

Dynamic Interaction of Crops with Environmental Conditions

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Notations

RBM with Duffing oscillator

Symbol

r	annual precipitation
t	time
u	pollen density in the atmospheric environment
v	index of the resource involving the plant growth
w	index of the plant biomass.
z	standard score of the annual mean temperature
I_h	period of harvesting season
J	annual olive yield in Jordan
M	annual yield
$R(A, B)$	correlation coefficient between the data series A and B
T	mean temperature T
Y	annual olive yield in Syria
α	pollination coefficient
β	release coefficient of pollen
γ	Duffing coefficient
ε	magnitude of external force $\mathbf{F}$
μ	mortality coefficient
ρ	decay coefficient of the pollen density
χ_p	indicator function for the pollination season
ω_e	angular frequency of external force
ω_s	angular frequency of biennial-bearing

Plant-nutrient interactions model

Symbol

f_{ZX}	interaction function for uptake of NO ₃ -N in the soil pore water by X
f_{XX}	interaction function for SNF of N ₂ gas diffusing from the atmosphere into the soil pore
f_{ZY}	interaction function for uptake of NO ₃ -N in the soil pore water by Y
f_{ZZ}	self-interaction function for NO ₃ -N in the soil pore water
B	constant representing the $\delta^{15}\text{N}$ -value of the legume plants grown solely with atmospheric N gas
K_S , K_X , K_Y , and K_Z	growth rates
R_{sample}	molar ratios of $^{15}\text{N}/^{14}\text{N}$ in the sample and the standard
R_{standard}	the molar ratios of $^{15}\text{N}/^{14}\text{N}$ in the standard
X	standardized (between 0 and 1) abundances of the leguminous plant
Y	standardized (between 0 and 1) abundances of the non-leguminous plant
Z	standardized (between 0 and 1) abundances of the nutrient resource
α_X and α_Y	absorption coefficients of the nutrients by the plants
β_X , β_Y , β_Z , and γ_X	shape parameters
$\delta^{15}\text{N}$	stable Nitrogen isotope in atmospheric air
$\delta^{15}\text{N}_{\text{leg}}$	$\delta^{15}\text{N}$ abundance of a legume plant sample
$\delta^{15}\text{N}_{\text{non-leg}}$	^{15}N abundance of the non-legume in the mixed quadrats with the legume.
$\delta^{15}\text{N}_{\text{reference}}$	$\delta^{15}\text{N}$ abundance of a non-legume plant that relies on soil mineral N
μ_X , μ_Y , and μ_Z	mortality or decay coefficients
%Ndfa	percent N derived from the air
%Ntr	percent N of a non-legume in a mixed quadrat transferred from legumes

CHAPTER 1: Introduction

1.1 Research background

Dynamical systems are characterized by difference or differential equations. For over two centuries, studies on differential equations focused on finding analytical solutions. However, analytical or closed-form solutions cannot always determine the behavior of a system, especially since it's challenging to obtain such solutions for nonlinear equations, except in special cases. At the end of the nineteenth century, the field of dynamical systems significantly advanced the understanding of certain nonlinear phenomena. The dynamics of a system are represented as either a discrete-time or continuous-time evolutionary process. The simplest continuous systems mathematical models consist of first-order differential equations. In first-order autonomous systems, which are explicit in time, the dynamics are highly constrained, as their motion is confined to the real line. In contrast, much richer dynamics are exhibited in simple non-autonomous systems (Layek, 2024).

Nonlinear dynamics play a significant role in natural and social sciences. For instance, in physical science it covers areas such as electronic circuits, Earth's atmosphere, fluid turbulence, lasers, and superconductivity; in chemistry it covers Brusselator model and Belousov–Zhabotinsky reaction; in biology it covers biochemical processes, cardiac systems and neural; in social sciences and ecology it covers market price fluctuations and spread of diseases. Generally, nonlinear systems are more difficult to solve compared to linear systems which follow superposition principle, which allows them to be broken down into manageable parts that can be solved separately and then combined to get the final solution. Despite their complexity, understanding solutions of linear systems can help in the analysis of nonlinear systems (Layek, 2024).

It is critical to understand the principle mechanism behind spatiotemporal chaos. Zhang et al. (2024) presented a study on spatiotemporal chaotic behaviors produced by the refuge

effect and their complex nature within a discrete Lotka-Volterra predator-prey model. The conditions for Neimark-Sacker and flip bifurcations approaching the stable equilibrium point were analyzed using the bifurcation parameter called the refuge effect. The complex system's dynamics are analyzed using maximum Lyapunov exponents, bifurcation diagrams, space-amplitude, spatial return maps, phase diagrams, Kolmogorov-Sinai entropy, and space-time diagrams. The system's stability was influenced by the refuge effect in two ways, that is, flip bifurcations and Neimark-Sacker leading to chaos. The unpredictability of chaotic attractors increases the system's chaotic nature. In addition, the characteristics of spatiotemporal dynamics leading to chaos are documented. The chaotic behavior of the system due to the two bifurcation types displays transitions among several typical features, such as pattern competition intermittency, defect chaotic diffusion patterns, frozen random patterns, and fully developed turbulence.

Ali et al. (2024) conducted a study using the modified generalized exponential rational functional method to examine the $(3+1)$ -dimensional Boussinesq equation. This research has revealed new wave solutions, including periodic singular, rational, singular, bright, rational, trigonometric, and other types of soliton solutions. In addition, specific transformations have transformed the equation into a dynamical system. They investigated bifurcation, Hamiltonian and phase portraits of the system. To aid the understanding of the long-term behavior of the system, the Hamiltonian function was used to ensure conservation over time and time-reversal symmetry and conservation. Poincaré maps were used to study the quasi-periodic behavior of the system. The propagation of solitons is represented in contour visualizations, 2-D and 3-D. To analyze optical pulse behavior, appropriate values were assigned to the free parameters and used to develop graphical representation, utilizing the analytically obtained solutions.

El-Sayed et al. (2024) studied two discrete system's chaos control and dynamics associated with singularly perturbed equation related to the Henon difference equation with continuous arguments. The two systems' local stability was examined. For both systems, the

state feedback control method was utilized. Numerical simulations were used to investigate their qualitative behavior. A halving-period bifurcation results from setting bifurcation parameter as the perturbation parameter. In addition, the Niemark-Sacker bifurcation is observed when the place of the two delays of the system are interchanged while not changing the other parameters. In addition, the system showed the existence of multiple periodic orbits, that is, 7-period, 8-period and 11- period orbits. The chaotic behaviors of the system were illustrated using maximal Lyapunov exponent diagrams. An on-off strategy was implemented to demonstrate the practicality of chaos control.

Reis and Savi (2024) examined a dissipative mechanical Duffing-type system by analyzing the chaos and spatiotemporal nonlinear dynamics under external stimulus. Nonlinear wave equation that has cubic nonlinearity governed the system's dynamics. System dynamics such as global and local behaviors were presented using perturbation methods. The evaluation of convective, local and mean exponents was defined using the Lyapunov exponents at different perturbations. They explored various dynamical regimes, while taking into consideration both heterogenous and homogenous stimuli. They observed different dynamical responses, chaotic, quasi periodic and periodic behaviors. The spatial transport of chaos is explained using a new concept of chaotic wave where its velocity is measured by the convective Lyapunov exponents.

In Reis and Savi (2022), the paper focused on investigating spatiotemporal dynamics where a partial differential equation that has cubic nonlinearity governs the conservative Duffing-type system, which corresponds to a network of Duffing oscillators. The research results could be used to improve the design of energy harvesting devices. The research explores general spatiotemporal dynamics and offers tools for classification of different responses. Tools used include a new perturbation approach, energetic analysis, and temporal and spatial Poincaré sections. Numerical simulations include spatial discretization which is analyzed using finite difference and time integration calculated using the fourth-order Runge–Kutta (RK4) method.

Chen (2025) focused on examining Turing bifurcation and used the amplitude equation to find its directions. To aid in understanding various spatiotemporal dynamics and patterns we need to define the direction of bifurcations. However, the direction of Turing bifurcation in reaction-diffusion systems involving predator-taxis has been explored by few researchers. The research thus focused on this approach within the diffusive Lotka-Volterra model through the weakly nonlinear analysis method.

One common approach to finding solutions to dynamical systems is through numerical methods. A significant challenge arises when conventional Runge-Kutta methods prove ineffective, a scenario frequently encountered in delay differential equations. Reviewing different approaches to such problems considered so far, He et al. (2024) proposed a new numerical method to solve the Lotka-Volterra predator-prey models with discrete delays. However, we still employ the conventional RK4 method in numerical analysis of the developed models, which do not explicitly include a delay term, because of its superiority over other time-marching methods in terms of accuracy against the computational cost (Burden and Faires, 2010).

1.2 Outline of dynamical models for crops and environment

Here, we develop two models for crops interacting with environmental conditions: 1) *Resource budget model with Duffing oscillator for dynamics of synchronized biennial-bearing olives in the Levant (Magero et al., 2024)*; and 2) *A dynamical system model for plant-nutrient interactions with symbiotic nitrogen fixation and Allee effect (Guo et al., 2025)*. Henceforth, we refer to the models 1) and 2) as “Resource budget model (RBM) with Duffing oscillator” and “Plant-nutrient interactions model”, respectively.

The RBM with Duffing oscillator is a temporally continuous spatially lumped resource budget model to explain the dynamics of synchronized biennial-bearing olives in the Levant, specifically focusing on Syria, the region’s foremost olive-producing country. The model is a

time-continuous counterpart of the celebrated resource budget model. It consists of a Duffing oscillator coupled with a dynamical model of pollination with an external force propelling olive growth by photosynthesis. The reference data are obtained from statistical databases of international organizations and our own observation systems in Jordan, a country neighboring Syria, providing a wealth of information to refine the model structure. An intensive review of Syria's modern history involving significant shifts in agricultural policy and social stability leads to a conclusion that the model should comprehend the anomaly of olive yield interacting with socio-political factors as an autonomous behavior. The conventional mathematical methodology analyzes the model's characteristics, such as solutions' non-negativity, boundedness, and stability. The system is stable during pollination off-season but may become unstable and unbounded during pollination on-season, which is a property that the time-discrete resource budget model cannot reproduce. A significant finding is that coupling individual fruit trees by anemophily is not essential in synchronization, overturning earlier studies in the literature. The values of model parameters that best fit the historical data of olive yield in Syria result in bounded chaos. With alternative parameter values, the model could exhibit periodic oscillation, instability, or blowing up, as clearly shown in bifurcation diagrams.

