where $C(i_0, i_1, \dots, i_\mu)$ is the multi-suffix of C_{2g+j} (see [4] for the definition): $k(j) = 1$, $C_\ell(j) = C(i_0, i_1, \dots, i_{\mu-1}, 1-i_\mu, 0, \dots, 0)$, $C_m(j) = C(i_0, i_1, \dots, i_\mu, 0, \dots, 0)$ and $C_n(j) = C(i_0, i_1, \dots, i_\mu, 0, \dots, 0)$. The coordinate ρ_j is now defined as follows:

$$(p_k(j), p_\ell(j), p_m(j), p_n(j)) = (0, 1, \infty, \rho_j),$$

where (a, b, c, d) means the cross ratio of a, b, c , and d .

We define a mapping ϕ by $\phi(\langle G \rangle) = \tau$. We note that if $\langle G \rangle \sim \langle \hat{G} \rangle$, then $\phi(\langle G \rangle) = \phi(\langle \hat{G} \rangle)$. We denote by $\mathcal{G}_g(\tilde{\Sigma}_0)$ the set

$$\mathcal{G}_g(\tilde{\Sigma}_0) = \{\tau = \phi(\langle G \rangle) \mid \langle G \rangle \in \mathcal{G}_g\}.$$

Then $\mathcal{G}_g(\tilde{\Sigma}_0) \cong \mathcal{G}_g$ and $\mathcal{G}_g(\tilde{\Sigma}_0) \subset D^{*g} \times (C \setminus \{0, 1\})^{2g-3}$. We call $\mathcal{G}_g(\tilde{\Sigma}_0)$ the Schottky space associated with $\tilde{\Sigma}_0$.

§3. Augmented Schottky spaces.

Let $\langle G_0 \rangle$ and $\tilde{\Sigma}_0$ be a fixed Schottky group and a fixed B.S.J.C. as in §2.

DEFINITION 5. We say $C_{2g+j} = C(i_1, \dots, i_\mu)$ (resp. $C_i = C(j_1, \dots, j_\sigma)$) is behind $C_{2g+l} = C(i'_1, \dots, i'_\nu)$ if $\nu < \mu$ and $i_k = i'_k$ ($k=1, 2, \dots, \nu$) (resp. $\nu < \sigma$ and $j_k = i'_k$ ($k=1, 2, \dots, \nu$)), and denote the fact $C_{2g+l} < C_{2g+j}$ (resp. $C_{2g+l} < C_i$). Otherwise, we say that C_{2g+j} (resp. C_i) is not behind C_{2g+l} and we denote the fact by $C_{2g+l} \nless C_{2g+j}$ (resp. $C_{2g+l} \nless C_i$).

We define the ordered cycle corresponding to α_i as follows.

We denote the shortest path from C_i to C_{g+i} on the tree of $\tilde{\Sigma}_0$ by

$$(1) \quad C_i, C_{2g+1}^{\delta(1)}(1), C_{2g+1}^{\delta(2)}(2), \dots, C_{2g+1}^{\delta(k)}(k), C_{g+i}$$

(see [4] and Fig. 1 on p.13 in this paper for trees.) Here $\delta(l)$ ($l=1,2,\dots,k$) are determined by $\delta(l) = +1$ or $\delta(l) = -1$ according as $C_{2g+l} < C_{g+i}$ or $C_{2g+l} < C_i$.

DEFINITION 6. The projection

$$(\alpha_i ; \gamma_i^{\delta(1)}(1), \dots, \gamma_i^{\delta(k)}(k))$$

of (1) onto $S_0 = \Omega(G_0)/\langle G_0 \rangle$ is called the ordered cycle corresponding to α_i , and is denoted by $L_{0,i}$.