The plant-nutrient interactions model deals with the mutual relationship between plant species and the environment, which exhibits outstanding behavior involving symbiotic nitrogen fixation. It is a nonlinear dynamical system model with three variables representing the standardized abundances of a leguminous plant, a non-leguminous plant, and a nutrient resource. We examine the unique existence and boundedness of solutions for initial value problems and delineate the equilibria of the model and their stability. An intensive field experiment was conducted to validate the model in real-world applications, using *Trifolium repens* (clover) and *Mentha × piperita* (mint) on a horizontally laid plot and an inclined plot. Stable isotope analysis for plant samples semi-quantifies the fate of nitrogen atoms involving isotope fractionation and verifies the model structure. Image processing of the plants' leaf areas

and photometer analysis for soil pore water samples determine the abundances of clover, mint, and nitrate-nitrogen in the soil. Repetitive use of the least square method identifies the values of the model parameters from the observed and standardized abundances of the three variables, revealing that a strong Allee effect exists in the interaction between clover and the soil environment. The model is comprehensive of different cases regarding asymptotic behaviors of the dynamical system, which may experience bifurcations.

CHAPTER 2: RBM with Duffing oscillator

2.1. Introduction

Biennial or alternate bearing refers to the well-known phenomenon in fruit trees where the fruit bearing increases or decreases every other year. The biennial bearing of individual fruit trees is often synchronized, referred to as masting for large seed crops (Lyles et al., 2009). Synchronized biennial-bearing becomes evident in statistics of large-scale fruit production, such as at a country level (Waller, 1979). Kelly (1994) defined types of masting behavior based on the statistical distribution of fruit production as a synchronized periodical bearing exhibits bimodal distribution. There are several evolutionary hypotheses to explain the rationale of synchronized biennial bearing: resource allocation hypothesis, economy of scale hypothesis in regard to pollination efficiency, predator satiation hypothesis, accessory costs hypothesis, and large seed hypothesis, as Nussbaumer et al. (2016) summarized the original notions of Kelly (1994). These hypotheses are often supported by evidence collected from wild plants under temperate and continental environments. However, the accessory costs hypothesis and the large seed hypothesis do not require synchronized fruit production (Nussbaumer et al., 2016).

Under the arid and semiarid environment, the East Mediterranean Region is historically recognized as the Levant and divided into several countries, such as Syria and Jordan, and disputed territories. Biennial-bearing tree crops having commercial or ecological significance in the Levant include *Olea europaea* L. (olive), *Prunus armeniaca* L. (apricot), *Pistacia vera* L. (pistachio), and *Quercus* species (oak). The olive is of paramount importance in the cuisine and landscape of the Levant. Aburuz et al. (2011) estimated that between 20%–50% of Jordanian households cultivate 10–20 olive trees each for commercial or ornamental purposes, considering them the source of pollen to cause allergic diseases. The cultivation of olive trees is predominantly rainfed in the Levant, though the percentage of irrigated olive trees is an increasing trend (Cafiero et al., 2009). The synchronized biennial bearing is more prevalent in

the statistics of fruit yields, particularly the olive, in the Levant countries rather than fluctuations associated with rainfall and other climatic conditions. These phenomena are thought to have been accepted in most fruit-producing regions since ancient civilizations, but decoding their mechanisms is still a debate in modern science. Yang and Midmore (2005) developed a time-continuous dynamical model to explain a mechanism of how plants allocate resources and partition growth at the whole-plant level, assuming nitrogen uptake in the root system and carbon fixations by photosynthesis in the shoot system are two dominant processes to regulate the morphological adjustment mechanism. The model consisted of ordinary differential equations (ODEs) with multiple dependent variables representing the biomass of partitioned plant systems and resource substrate pools. However, the resource allocation hypothesis detailed in the model was dominantly linear and not linked with the synchronized biennial bearing. The significance of nonlinearity as a fundamental component in ecosystems is broadly recognized. Notably, even a time-discrete single population model with density-dependent nonlinearity can display behaviors characterized by bifurcating chaos (Duarte et al., 2012).

The celebrated resource budget model (RBM) is a nonlinear deterministic time-discrete dynamical system explaining the mechanism of the biennial bearing, which is possibly synchronized due to coupling by anemophily (Isagi et al., 1997). A RBM represents the stock and flow of photosynthate as a resource in an individual plant and can capture oscillation phenomena; the global coupling of the oscillators well explains the synchronization of fruit production in such a case that the dominant cause of the coupling is pollination (Esmaceli et al., 2021; Satake and Iwasa, 2000; Tachiki et al., 2010). The RBM by Prasad et al. (2017) detailed the masting of fruit crops, such as acorns of anemophilous oak trees, demonstrating that a bifurcation point divided the states of the resource into a one-year periodic regime and a multi-band chaotic regime. Lyles et al. (2015) conducted numerical experiments of RBM to attribute masting to stochastic environmental noise. Depending on anemophilous pollination, like oak

trees, xenogamy is a well-known characteristic of olive trees. Mazzeo et al. (2014) conducted intensive research on the link between the amount and quality of olive pollen with biennial bearing. However, those earlier studies in the literature did not present convincing evidence that anemophilous pollination eventually contributes to the global coupling of biennial-bearing olive trees.

This study aims to develop a time-continuous counterpart of the RBM, which statistically reproduces the behavior of annual olive yield at a country level in the Levant. One of the key components of the model is the Duffing oscillator, which exhibits a wide range of anomalous phenomena, such as coexisting attractors, chaotic and regular motions, global and local bifurcations, and fractal and regular basin boundaries, due to its nonlinearity (Alves et al., 2003; Ueda, 1991). Luo and Min (2011) reviewed the history of synchronization research since Huygens in the 17th century (Willms et al., 2017), then analyzing synchronization for chaos in the coupled master and slave systems of a periodically forced Duffing oscillator with a periodically excited pendulum. Under specific parameters, the two systems can be partially and fully synchronized. The forced pendulum has rotational and librational chaotic motions, and the synchronization of the periodically forced Duffing oscillator can only occur during the librational chaotic motions of the pendulum. The model developed here extends the Duffing oscillator to include the effect of pollination on temporarily destabilizing the system, not on coupling, and its state variables are the pollen density in the atmospheric environment, an index of the resource involving the plant growth, and an index of the plant biomass. Assigning those biophysical quantities to the state variables implies that they are non-negative but possibly unbounded or unstable. Nonlinearity involving square root terms, as common in modeling symbiotic phenomena (Ajraldi et al., 2011), is assumed on the deposit of the pollen to achieve fast decay. The extended Duffing oscillator as a novel RBM also comprehends the anomaly of olive yield interacting with socio-political factors, such as agrarian reforms, subsidies, and unrest, as an autonomous behavior: it can autonomously reproduce the feedback mechanism

from national olive yield to political decision making and social movement. Then, we examine the non-negativity, boundedness, and stability of the model's solutions that represent the dynamics of the state variables, using the conventional mathematical methodology for nonlinear systems (Sell, 1963; Sell, 1964). It is noteworthy that synchronization occurs even in the absence of coupling during the pollination off-season, while the model is consistent with the spatially distributed analogue considering the dispersion of pollen grains by the wind. This fact is in contrast with instances of spatially evolving phenomena, such as the diffusive predator-prey system (Xiao and Xia, 2024) and vegetation growth systems (Consolo and Grifó, 2023; Klausmeier, 1999), which demonstrate remarkable pattern formation. Finally, we present illustrative examples based on numerical approaches. The RK4 method numerically attempts to solve the model as a system of ordinary differential equations. The least square method is applied to identify the values of model parameters so that the model better mimics the real-world phenomena seen in statistical data. Then, we address the chaotic nature of the dynamics and bifurcation with respect to the critical parameters.

2.2. Preliminary analysis

Although biennial-bearing is a notable phenomenon in olive production, olives are harvested once a year in autumn. Accordingly, annual data provide the highest resolution for olive production statistics. **Fig. 1** presents the charts of annual production (P), area harvested (A), and yield (Y) of olives in Syria, spanning from 1961 through 2021. The data, sourced from the statistical database of FAOSTAT (FAO, 1997), are normalized to the base year values of 1961. The olive industry is predominant in the southern and western regions, with Aleppo leading, followed by Idlib, Latakia, Dara'a, and rural Damascus (FAO, 2021). The area harvested A was in a constant increase trend until 2011, when it plateaued with the outbreak of the civil war. The synchronized biennial bearing is evident, as shown by the temporal fluctuations of yield and production. The yield, the efficiency of production, improved during 1990-2006. However,

the peak of the yield and production in 2006 followed by the sudden drop in 2007 seems an anomaly; the yield returned to levels seen before 1990. The effect of the civil war on olive production was minor compared with the other crops, such as wheat and cotton (Mohammed et al., 2020).

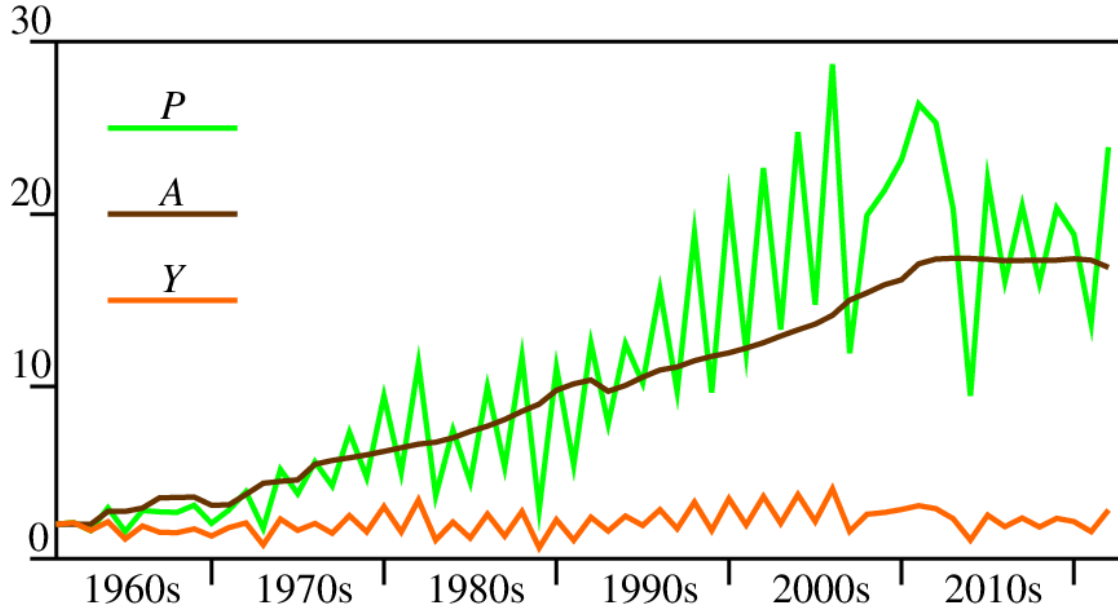


Fig. 1. Normalized charts of annual production (P), area harvested (A), and yield (Y) of olives in Syria

The annual data on olive production reflects the year-round farming activities, which continuously change over time. To capture their dynamics, we develop a time-continuous dynamical model, necessitating insights into the infinitesimal behavior of relevant state variables. To collect meteorological data, we established three observation sites on campuses of Jordanian universities, summarized in **Table 1**, setting up data loggers with multi-weather sensors for recording time series data with a logging interval of 10 minutes (Albalasmeh et al., 2023). However, at the Rabba site, burst time series data with a logging interval of 2 minutes are also recorded during rainstorm events. The social turmoil in the Levant region, except for Jordan, significantly hindered extensive field measurements during the course of this research. The Mazrah site, situated in the Jordan Rift Valley adjacent to the Dead Sea, experiences a hot desert climate markedly different from the other regions of the Levant (Unami et al., 2020;

Unami et al., 2021). The other two sites are in areas of intensive olive cultivation, but the environment tends to change from Mediterranean to steppe climates.

Table 1. Observation sites for meteorological observation in Jordan.

Site name	Location	Latitude	Longitude	Altitude	Köppen climate classification
Mazrah	Mutah University	31°15'33.3"	35°29'20.2"	−364	BWh
Rabba	Mutah University	31°16'58.1"	35°44'14.5"	929	Csa/BSk
JUST	Jordan University of Science and Technology	32°28'00.3"	35°58'29.3"	613	Csa/BSh

2.2.1 Correlation analysis of time-discrete data

Sato et al. (2024), examining the causality among meteorological elements and the annual olive productions in Syria and Jordan from 1965 through 2020, revealed the presence of synchronized biennial bearing and a lagged correlation between the annual mean temperature and the annual olive production. Here, we present a supplemental correlation analysis using the up-to-date time series data spanning from 1961 through 2021: annual olive yield Y in Syria and J in Jordan retrieved from the FAOSTAT (FAO, 1997), and annual precipitation r and mean temperature T at Aleppo, a large city in the olive production area of Syria, retrieved from Climate Change Knowledge Portal of the World Bank (World Bank, 2024). Considering Y as the response variable, we performed correlation analysis to identify the dominant explanatory variables among r , T , J , and Y itself, including lagged ones, as shown in **Table 2**. Notations of the superscripts “-” and “--” represent one-year and two-year lagged time series, respectively. The correlation coefficient between each pair of two variables is utilized to evaluate the dominance. In both countries, low auto-correlations in annual precipitation r and average temperature T indicate significant variability in these meteorological elements. The variability of annual precipitation r does not markedly influence annual olive yield Y in Syria. Conversely, the time series of one-year lagged annual average temperature T^- , rather than T and T^{--} , shows a moderate correlation with annual olive yield Y in Syria. A notable auto-correlation exists

between annual olive yield Y and two-year lagged annual olive yield Y^- , providing strong evidence of biennial bearing. Based on the analysis above, our modeling strategy is to select the best parameter set representing the effect of one-year lagged annual average temperature and to include the biennial bearing as a normal mode of the dynamical system.

Table 2. Correlation coefficients between the variables.

	r^-	r^-	r	T^-	T^-	T	J^-	J^-	J	Y^-	Y^-	Y
r^-	1.000	-0.016	-0.164	-0.117	-0.066	0.059	0.122	0.031	-0.016	0.036	-0.094	-0.166
r^-		1.000	-0.010	-0.005	-0.142	-0.087	-0.183	0.088	0.006	-0.044	0.049	-0.109
r			1.000	0.177	-0.014	-0.141	-0.135	-0.184	0.076	-0.145	-0.041	0.045
T^-				1.000	0.385	0.395	0.398	0.580	0.544	0.117	0.330	0.181
T^-					1.000	0.415	0.604	0.429	0.596	0.256	0.091	0.347
T						1.000	0.508	0.614	0.476	0.191	0.246	0.107
J^-							1.000	0.384	0.733	0.577	-0.128	0.515
J^-								1.000	0.421	-0.063	0.543	-0.099
J									1.000	0.528	-0.085	0.558
Y^-										1.000	-0.246	0.733
Y^-											1.000	-0.255
Y												1.000

2.2.2 Environmental background

Sunlight is the energy source for photosynthetic organisms, including olives. Evidence of the dominant role of sunlight for olive yield is the broad recognition that pruning is one of the most essential practices in grove management, maintaining well-lit leaf surfaces and balancing vegetative growth and reproductive functions (Lodolini et al., 2023). The high correlation between T^- and Y , as shown in the previous subsection, implies that the annual mean air temperature represents the energy of sunlight that produces the photosynthate for the growth of the olive. To verify that the air temperature well represents the energy of sunlight, statistical analysis is performed on time series data of air temperature and solar radiation observed at the Rabba site from February 2nd, 2022, through February 10th, 2024, as shown in **Fig. 2**. There is a high correlation between the solar radiation and the 42 days lagged air temperature with a correlation coefficient 0.80.

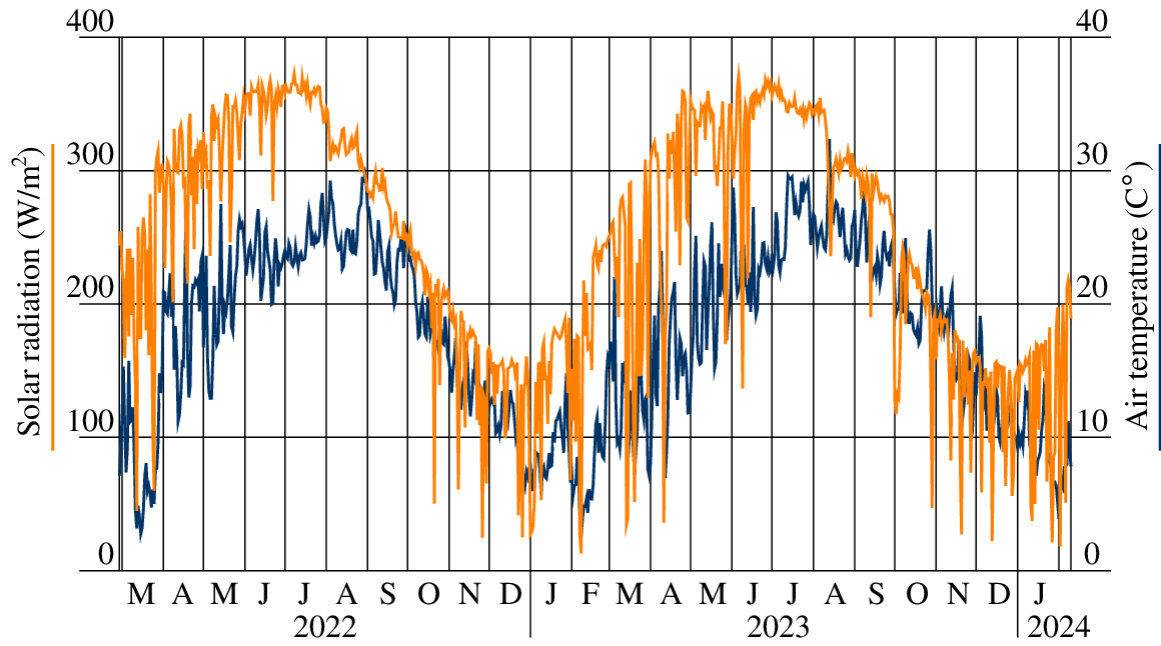


Fig. 2. Air temperature and solar radiation observed at the Rabba site.

2.2.3 Effect of pollination

A detailed regression model for olive pollen suggested a complex situation involving seasonal temperature variations and plant growth stages such as flowering and pollination in Italy (Fornaciari et al., 1998). Koubouris et al. (2009) discussed the principal role of temperature impacting olive pollen performance. Galán et al. (2004) identified a close relationship between the annual pollen production and subsequent olive yield in Spain. Rojo et al. (2016) investigated the physical dynamics of olive pollen distributed in western Mediterranean countries. Pollen grains released from the stamens of olive flowers travel in the air, where the wind velocity field determines their drift and diffusion (Hernández-Ceballos et al., 2011). By integrating the Kolmogorov forward equation associated with the diffusion process of particles transported in turbulence, Yoshioka et al. (2012) derived a linear parabolic partial differential equation governing the particle concentration under the assumption that the particles do not interact with one another. The drift coefficient vector and diffusion coefficient tensor were estimated from time series data of the velocity field. Applying this framework to pollen grains treated as particles transported in air dominated by horizontal turbulence, the horizontal drift coefficient

vector $\mathbf{W}$ and diffusion coefficient tensor D near the ground are estimated from observed anemometer data. **Table 3** shows those coefficients identified from the anemometer data observed at the three Jordanian sites. The large magnitudes of $\mathbf{W}$ and D imply that a pollen grain, once released from the source, can travel the whole Levant region in a few days. Vertical transport and deposition of pollen grains involve other factors such as the three-dimensional wind field structure, precipitation, humidity, topography, pollen size and shape, etc.

However, it is simplistic to conclude that these facts support the hypothesis in accordance with the xenogamy of olives and with empirical evidence. As it is common knowledge that olive pollen significantly contributes to the onset and deterioration of allergic diseases, the relationship between the dispersion of olive pollen and the prevalence of allergy patients is worth mentioning. Based on the observation of Jordanian allergic rhinitis patients attending outpatient clinics in Amman City, Aburuz et al. (2011) regarded grasses, olive pollen, and thistleweed as significant allergens, which are seasonal in countries with arid, semiarid, and Mediterranean climates. Al-Zayadneh et al. (2019) reported the highest frequency of sensitization on olive pollen in asthmatic children from southern Jordan. Abu-Ekteish et al. (2009) concluded that there was no significant difference in the prevalence of asthma diagnosed by a physician between schoolchildren in Amman City and those of Bedouins from northern Jordan. These medical studies support the inference from the anemometer data that spatial dispersion of pollen grains is intense throughout the region. On the contrary, Kirmaz et al. (2005) found that symptoms of olive pollen allergy were not seasonal, even in patients monosensitized to olive pollen, which was difficult to explain. It might be hypothesized that the pollen grains, though biologically inviable, remain in the ambient even after the pollination season.