Let I be a subset of $\{1,2,\dots,g\}$ and J a subset of $\{1,2,\dots,2g-3\}$. We denote by $|I|$ and $|J|$ the cardinality of I and J , respectively. Let $L_{0,j}(1), L_{0,j}(2), \dots, L_{0,j}(t)$ be the complete list of cycles containing γ_j^{δ} , and let $\alpha_{0,k}$ be the " α -loops" contained in $L_{0,k}$ ($1 \leq k \leq t$), where $t = t(j)$ depends on j . We define the subset $I(J)$ of $\{1,2,\dots,g\}$ by

$$I(J) = \{i \in \{1,2,\dots,g\} \mid \alpha_{0,i} \text{ is contained in } L_{0,j}(k) \text{ for some } k (1 \leq k \leq t(j)) \text{ and for some } j \in J\}.$$

Remark. If $\tilde{\Sigma}_0$ is a S.S.J.C., then $I(J) = \emptyset$.

We define the following sets $X = \delta^{I,J} \mathcal{G}_g(\tilde{\Sigma}_0)$ with $I \supset I(J)$:

(i) When $I = J = \emptyset$, we define X as $\mathcal{G}_g(\tilde{\Sigma}_0)$, the Schottky space associated with $\tilde{\Sigma}_0$.

(ii) When $I \neq \emptyset, j = \emptyset$, we define X as follows:

$$\delta^{I, \emptyset} \mathcal{G}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \dots, t_g, \rho_1, \dots, \rho_{2g-3}) \mid t_i = 0 \ (i \in I), t_i \neq 0 \ (i \notin I), \rho_j \neq 1 \ (j=1, \dots, 2g-3), \text{ and } \tau \text{ represents a Riemann surface with nodes such that only } \alpha_i \ (i \in I) \text{ are nodes} \}.$$

(iii) When $I = \emptyset, J \neq \emptyset$, we define X as follows:

$$\delta^{\emptyset, J} \mathcal{G}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \dots, t_g, \rho_1, \dots, \rho_{2g-3}) \mid t_i \neq 0 \ (i=1, \dots, g), \rho_j = 1 \ (j \in J), \rho_j \neq 1 \ (j \notin J) \text{ and } \tau \text{ represents a Riemann surface with nodes such that only } \gamma_j \text{ are nodes} \}.$$

(iv) When $I \supset I(J) \neq \emptyset$, X is defined as follows:

$$\delta^{I, J} \mathcal{G}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \dots, t_g, \rho_1, \dots, \rho_{2g-3}) \mid t_i = 0 \ (i \in I), t_i \neq 0 \ (i \notin I), \rho_j = 1 \ (j \in J), \rho_j \neq 1 \ (j \notin J) \text{ and } \tau \text{ represents a compact Riemann surface such that only } \alpha_i \ (i \in I) \text{ and } \gamma_j \ (j \in J) \text{ are nodes} \}.$$

DEFINITION 7.

$$\widehat{\mathcal{G}}_g^*(\tilde{\Sigma}_0) = \bigcup \{ \delta^{I, J} \mathcal{G}_g(\tilde{\Sigma}_0) \mid I \subset \{1, 2, \dots, g\}, J \subset \{1, 2, \dots, 3g-3\} \text{ with } I \supset I(J) \}$$

is called the augmented Schottky space associated with $\tilde{\Sigma}_0$.

Remark. Let $S(\tau)$ be the Riemann surface represented by τ . $\{S(\tau) \mid \tau \in \widehat{\mathcal{G}}_3^*(\tilde{\Sigma}_0)\}$ is the sets of all Riemann surfaces in Fig.2 and Fig.3 in the cases of Example 1 and Example 2, respectively.

§ 4. Interchange operators.

For simplicity, we will only consider interchange operators in the case of Example 1 (see Fig.4). For detail, see Sato [5]. Choose j with $I(\{j\}) \neq \emptyset$. Let $i \in I(\{j\})$. For these i and j , we introduce the interchange operators $I_g(i,j)$.

Remark. Since $I(J)$ is always empty in the case where $\tilde{\Sigma}$ is a S.S.J.C., we can not define an interchange operator in this case.