Table 3. Advection and diffusion coefficients identified from the Jordanian sites.

Site	Observation periods	Interval	Number of measurements	$\mathbf{W}$ (m/s)	$\ \mathbf{W}\ $ (m/s)	D (m ² /s)	$\ D\ $ (m ² /s)
Mazrah	From 13:40 27SEP2014 To 15:10 08FEB2020 & From 08:30 19SEP2022 To 08:00 12FEB2024	10 min	347652	$\begin{pmatrix} 0.196 \\ -0.246 \end{pmatrix}$	0.315	$\begin{pmatrix} 302.908 & 100.349 \\ 100.349 & 884.952 \end{pmatrix}$	901.768
Rabba	From 07:30 27FEB2022 To 15:10 11FEB2024	10 min	102863	$\begin{pmatrix} 1.295 \\ 0.161 \end{pmatrix}$	1.305	$\begin{pmatrix} 860.796 & -105.957 \\ -105.957 & 233.422 \end{pmatrix}$	878.207
Rabba	From 18:48 10APR2023 To 07:18 11APR2023	2 min	376	$\begin{pmatrix} 1.204 \\ 0.457 \end{pmatrix}$	1.287	$\begin{pmatrix} 182.148 & -37.939 \\ -37.939 & 60.058 \end{pmatrix}$	192.977
JUST	From 09:10 07AUG2022 To 12:30 15FEB2024	10 min	80229	$\begin{pmatrix} 0.122 \\ -0.126 \end{pmatrix}$	0.176	$\begin{pmatrix} 53.139 & -48.601 \\ -48.601 & 64.700 \end{pmatrix}$	107.863

2.2.4 Socio-political background

Agrarian reforms in Syria began in 1958 when the United Arab Republic (UAR) was formulated as a political union between Syria and Egypt based on Arab pan-Arabism. Socialism as an ideological facilitator for Arab unity gradually intertwined with nationalist principles in Syria. Adopting structures resembling Soviet model regimes, such as centralized economic planning, turned out to be a practical measure for achieving bureaucratic growth and political control accompanied by social welfare (Ayubi, 1992). After the collapse of UAR and the restoration of the Syrian Arab Republic in 1961, the Arab Socialist Ba'ath Party assumed power in 1963 and planned to establish state farms, peasant cooperatives, and state-led water and agricultural resource management (Daoudy, 2020). The government promoted the transition from laissez-faire rainfed to collective irrigated agriculture for decades through large-scale public investment. By the late 1970s, the agrarian reform transformed a large part of the landless proletariat into a small-holding peasantry (Hinnebusch, 1995). According to the Syrian government report (Mohammad, 2009), olive cultivation had several achievements, including public initiatives focused on land reclamation, subsidized olive seedling distribution, free extension services, and promotion of "Integrated Pest Management" to reduce pesticide residues. One significant initiative promoting olive cultivation was the "Green Belt" project launched in 1980, where participating farmers received long-term, low-interest loans and complimentary services to reclaim land before planting trees (Cafiero et al., 2009). The project primarily targeted the eastern parts of Homs and Hama Governorates, where rainfall is relatively low, but olive cultivation is still rainfed. At the same time, however, different factors, such as the crisis in the domestic economy, pressures from the international security system, and power struggles, drove economic liberalization, appropriately called *infatah* (opening), with chaotic fluctuation of state policies and socio-economical responses (Hinnebusch, 2000). Around the hereditary succession of power in 2000, the Syrian government started on the path to the so-called "social market economy," including liberalization of the agricultural sector

(Rocchi et al., 2013). The social market economy sought to promote privatization, which would eradicate corruption in the state farms, while maintaining social welfare programs and the government's initiatives to combat unemployment and poverty (Ababsa, 2015). Paradoxically, it resulted in challenges such as a labor shortage and high prices of raw materials and production tools (Abboud, 2015). In September 2007, the government announced the decision to reduce subsidies for oil products, and all fuel subsidies ceased abruptly in May 2008. This decision severely affected rural agricultural populations, and Daoudy (2020) considered it one of the causes of the civil war.

The overview above shows that the relationship between crop production and socio-politics in Syria is bilateral, unlike the causality between solar radiation and photosynthesis.

2.3. Extension of Duffing oscillator for synchronized biennial bearing

2.3.1 Model development

We hypothesize a temporally continuous spatially lumped RBM

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\rho\sqrt{u} + \beta \exp(v)\chi_p \\ -w + \alpha\rho\sqrt{u} \\ \omega_s^2 v + \gamma v^3 - \mu w \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon d \sin(\omega_e t) \\ 0 \end{pmatrix} = \mathbf{f}(t, \mathbf{u}) + \mathbf{F}(t) \quad (1)$$

where t is the time, u is the pollen density in the atmospheric environment, v is an index of the resource involving the plant growth, w is an index of the plant biomass. The Greek symbols represent the relevant non-negative parameters; ρ is the decay coefficient of the pollen density, β is the release coefficient of pollen, χ_p is the indicator function for the pollination season assumed to be from Day 15 through Day 24 counted from the spring equinox of each year, α is the pollination coefficient, ε is the magnitude of external force $\mathbf{F}$, ω_e is the angular frequency of external force, ω_s is the angular frequency of biennial-bearing, γ is the Duffing coefficient, and μ is the mortality coefficient. The external force drifts with the parameter d , which is prescribed as

$$d = \begin{cases} 1 + |z| & \text{if } z \sin(\omega_e t) > 0 \\ 1 & \text{otherwise} \end{cases} \quad (2)$$

where z is the standard score of the annual mean temperature. The model's biological assumptions are as follows: the plant's resource consumption rate is proportional to its biomass, and both pollination and solar radiation, represented as the external force, intensify photosynthesis to nourish the resource. Additionally, the dependency of plant growth on the resource is nonlinear. We select the Duffing type of nonlinearity with the cubic term because it is the simplest form that achieves boundedness when unforced. The annual yield M estimated from this model is calculated as

$$M = \int_{I_h} w dt \quad (3)$$

where I_h is the period of harvesting season assumed to be from Day 205 through Day 234 counted from the spring equinox of each year.

The pollen density u in the atmospheric environment decays proportionately to the square root of u itself. It increases proportionally to $\exp(v)$, a positive indicator of the resource abundance producing viable pollen during the specified pollination season χ_p . Firstly, we show the non-negativity of u , by considering the initial value problem of an ordinary differential equation

$$\begin{cases} \frac{du}{dt} = -\rho \operatorname{sgn}(u) \sqrt{|u|} + v^+ \\ u(0) = u_0 \geq 0 \end{cases} \quad (4)$$

where v^+ is non-negative variable of time. Assume that there exists $t_1 \in (0, \infty)$ such that $u(t_1) < 0$. Let $t_0 \in [0, t_1)$ be the first exit time of u from $[0, \infty)$, implying $u(t_0) = 0$ and that there exists $t_2 \in (t_0, t_1]$ such that $u(t) < 0$ for all $t \in (t_0, t_2]$. Then, because of the mean value theorem, there exists $t_m \in (t_0, t_2]$ such that

$$\left. \frac{du}{dt} \right|_{t=t_m} = -\rho \operatorname{sgn}(u(t_m)) \sqrt{|u(t_m)|} + v^+ = \rho \sqrt{|u(t_m)|} + v^+ < 0 \quad (5)$$

leading to a contradiction. As $\text{sgn}(u)\sqrt{|u|} = \sqrt{u}$ when $u \geq 0$, u as a solution to (1) with a non-negative initial value is non-negative.

Remark 2.1

A vanishing $C_B^1[0, \infty)$ function

$$u = \begin{cases} \left(\sqrt{u_0} - \rho t/2\right)^2 & \text{if } 0 \leq t \leq 2\sqrt{u_0}/\rho \\ 0 & \text{if } 2\sqrt{u_0}/\rho < t \end{cases} \quad (6)$$

solves (4) when $v^+ = 0$.

2.3.2 Model analysis

Boundedness, stability, and uniqueness of $\mathbf{u}$ are the critical issues of the model to be analyzed.

The energy function

$$H(u, v, w) = \frac{2\rho u^{3/2}}{3} + \frac{\omega_s^2 v^2}{2} + \frac{\gamma v^4}{4} + \frac{w^2}{2} \geq 0 \quad (7)$$

satisfies

$$\begin{aligned} \frac{dH}{dt} &= \frac{d}{dt} \left(\frac{2\rho u^{3/2}}{3} + \frac{\omega_s^2 v^2}{2} + \frac{\gamma v^4}{4} + \frac{w^2}{2} \right) = \rho\sqrt{u} \frac{du}{dt} + (\omega_s^2 v + \gamma v^3) \frac{dv}{dt} + w \frac{dw}{dt} \\ &= -\rho^2 u + \beta\rho\sqrt{u} \exp(v)\chi_p + \left(\frac{dw}{dt} + \mu w \right) \frac{dv}{dt} + \left(\alpha\rho\sqrt{u} + \varepsilon d \sin(\omega_e t) - \frac{dv}{dt} \right) \frac{dw}{dt} \\ &= -\rho^2 u + \beta\rho\sqrt{u} \exp(v)\chi_p + \mu w \frac{dv}{dt} + \left(\alpha\rho\sqrt{u} + \varepsilon d \sin(\omega_e t) \right) \frac{dw}{dt} \\ &= -\rho^2 u + \beta\rho\sqrt{u} \exp(v)\chi_p - \mu w^2 + \left(\alpha\rho\sqrt{u} + \varepsilon d \sin(\omega_e t) \right) \left(\mu w + \frac{dw}{dt} \right) \\ &= -\rho^2 u - \mu w^2 + \left(\alpha\rho\sqrt{u} + \varepsilon d \sin(\omega_e t) \right) (\omega_s^2 v + \gamma v^3) + \beta\rho\sqrt{u} \exp(v)\chi_p \end{aligned} \quad (8)$$

which is non-positive when $\alpha = \beta\chi_p = \varepsilon = 0$, implying the boundedness of the conventional unforced Duffing oscillator in the absence of pollination and solar radiation. When $\beta\chi_p = \varepsilon = 0$ in the absence of pollination and solar radiation, there are trivial two stable equilibria

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \pm \sqrt{\omega_s^2/\gamma} \\ 0 \end{pmatrix} \quad (9)$$

with the eigenvalues $\lambda = \left(-\mu \pm \sqrt{\mu^2 - 16\omega_s^2}\right)/2$. Boundedness, as well as stability, is not guaranteed for the cases of active pollination with the external force, as can be seen from the locally linearized system of (1)

$$\frac{d\delta\mathbf{u}}{dt} = \frac{d}{dt} \begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \delta\mathbf{u} = \begin{pmatrix} -\frac{\rho}{2\sqrt{u}} & \beta \exp(v)\chi_p & 0 \\ \frac{\alpha\rho}{2\sqrt{u}} & 0 & -1 \\ 0 & \omega_s^2 + 3\gamma v^2 & -\mu \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix}. \quad (10)$$

The characteristics polynomial of the coefficient matrix of (10) is

$$\lambda^3 + (\hat{\rho} + \mu)\lambda^2 + (\hat{\rho}\mu + \hat{\gamma} - \alpha\hat{\beta}\hat{\rho})\lambda + \hat{\gamma}\hat{\rho} - \alpha\hat{\beta}\mu\hat{\rho} = 0 \quad (11)$$

where $\hat{\beta} = \beta \exp(v)\chi_p$, $\hat{\rho} = \rho/2\sqrt{u}$, and $\hat{\gamma} = \omega_s^2 + 3\gamma v^2$, whose roots indicates the local stability.

When u is very large relative to ρ ,

$$\lim_{u \rightarrow \infty} \lambda = \lim_{\hat{\rho} \rightarrow 0} \lambda = 0, \frac{-\mu \pm \sqrt{\mu^2 - 4\hat{\gamma}}}{2} \quad (12)$$

whose real parts are non-positive if $\mu \geq 0$. When u approaches to 0,

$$\lim_{u \rightarrow 0} \lambda = \lim_{\hat{\rho} \rightarrow \infty} \lambda = -\infty, \frac{\alpha\hat{\beta} - \mu \pm \sqrt{(\alpha\hat{\beta} + \mu)^2 - 4\hat{\gamma}}}{2} \quad (13)$$

whose real parts are indefinite during the pollination season and negative during the pollination off-season if $\mu > 0$. The uniqueness of $\mathbf{u}$ has a significant implication regarding the synchronization, as in **Remark 2.2** below.

Remark 2.2

Assume that $\mathbf{u}_0$ and $\mathbf{u}_1$ solves (1) with different initial values $\mathbf{u}_0(0)$ and $\mathbf{u}_1(0)$ as

$$\begin{cases} \frac{d\mathbf{u}_0}{dt} = \mathbf{f}(t, \mathbf{u}_0) + \mathbf{F}(t) \\ \frac{d\mathbf{u}_1}{dt} = \mathbf{f}(t, \mathbf{u}_1) + \mathbf{F}(t) . \\ \mathbf{u}_0(0) \neq \mathbf{u}_1(0) \end{cases} \quad (14)$$

Then, the difference between $\mathbf{u}_0$ and $\mathbf{u}_1$ solves

$$\begin{aligned} \frac{d(\mathbf{u}_1 - \mathbf{u}_0)}{dt} &= \frac{d}{dt} \begin{pmatrix} u_1 - u_0 \\ v_1 - v_0 \\ w_1 - w_0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\rho}{\sqrt{u_0} + \sqrt{u_1}} & \beta(\exp(v_1 - v_0) - 1)\exp(v_0)\chi_p & 0 \\ \frac{\alpha\rho}{\sqrt{u_0} + \sqrt{u_1}} & 0 & -1 \\ 0 & \omega_s^2 + \gamma(v_0^2 + v_0v_1 + v_1^2) & -\mu \end{pmatrix} (\mathbf{u}_1 - \mathbf{u}_0), \end{aligned} \quad (15)$$

which has the same structure as (10), implying that synchronization occurs even without coupling such as the dispersion term during the pollination off-season, if the external force $\mathbf{F}$ is common.

2.3.3 Spatially distributed model

This subsection examines the spatially distributed analogue of (1) considering the dispersion of pollen, written as

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \begin{pmatrix} \nabla \cdot (\nabla D \tilde{u} - \mathbf{W} \tilde{u}) - \tilde{\rho} \sqrt{\tilde{u}} + \tilde{\beta} \exp(\tilde{v}) \chi_p \\ -\tilde{w} + \alpha \tilde{\rho} \sqrt{\tilde{u}} \\ \omega_s^2 \tilde{v} + \tilde{\gamma} \tilde{v}^3 - \mu \tilde{w} \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon d \sin(\omega_e t) \\ 0 \end{pmatrix} = \tilde{\mathbf{f}}(t, \tilde{\mathbf{u}}) + \mathbf{F}(t) \quad (16)$$

where the tilde represents the analogue of the variables and parameters in (1). The state variables in (1) are supposed to be averaged in the sense that

$$\mathbf{u} = \frac{1}{|\Omega|} \int_{\Omega} \tilde{\mathbf{u}} d\Omega = \frac{1}{|\Omega|} \int_{\Omega} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} d\Omega. \quad (17)$$

The parameters are bounded as $\tilde{\beta} \leq \beta$, $\tilde{\rho} \geq \rho$, and $\tilde{\gamma} \leq \gamma$ if the higher moments are small enough, because

$$\begin{aligned} \frac{1}{|\Omega|} \int_{\Omega} \exp(\tilde{v}) d\Omega &= \frac{1}{|\Omega|} \int_{\Omega} \exp(v + (\tilde{v} - v)) d\Omega = \frac{1}{|\Omega|} \int_{\Omega} \exp(\tilde{v} - v) \exp(v) d\Omega \\ &= \frac{1}{|\Omega|} \int_{\Omega} \exp(\tilde{v} - v) d\Omega \exp(v) = \frac{\beta}{\tilde{\beta}} \exp(v) \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{1}{|\Omega|} \int_{\Omega} \sqrt{\tilde{u}} d\Omega &= \frac{1}{|\Omega|} \int_{\Omega} \sqrt{u + (\tilde{u} - u)} d\Omega \\ &\approx \frac{1}{|\Omega|} \int_{\Omega} \left(\sqrt{u} + \frac{\tilde{u} - u}{2\sqrt{u}} - \frac{(\tilde{u} - u)^2}{4u\sqrt{u}} \right) d\Omega = \sqrt{u} - \frac{1}{4u\sqrt{u}|\Omega|} \int_{\Omega} (\tilde{u} - u)^2 d\Omega, \\ &= \left(1 - \frac{1}{4|\Omega|} \int_{\Omega} \left(\frac{\tilde{u}}{u} - 1 \right)^2 d\Omega \right) \sqrt{u} = \frac{\rho}{\tilde{\rho}} \sqrt{u} \end{aligned} \quad (19)$$

and

$$\begin{aligned} \frac{1}{|\Omega|} \int_{\Omega} \tilde{v}^3 d\Omega &= \frac{1}{|\Omega|} \int_{\Omega} (v + (\tilde{v} - v))^3 d\Omega \\ &\approx \frac{1}{|\Omega|} \int_{\Omega} \left(v^3 + 3v^2(\tilde{v} - v) + 3v(\tilde{v} - v)^2 + (\tilde{v} - v)^3 \right) d\Omega \\ &= v^3 + \frac{3v^2}{|\Omega|} \int_{\Omega} (\tilde{v} - v) d\Omega + \frac{3v}{|\Omega|} \int_{\Omega} (\tilde{v} - v)^2 d\Omega + \frac{1}{|\Omega|} \int_{\Omega} (\tilde{v} - v)^3 d\Omega \\ &= \left(1 + \frac{3}{|\Omega|} \int_{\Omega} \left(\frac{\tilde{v}}{v} - 1 \right)^2 d\Omega + \frac{1}{|\Omega|} \int_{\Omega} \left(\frac{\tilde{v}}{v} - 1 \right)^3 d\Omega \right) v^3 = \frac{\gamma}{\tilde{\gamma}} v^3 \end{aligned} \quad (20)$$

Therefore, averaging due to pollen dispersion may increase the fictitious release coefficient of pollen, decrease the fictitious decay coefficient of the pollen density, and amplify the nonlinearity of oscillation.

2.4. Illustrative numerical examples

Numerical examples are presented in this section to validate the model's usefulness for better understanding the dynamics of synchronized biennial-bearing. The unit for time is taken as the year, and the parameters ω_s and ω_e are fixed at π and 2π , respectively.

2.4.1 Parameter identification

The parameter values are identified to fit the observed data in the sense of minimizing the performance index

$$\begin{aligned}
 J = & \left| R(T^{--}, Y) - R(T^{--}, M) \right|^2 + \left| R(T^-, Y) - R(T^-, M) \right|^2 + \left| R(T, Y) - R(T, M) \right|^2 \\
 & + \left| R(T^{--}, Y) - R(T^-, Y) - R(T^{--}, M) + R(T^-, M) \right|^2 \\
 & + \left| R(T^-, Y) - R(T, Y) - R(T^-, M) + R(T, M) \right|^2
 \end{aligned} \tag{21}$$

where $R(A, B)$ is the correlation coefficient between the data series A and B . Global sequential searches and local gradient descent identified the parameter set $(\rho, \beta, \alpha, \mu, \gamma, \varepsilon) = (4.000, 40.00, 0.333, 0.0898, 0.607, 5.000)$ to be close to the optimal. The parameters ρ , β , and ε are less sensitive than the other parameters α , μ , and γ , which were determined with the gradient descent.

Fig. 3 show the recurrence plots of the annual olive yield Y , observed in Syria, and M , estimated from the model with the identified parameter values. According to Marwan et al. (2007), recurrence plots qualitatively visualize recurring structures of time series, such as periodic motions, chaotic systems, and uniformly distributed noises, by delineating the repetition of two close values. The two recurrence plots in **Fig.3** show the dynamical similarities between Y and M , as both are regarded as dominantly chaotic with limited stochastic disturbances.