For simplicity, we only consider $I_g(1,2)$, which is defined as follows (see Fig.4 on p.15): For a B.S.J.C. $\tilde{\Sigma}$,

$$I_g(1,2)(\tilde{\Sigma}) = \tilde{\Sigma}^* = \{c_1^*, c_2^*, \dots, c_6^*; c_7^*, c_8^*, c_9^*\},$$

where $c_1^* = A_1^{-1}(c_8)$, $c_2^* = A_1^{-1}(c_2)$, $c_3^* = c_3$, $c_4^* = c_8$, $c_5^* = c_5$, $c_6^* = c_6$, $c_7^* = c_7$, $c_8^* = c_1$, and $c_9^* = c_9$.

For a B.S.L. $\Sigma = \{\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2, \gamma_3\}$, $I_g(1,2)(\Sigma) = \{\alpha_1^*, \alpha_2^*, \alpha_3^*; \gamma_1^*, \gamma_2^*, \gamma_3^*\}$, where $\alpha_1^* = \gamma_2$, $\alpha_2^* = \alpha_2$, $\alpha_3^* = \alpha_3$, $\gamma_1^* = \gamma_1$, $\gamma_2^* = \alpha_1$, $\gamma_3^* = \gamma_3$.

For ordered cycles L_1, L_2 and L_3 , $L_1^* = I_g(1,2)(L_1) = (\alpha_1^*; \gamma_2^*, \gamma_1^*)$, $L_2^* = I_g(1,2)(L_2) = (\alpha_2^*; \gamma_2^*, \gamma_1^*, \gamma_3^*)$ and $L_3^* = I_g(1,2)(L_3) = (\alpha_3^*; \gamma_3^{*-1}, \gamma_1^{*-1})$, where we write γ_j^* for γ_j^{*+1} for simplicity.

For a marked Schottky group $\langle G \rangle = \langle A_1, A_2, A_3 \rangle$, $\langle G^* \rangle = I_g(1,2)(\langle G \rangle) = \langle A_1^*, A_2^*, A_3^* \rangle$, where $A_1^* = A_1$, $A_2^* = A_2 A_1$, $A_3^* = A_3$.

We obtain Theorem 1 by using interchange operators. See Sato [5] for details.

§ 5. Relations between limits of Schottky groups and limits of Riemann surfaces.

Here we will consider Problem 2. Let S be a compact Riemann surface of genus g with or without nodes. We denote by $N(S)$ the set of all nodes on S . We assume that each component of $S \setminus N(S)$ has the Poincaré metric. The Poincaré metric $\lambda(z)|dz|$ on S is defined as the Poincaré metric on each component of $S \setminus N(S)$.

DEFINITION 8. If the following conditions are satisfied, a sequence of Riemann surfaces $\{S_n\}$ converges to a surface S as marked surfaces: There exists a locally quasiconformal mapping $\phi_n : S \setminus N(S) \rightarrow S_n \setminus P(S_n)$ such that (i) $\lambda_n(\phi_n(z))|d\phi_n(z)|$ uniformly converges to $\lambda(z)|dz|$ on every compact subset of $S \setminus N(S)$, where $\lambda_n(z)|dz|$ and $\lambda(z)|dz|$ are the Poincaré metrics on S_n and S , respectively, (ii) ϕ_n maps a deleted neighborhood $N(\alpha_i) \setminus \{\alpha_i\}$ (resp. $N(\gamma_j) \setminus \{\gamma_j\}$) of α_i (resp. γ_j) to a deleted neighborhood $N(\alpha_{i,n}) \setminus \{\alpha_{i,n}\}$ (resp. $N(\gamma_{j,n}) \setminus \{\gamma_{j,n}\}$) of $\alpha_{i,n}$ (resp. $\gamma_{j,n}$) if $\alpha_i \in N(S)$ (resp. $\gamma_j \in N(S)$), and (iii) ϕ_n maps a neighborhood $N(\alpha_i)$ (resp. $N(\gamma_j)$) of α_i (resp. γ_j) to a neighborhood $N(\alpha_{i,n})$ (resp. $N(\gamma_{j,n})$) of $\alpha_{i,n}$ (resp. $\gamma_{j,n}$) if $\alpha_i \notin N(S)$ (resp. $\gamma_j \notin N(S)$), where $P(S_n) = f_n^{-1}(N(S))$ and $f_n : S_n \rightarrow S$ is a deformation.