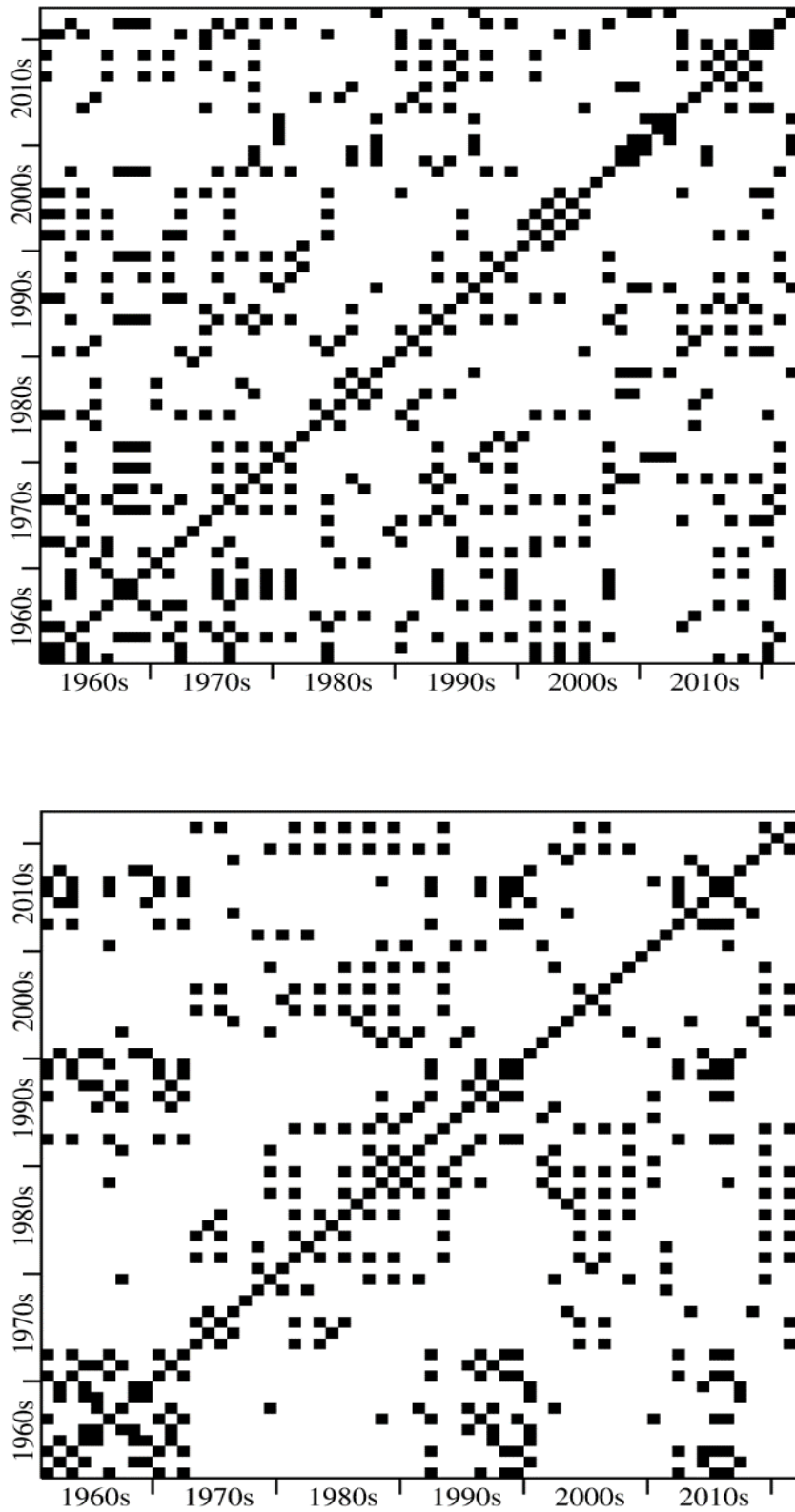


Fig. 3. Recurrence plots of annual olive yield Y (upper panel), observed in Syria, and M (lower panel), estimated from the model with the identified parameter values.

2.4.2 Sensitivity analysis based on bifurcation diagrams

Bifurcation is the most significant feature of the effect of the sensitive parameters on the model dynamics. The cases where the numerical solution of (1) with the RK4 method fails are regarded as indicating unboundedness or instability. Figs. 4, 5, and 6 show the bifurcation diagrams of M without a drift of external force plotted with respect to α , μ , and γ , respectively. In each of these figures, the identified parameter value is indicated with a vertical line in blue, on which the visited M -values are dotted green. Omitting the drift of external force implies that the system is entirely deterministic. As discussed in Subsection 2.3.2, the system at each $(t, \mathbf{u})$ is possibly unstable during the pollination season and stable otherwise, resulting in transitions of global behaviors among stable oscillation, chaos, unstable oscillation, and blowing up, depending on the parameter values. **Figs. 4 and 5** indicate that the larger efficiency of pollination, represented by α , and the smaller mortality, represented by μ , destabilize the olive yield. **Fig. 6** shows that the nonlinearity of plant growth in response to resource abundance also contributes to the stability of yield. However, the stabler yield entails a lower average yield. The identified parameter set achieves bounded chaos with a higher average yield.

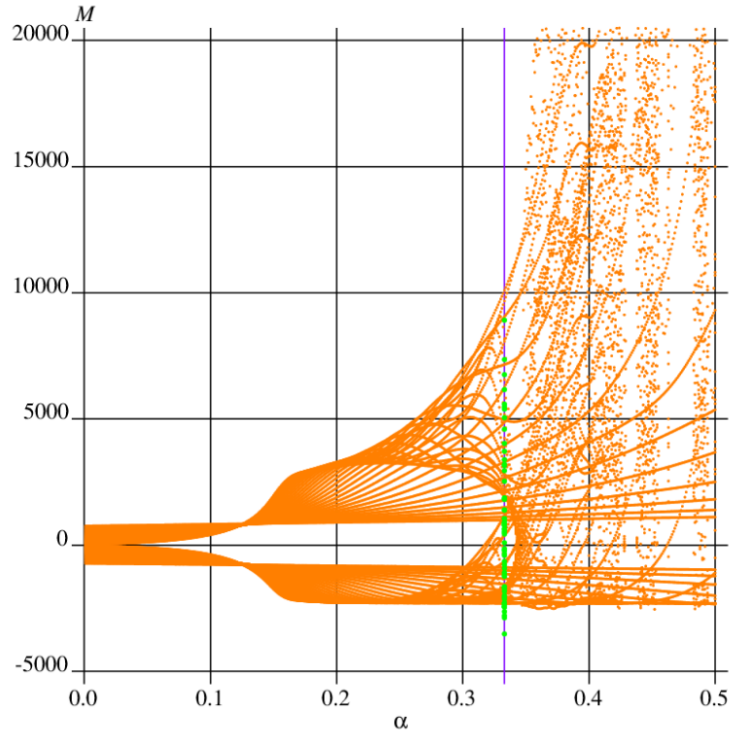


Fig. 4. Bifurcation diagram of M without a drift of external force plotted with respect to α . The sub-optimal values $(\rho, \beta, \mu, \gamma, \varepsilon) = (4.000, 40.00, 0.0898, 0.607, 5.000)$ are used for the other parameters.

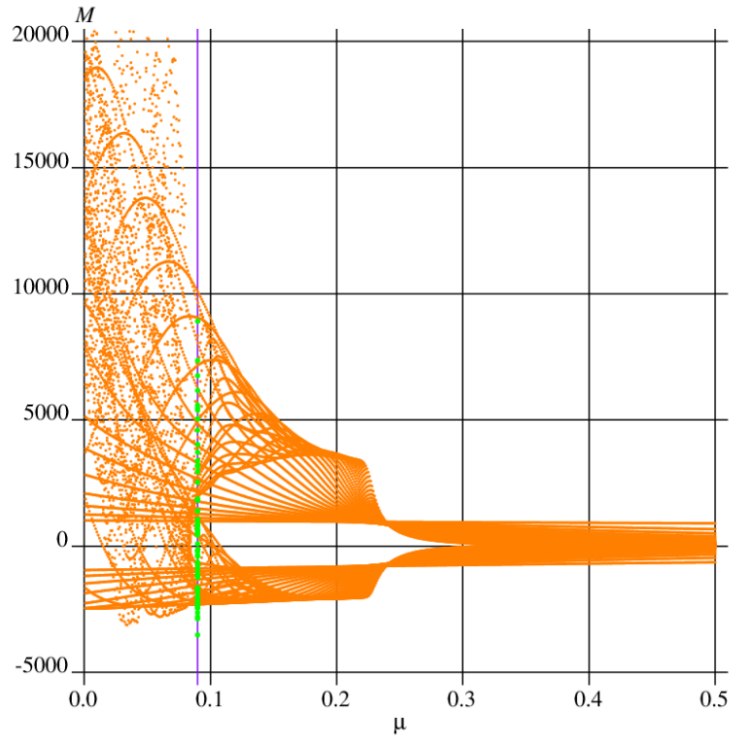


Fig. 5. Bifurcation diagram of M without a drift of external force plotted with respect to μ . The sub-optimal values $(\rho, \beta, \alpha, \gamma, \varepsilon) = (4.000, 40.00, 0.333, 0.607, 5.000)$ are used for the other parameters.

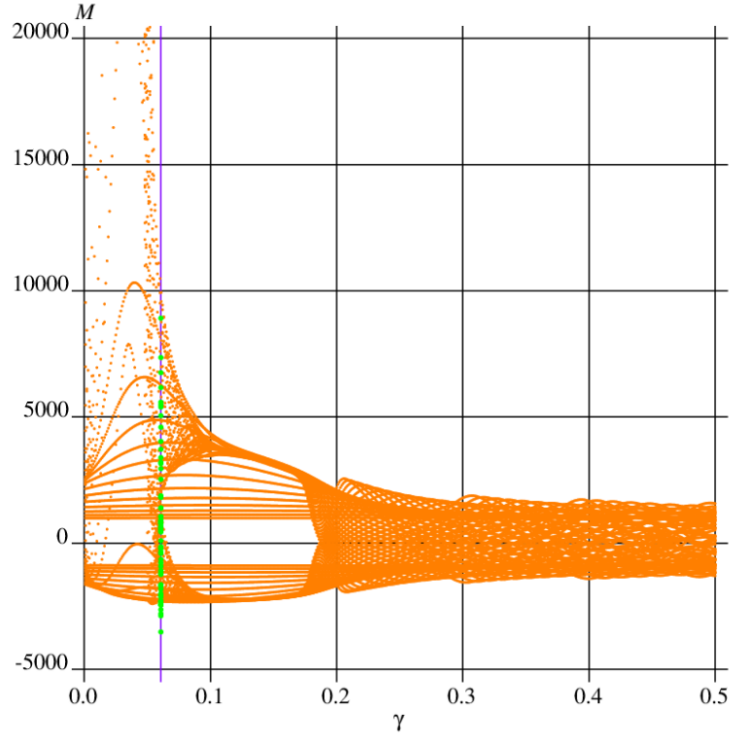


Fig. 6. Bifurcation diagram of M without a drift of external force plotted with respect to γ . The sub-optimal values $(\rho, \beta, \alpha, \mu, \varepsilon) = (4.000, 40.00, 0.333, 0.0898, 5.000)$ are used for the other parameters.