By constructing locally quasiconformal mappings, we have Theorem 2. See Sato [6] for details.

Let $\langle G_0 \rangle$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed B.S.J.C. for $\langle G_0 \rangle$, respectively. Set $S_0 = \Omega(G_0)/\langle G_0 \rangle$. Given a point $\tau \in \delta^{I,J} \mathcal{G}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Then $S(\tau)$ is a compact Riemann surface with $|I| + |J|$ nodes of genus g . We define the following sets: $J_1 = \{j \in J \mid \gamma_j \text{ is a dividing loop on } S_0\}$, $J_2 = \text{any subset of } J \setminus J_1$, $\tilde{\Sigma}_1 = I_g(i_{k(1)}, j_{\ell(1)})(\tilde{\Sigma}_0)$ with $i_{k(1)} \in I(\{j_{\ell(1)}\})$, $j_{\ell(1)} \in J_2$ and $J_{21} = J_2 \setminus \{j_{\ell(1)}\}$. Choose $j_{\ell(2)} \in J_{21}$ such that $I_1(\{j_{\ell(2)}\}) \cap (I(J_2) \setminus \{i_{k(1)}\}) \neq \emptyset$. Set $\tilde{\Sigma}_2 = I_g(i_{k(2)}, j_{\ell(2)})(\tilde{\Sigma}_1)$ with $i_{k(2)} \in I_1(\{j_{\ell(2)}\})$, $i_{k(2)} \neq i_{k(1)}$. We set $J_{22} = J_{21} \setminus \{j_{\ell(2)}\} = J_2 \setminus \{j_{\ell(1)}, j_{\ell(2)}\}$. By the same way, we determined the following: $j_{\ell(3)}, i_{k(3)}, J_{23}, \tilde{\Sigma}_3, I_3(J_{23}); \dots$ $\dots; j_{\ell(s)}, i_{k(s)}, J_{2,s}, \tilde{\Sigma}_s$: Here s is the integer satisfying the following (i) and (ii): (i) $I_{s-1}(\{j_{\ell(s)}\}) \cap I(J_2) \setminus \{i_{k(1)}, i_{k(2)}, \dots, i_{k(s-1)}\} \neq \emptyset$, (ii) $I_s(\{j\}) \subseteq \{i_{k(1)}, \dots, i_{k(s)}\}$ for any $j \in J_2 \setminus \{j_{\ell(1)}, j_{\ell(2)}, \dots, j_{\ell(s)}\}$.

We set $J_3 = J \setminus (J_1 \cup J_2)$, $J_4 = \{j_{\ell(1)}, j_{\ell(2)}, \dots, j_{\ell(s)}\}$, $J_5 = J_2 \setminus J_4$, $I_1 = I \setminus I(J)$, $I_4 = \{i_{k(1)}, i_{k(2)}, \dots, i_{k(s)}\}$, $I_3 = I_s(J_3)$, $I_5 = I \setminus (I_1 \cup I_3 \cup I_4)$, $I_6 = \text{a subset of } I_5$, $I_7 = I_5 \setminus I_6$, $I^* = I \setminus I_7$ and $J^* = J \setminus J_4$. Then we have Theorem 3. See Sato [6] for the proof.

COROLLARY. Given $\tau \in \delta^{I,J} \mathcal{G}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Then there exists a sequence of points $\{\tau_n\} \subset \mathcal{G}_g(\tilde{\Sigma}_0)$ such that (i) $\tau_n \rightarrow \tau$ as $n \rightarrow \infty$ and (ii) $S(\tau_n)$ does not converge to $S(\tau)$ as marked surfaces.

Remark. By a similar method to the proof of Theorem 2, we

have the following. If $\tilde{\Sigma}_0$ is a S.S.J.C., then $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces for any point $\tau \in \hat{\mathcal{G}}_g^*(\tilde{\Sigma}_0)$ and for any sequence of points $\{\tau_n\} \subset \mathcal{G}_g(\tilde{\Sigma}_0)$ with $\tau_n \rightarrow \tau$.

§6. Appendices.

We will consider the following in the forthcoming papers [7,8].

1. Properties of interchange operators. There are five kind of interchange operators as follows: (1) $I_g(\alpha_i, \alpha_i^{-1}) = I_g(C_i, C_{g+i})$, (2) $I_g(\alpha_i, \alpha_j) = I_g(C_i, C_j)$, (3) $I_g(\gamma_j, \gamma_j^{-1}) = I_g(C_{2g+j}^+, C_{2g+j}^-)$, (4) $I_g(\gamma_i, \gamma_j) = I_g(C_{2g+i}, C_{2g+j})$ and (5) $I_g(\alpha_i, \gamma_j) = I_g(C_i, C_{2g+j})$. Here we only considered and used interchanged operators in case (5).

2. Relations between Nielsen isomorphisms and interchange operators. Here Nielsen isomorphisms are

$$N_1(A_1, A_i) : \langle A_1, A_2, \dots, A_i, \dots, A_g \rangle \rightarrow \langle A_i, A_2, \dots, A_1, \dots, A_g \rangle .$$

$$N_2(A_1, A_1^{-1}) : \langle A_1, A_2, \dots, A_g \rangle \rightarrow \langle A_1^{-1}, A_2, \dots, A_g \rangle .$$

$$N_3(A_1, A_2) : \langle A_1, A_2, A_3, \dots, A_g \rangle \rightarrow \langle A_1, A_1 A_2, A_3, \dots, A_g \rangle .$$

3. Boundary behavior of the space of marked Schottky groups of real type of genus 2. We say $\langle G \rangle = \langle A_1, A_2 \rangle$ a schottky group of real type if $A_1, A_2 \in \text{SL}(2, \mathbb{R})$.

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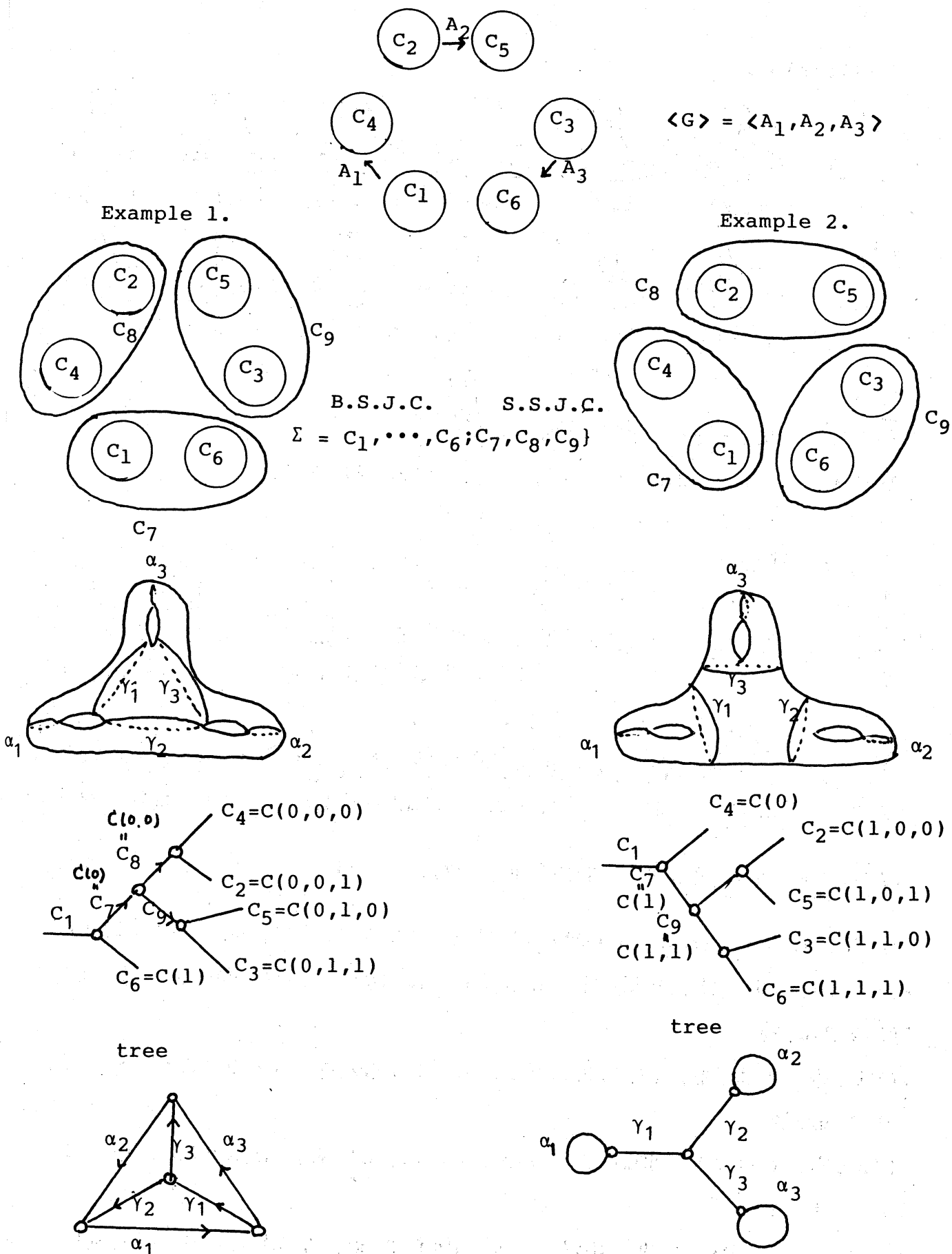