2.5. Conclusions

The motivation of this study was to construct a mathematical model that could holistically elucidate the dynamics of synchronized biennial-bearing olives in the Levant. The preliminary analysis surveyed various factors influencing the peculiar behavior observed in the time series data of annual olive yield, which demonstrated compelling evidence of synchronized biennial-bearing in Syria and Jordan. Given the moderate correlation between the one-year lagged annual mean temperature and the annual olive yield, we inferred that temperature could be considered an external force propelling olive growth, though it is biologically linked to photosynthesis originating from solar radiation. Consequently, we also assessed the relationship between air temperature and solar radiation using the detailed data observed in Jordan. Another biological factor, pollination, was examined using an aerodynamic approach and information on allergy patients. Syria, the foremost olive-producing country in the Levant, underwent significant shifts in agricultural policy and social stability. Upon reviewing Syria's

modern history, we concluded that the relationship between crop production and socio-political factors was bilateral, unlike the causality between solar radiation and photosynthesis. Therefore, the model should reproduce anomalies, such as the decline in olive yield impacted by the civil war, not as an external force but as an inherent chaotic nature of the dynamics. The extended Duffing oscillator, a time-continuous counterpart of RBM, is a model that effectively fulfills this purpose: its linear part generates a biennial oscillation, while the cubic term contributes to the emergence of chaos. The properties of the model's solutions, specifically non-negativity (as highlighted in **Remark 2.1**), boundedness, and stability, were analyzed with the conventional mathematical methodology for nonlinear systems. The model was further integrated as a spatially distributed model to incorporate multiple individual fruit trees coupled with each other by anemophily, which is not essential in synchronization, as stated in **Remark 2.2**. The illustrative numerical examples validated the model's applicability to the real-world phenomena in Syria. The RK4 method was employed to derive numerical solutions to the model as a system of ordinary differential equations, provided they were bounded and stable. The recurrence plots delineated dominant chaos with limited stochastic disturbances when the parameters were set to their best-fit values. Bifurcation diagrams, exhibiting asymmetry due to the nonlinearity, indicated that the best-fit parameter values achieved the intended chaotic nature of the dynamics. With alternative parameter values, the model could exhibit periodic oscillation, instability, or blowing up that a conventional RBM cannot consider.

CHAPTER 3: Plant-nutrient interactions model

3.1. Introduction

Modeling agroecological systems with ordinary differential equations (ODEs) is of great interest to agricultural scientists and mathematicians. A classic but still emerging research topic is dynamical systems to model the populations of two biological species in a predator-prey relationship, developed from the celebrated Lotka-Volterra model (Ajraldi et al., 2011). For competitive two plant species, a limitation of Lotka-Volterra type models with two dependent variables is that the behavior of the resource under competition, such as water, light, and nutrients, is not well represented. Louie et al. (2002) overcame that issue by introducing a delay effect into a classic growth model of ryegrass–clover mixtures. Fractional calculus is a powerful tool for considering different phenomena with delay or memory effects (Unami, 2023). Das and Gupta (2011) addressed the time-fractional Lotka-Volterra model and their numerical solutions. Time-fractional derivatives can represent spatially distributed phenomena in lumped models (Adel and Khader, 2023). Another significant approach is the explicit inclusion of the resource in the dynamical system, where the biological species and the resource may be spatially distributed. In such a dynamical system, the biological species and the resource interact with each other depending on their populations or concentrations and their gradients, inducing taxis, tropism, and diffusion. The Klausmeier model is a spatially distributed dynamical system of nonlinear plant growth and unilateral water current, explaining the bifurcation of vegetation patterns, which are possibly nonstationary (Klausmeier, 1999). Including diffusion of plants and water creates more complex and realistic patterned plant distributions (Wang et al., 2021). Other studies attempted to increase the number of dependent variables in lumped dynamical systems. Kavadia et al. (2008) developed a two-prey–one-predator system involving nitrogen (N) fixation. Aliyu and Mohd (2021) examined a system consisting of resource, competitor, exploiter, and mutualist species. Aljahdaly (2023) analyzed

a two-predator-one-prey system during the mating period, discussing its positivity, boundedness, and stability. The extended resource budget model by Magero et al. (2024) successfully reproduced chaotic dynamics of synchronized biennial-bearing olives. In this study, we develop a lumped dynamical system model with three dependent variables representing two plant species and a nutrient resource, considering the unique existence of a bounded non-negative stable solution to each of realistic initial value problems.

The Allee effect is one of the essential phenomena in population dynamics. It is based on the notion that populations at low numbers are affected by a positive relationship between population growth rate and density, which increases their likelihood of extinction (Courchamp et al., 2008). According to Sen et al. (2021), there are two main types of Allee effects: strong Allee effect and weak Allee effect. A strong Allee effect involves a population threshold below which the per capita growth rate is negative, causing the population to decline, but above which the growth rate becomes positive, leading to population growth and eventual stabilization near the carrying capacity. A weak Allee effect does not require a specific threshold for population survival. The mechanisms inducing Allee effects include resource competition, social interaction, and environmental conditions. The recent emerging development in studies on dynamical system models indicates the significance of Allee effects. Verdy (2010) developed a structured population model with delay, exhibiting chaotic behaviors. Xiao and Xia (2024) found rich phenomena involving bifurcations in a spatially distributed diffusive predator-prey model. Including nonlinear cross-diffusion to represent taxis enables the model to reproduce complex spatio-temporal population patterns (Han and Dai, 2019). A stochastic counterpart problem helps achieve consistency with real-world data (Saha et al., 2013). However, linking complex phenomena observed in field experiments with Allee effects is still challenging (Albalasmeh et al., 2023).

N is one of the most essential plant nutrients. When considering N as the dominant nutrient resource, symbiotic N fixation (SNF) becomes the critical factor affecting the

dynamics of the agroecological system, possibly involving an Allee effect. SNF is generally accepted as a product of mutualism established between legumes and rhizobia (Moyano et al., 2017). In 1980, Jensen and Sørensen (1987) suggested that the cropping of a legume stimulates the growth of its homologous rhizobia. Therefore, the SNF in legumes is a fundamental process for maintaining soil fertility and the continued productivity of organic cropping systems. The N fixation rates by agricultural legumes vary from 50 to 200 kg N ha⁻¹ yr⁻¹, depending on many factors, including available soil mineral N, legume species, interactions with adjacent non-legumes, weather, and harvest management (Reilly et al., 2022). The SNF can occur even under harsh arid conditions (Zahran, 1999). Robert and Schmidt (1983) reported that the rhizobia could survive well in fallow soil. A better understanding of SNF regulation in cropping systems can quantify its contribution to the farmland N budget and provide insight into the transfer of N between legume and non-legume species growing together. In the context of dynamical system modeling, Zhao et al. (2021) analyzed the stability of an all-inclusive symbiotic system with reaction-diffusion-advection and delay.

Stable isotopes are often used as natural labels to quantify the uptake of individual components from a mixture of substrates (Peoples et al., 2015). Dynamic isotopic equations combined with traditional ecological models may provide new insight into internal relationships in ecosystems. Vavilin et al. (2014) reported that isotope fractionation can be quantified with biomass's initial N isotopic signature. They evaluated nonlinear dynamics of N isotopic signature during uptake and assimilation of ammonium, nitrate, and urea by a culture of *Thalassiosira pseudonana*, describing complex substrate kinetics in N-sufficient and N-deficient conditions reasonably well. Tanaka et al. (2015) conducted a long-term and continuous ¹⁵N-tracer experiment using ¹⁵N-labelled nitrate (¹⁵NO₃-N) to construct a simple mixing model of internally-circulating N and externally-acquired N for algal endosymbionts in corals.

Many researchers have taken advantage of the small but measurable differences in ^{15}N abundance that often occur naturally between legumes, non-legumes, and atmospheric N_2 to determine the reliance upon SNF for growth in various plants. Kohl et al. (1971) reported that nonsymbiotic N fixation does not significantly fractionate the isotopes, while denitrification favors ^{14}N , leaving soil N enriched in ^{15}N , resulting in the fractionation of N isotopes through physical and biochemical processes, such as evaporation, denitrification, and nitrification.

In this study, we aim to build upon these methods to investigate nitrogen dynamics in an agroecosystem. The primary objective is to include the dynamical relationships among a leguminous plant, a non-leguminous plant, and a nutrient resource in a lumped dynamical system model of three ODEs without delay effect. We conducted an intensive field experiment to calibrate the model and to explore the influence of spatial diffusion and advection on the dynamics of N, which the model does not explicitly include. Among several measurement approaches we implement, the stable isotope analysis for plant samples from the experiment verifies the model structure and provides deeper insights into the nitrogen cycling and interactions between the plant species in the system.

3.2. Field experiment and measurements

The field experiment was conducted at Ehime University for 25 weeks from mid-April through mid-October 2022. Biological species used in the experiments are the leguminous plant *Trifolium repens* (white clover, clover henceforth), the non-leguminous plant *Mentha × Piperita* (peppermint, mint henceforth), and the rhizobia (Konryukin Mame-Zo, details not publicized), which was commercially available from the Tokachi Federation of Agricultural Cooperatives.

3.2.1 Experimental setup

Fig. 7 shows the experimental setup for cropping the biological species in a greenhouse at

33°50'18.8" N 132°47'33.2" E in the Tarumi Campus of Ehime University, Japan. There are two plots of identical layout: horizontal and inclined (10°), facing almost toward the true south. Each plot is a tank having internal dimensions of 1000 mm long × 190 mm high × 400 mm wide to contain soil with impermeable bottom and side walls. The soil was taken from the site at 33°19'14.1" N 132°31'21.0" E, where a landslide occurred from June 28 to July 8, 2018, to devastate the orchards of *Citrus unshiu*. The soil was heated at 100 °C for 24 hours to combust organic matter, but inorganic N compounds might remain. The soil density is 2.684 g/cm³, and saturated hydraulic conductivity ranges from 1.0×10^{-4} to 10^{-6} m/s. Each plot is divided into five areas: an uncultivated area, a clover area, a mint area, a mixture area, and another uncultivated area, which are referred to as H1, H2, H3, H4, and H5 in the horizontal plot; and I1, I2, I3, I4, and I5 in the inclined plot, where the elevation is the highest at I1. The plant seeds were sown on April 20, 2022, immediately after being thoroughly mixed with the suggested amount of rhizobia. To holistically monitor the ambient environment, meteorological data was collected using a HOBO weather station (Onset Computer Corp.) installed outside the greenhouse, measuring solar radiation, wind direction, wind speed, precipitation, air temperature, barometric pressure, and relative humidity every 10 minutes. The greenhouse, constructed as a tunnel with a north-to-south axis, shielded the plots from the precipitation and the wind while allowing ventilation through its open ends. Soil moisture at a depth of 10 cm was observed every 10 minutes using a soil water potential sensor (TEROS21, METER Group, Inc.) and soil moisture sensors (TEROS12, METER Group, Inc.; UIZ-ML3-LR, UIZIN Co., LTD). The data obtained from these sensors are utilized to verify that the physical environment is not constraining the plant growth as much as N is. An in-situ root imager (CI-600, CID Bio-Science, Inc.) inserted into the tube shown in **Fig. 7** was operated weekly to capture images of plant roots developing in the soil.