Fig.1

Example 1.

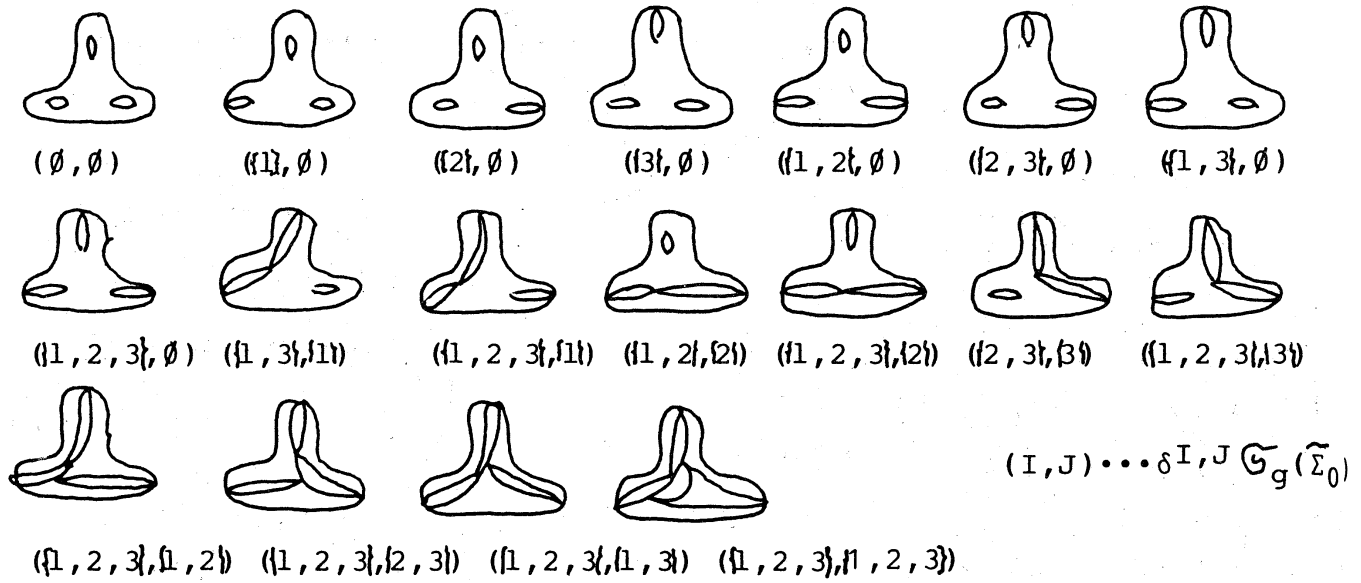


Fig. 2.

Example 2.

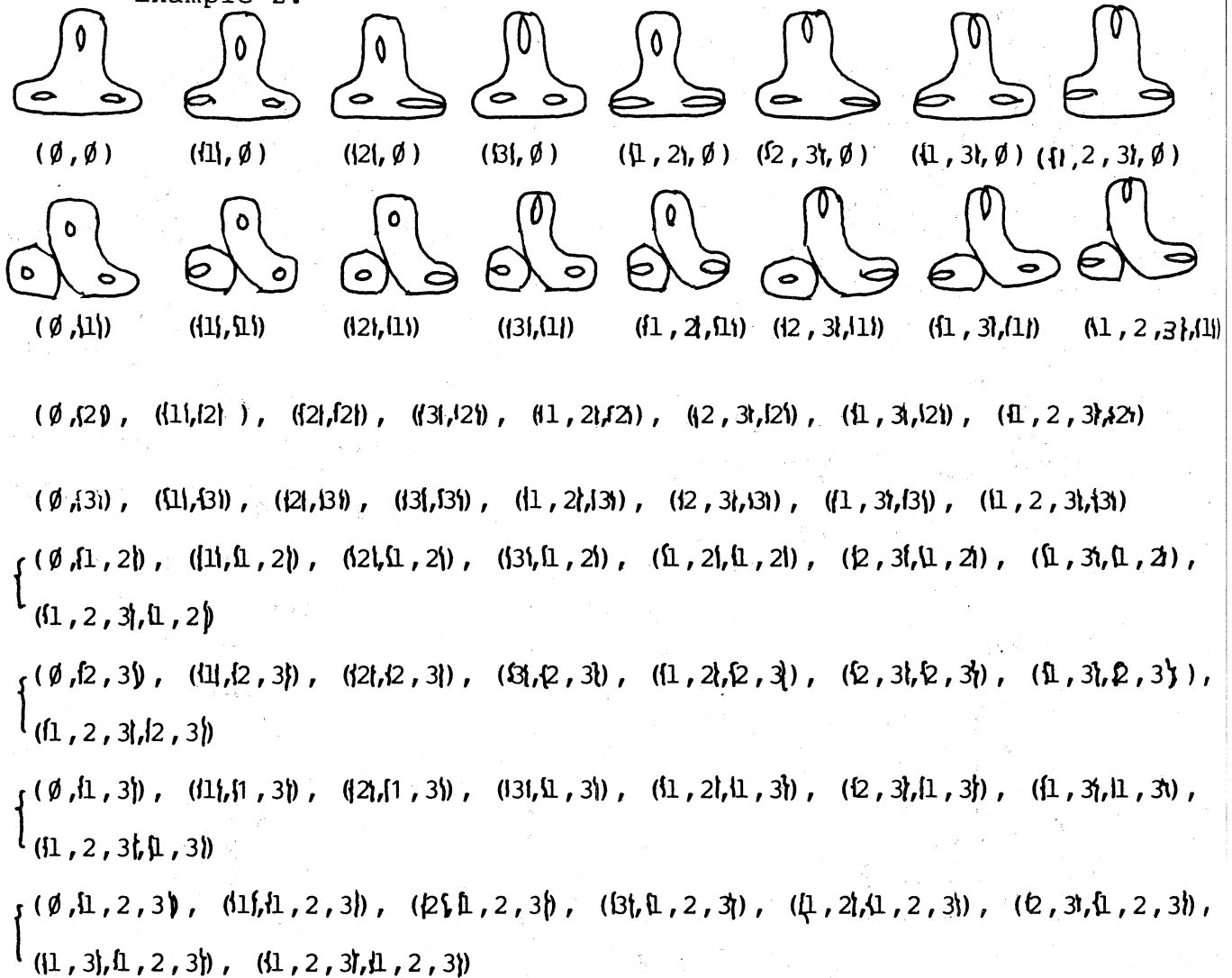
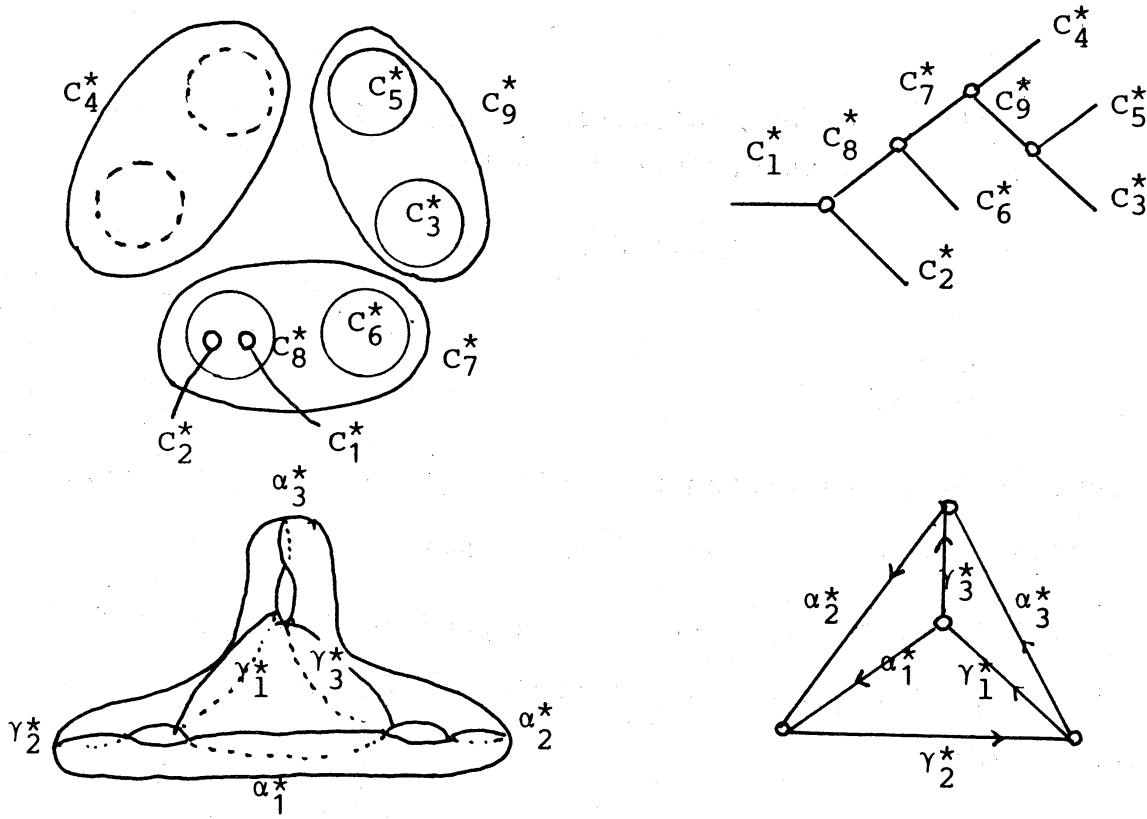


Fig. 3.



$$\tilde{\Sigma}^* = I_g(1,2)(\tilde{\Sigma}) = \{c_1^*, \dots, c_6^*; c_7^*, c_8^*, c_9^*\}$$

$$\Sigma^* = I_g(1,2)(\Sigma) = \{\alpha_1^*, \alpha_2^*, \alpha_3^*; \gamma_1^*, \gamma_2^*, \gamma_3^*\}$$

Fig,4.