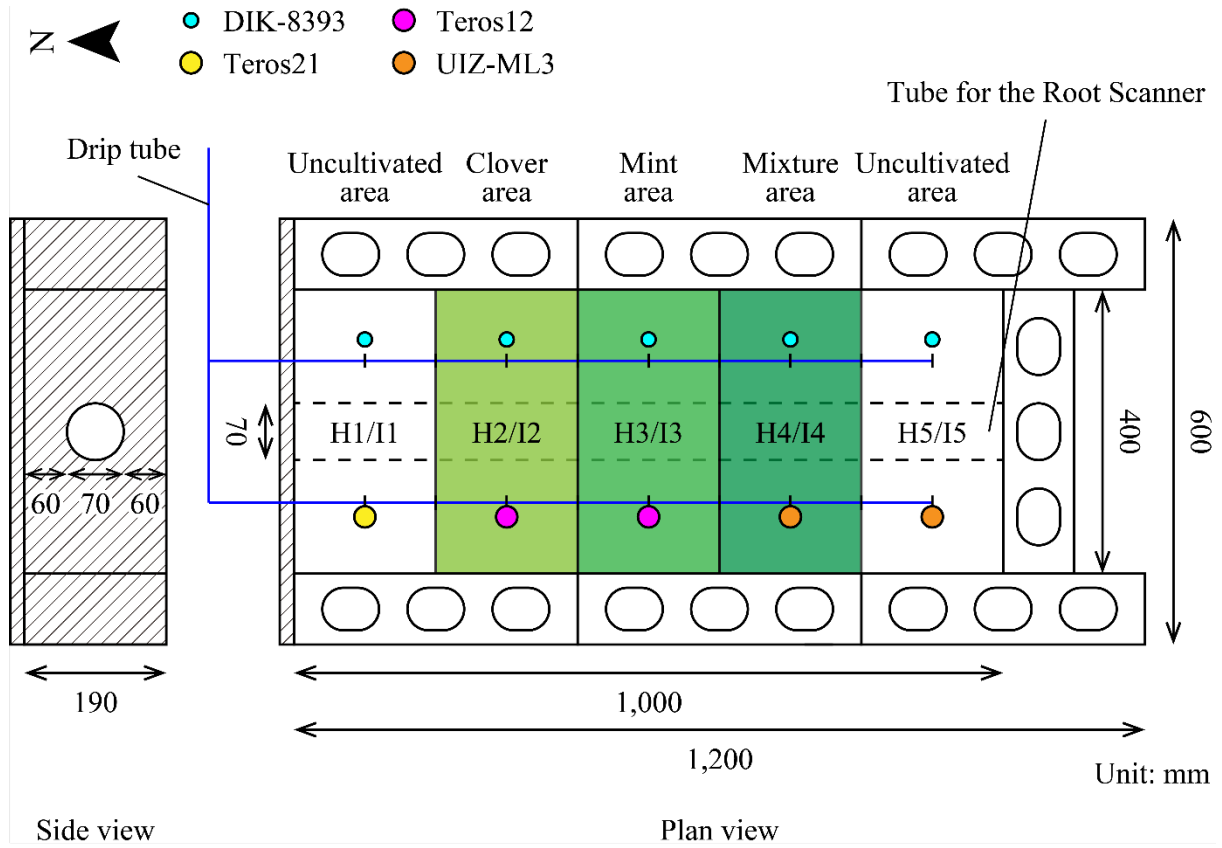


Fig. 7. Schematic sketch of each plot in the field experiment.

3.2.2 Measurements

3.2.2.1 Stable isotope analysis of N

To indicate the natural ^{15}N abundance, Peoples et al. (2015) reported the stable isotope data as the isotope ratio $\delta^{15}\text{N}$ of the sample relative to the atmospheric air standard (in per mill (‰)):

$$\delta^{15}\text{N} (\text{‰}) = \frac{R_{\text{sample}} - R_{\text{standard}}}{R_{\text{standard}}} \times 1000 \quad (22)$$

where R_{sample} and R_{standard} are the molar ratios of $^{15}\text{N}/^{14}\text{N}$ in the sample and the standard, respectively; therefore, the $\delta^{15}\text{N}$ of the atmospheric air is 0.000, referred to as $\delta^{15}\text{N}_{\text{air}}$. Donahue et al. (2020) proposed an equation to describe the percent N derived from the air (%Ndfa) for the legume plant:

$$\% \text{Ndfa} = \frac{\delta^{15}\text{N}_{\text{reference}} - \delta^{15}\text{N}_{\text{leg}}}{\delta^{15}\text{N}_{\text{reference}} - B} \times 100 \quad (23)$$

where $\delta^{15}\text{N}_{\text{reference}}$ is the $\delta^{15}\text{N}$ abundance of a non-legume plant that relies on soil mineral N, $\delta^{15}\text{N}_{\text{leg}}$ is the $\delta^{15}\text{N}$ abundance of a legume plant sample, and B is a constant representing the $\delta^{15}\text{N}$ -value of the legume plants grown solely with atmospheric N gas. Jacot et al. (2000) have determined the value of B to be -1.66 for legume species. Vinther and Jensen (2000) suggested that %Ndfa in grass-clover mixtures of a grazed organic cropping system showed seasonal variations. Besides, Reilly et al. (2022) studied the percent N of a non-legume in a mixed quadrat transferred from legumes (%Ntr) with the following equation:

$$\% \text{Ntr} = \frac{\delta^{15}\text{N}_{\text{reference}} - \delta^{15}\text{N}_{\text{non-leg}}}{\delta^{15}\text{N}_{\text{reference}} - B} \times 100 \quad (24)$$

where $\delta^{15}\text{N}_{\text{non-leg}}$ represents the ^{15}N abundance of the non-legume in the mixed quadrats with the legume.

The plants in each area of the experimental plots were sampled randomly on June 24 in Week 9, July 11 in Week 12, August 17 in Week 17, and September 7 in Week 20. The plant samples were dried at $65\text{ }^{\circ}\text{C}$ in a forced air oven for 48h. The dried samples were weighed and ground into ultra-fine powder using a mortar and a ball mill. After redrying ($65\text{ }^{\circ}\text{C}$ for 24 h), the samples (10-15 mg) were weighed again in tin caps (40 ml) (Jacot et al., 2000). The samples were analyzed for $^{15}\text{N}/^{14}\text{N}$ ratio by an Isotope Ratio Mass Spectrometry (IRMS) in the Laboratory of Soil Science of Kyoto University.

3.2.2.2 Photometer analysis of soil pore water quality

From April 18-19, 2022 (the night before the sowing day), to July 18-19, 2022 (the end of the vegetative stage), the soil pore water samples were collected from the ten areas once a week during the night from Monday to Tuesday, using vacuum soil water samplers (DIK-8393, Daiki Rika Kogyo Co., Ltd.). The concentration of $\text{NO}_3\text{-N}$ in each sample was determined by Photometer analysis using YSI nitrate (nitratetest) reagent and a YSI 9500 Photometer.

3.2.2.3 Image processing to determine leaf areas

The leaf area (LA) is the total area of leaves projected to the horizontal plane and indicates the solar radiation the plants receive for photosynthesis. Therefore, the LA is utilized as an index of crop growth. Vertically downward shots were obtained by a digital camera to determine the weekly LA of each plant species grown in H2, H3, H4, I2, I3, and I4 areas, using the image processing software ImageJ.

3.3. Model development and analysis

We hypothesize that the dynamics of the system consisting of the leguminous plant (clover), the non-leguminous plant (mint), and the nutrient resource (NO₃-N) is modeled by the system of ODEs

$$\begin{cases} \frac{dX}{dt} = f_X(X, Y, Z) = \alpha_X f_{ZX} + f_{XX} - \mu_X X \\ \frac{dY}{dt} = f_Y(X, Y, Z) = \alpha_Y f_{ZY} - \mu_Y Y \\ \frac{dZ}{dt} = f_Z(X, Y, Z) = -f_{ZX} - f_{ZY} + f_{ZZ} - \mu_Z Z \end{cases} \quad (25)$$

where X , Y , and Z represent the standardized (between 0 and 1) abundances of the leguminous plant, the non-leguminous plant, and the nutrient resource, respectively; t is the time; α_X and α_Y are the absorption coefficients of the nutrients by the plants; μ_X , μ_Y , and μ_Z are mortality or decay coefficients; and f_{ZX} , f_{XX} , f_{ZY} , and f_{ZZ} are the interaction functions prescribed as

$$\begin{cases} f_{ZX} = K_X(1 - X - Y)Z(\beta_X - Z)X \\ f_{XX} = K_S(1 - X - Y)(Z + \gamma_X)(\beta_Z - Z)X \\ f_{ZY} = K_Y(1 - X - Y)Z(\beta_Y - Z)Y \\ f_{ZZ} = K_Z Z(1 - Z) \end{cases} \quad (26)$$

where K_S , K_X , K_Y , and K_Z are the growth rates; and β_X , β_Y , β_Z , and γ_X are the shape parameters. Each entry of the Jacobian matrix of (25) is presented in **Appendix A** and is finite for finite X , Y , and Z . With the explanatory chart **Fig. 8**, the agroecological rationale of (25)

and (26) is provided as follows. The leguminous clover relies on N nutrients by two different processes: uptake of $\text{NO}_3\text{-N}$ in the soil pore water (f_{ZX}) and SNF of N_2 gas diffusing from the atmosphere into the soil pore (f_{XX}). The non-leguminous mint relies only on the uptake of $\text{NO}_3\text{-N}$ in the soil pore water (f_{ZY}). Both plants geometrically undergo growth or death with the mortality rates. The $\text{NO}_3\text{-N}$ in soil pore water increases by nitrification of $\text{NH}_4\text{-N}$ and decreases by plant uptake. The carrying capacity, standardized as 1, suppresses the rates of uptake, SNF, and nitrification as the total abundance of the two plants approaches it, resulting in the nonlinear interaction functions.

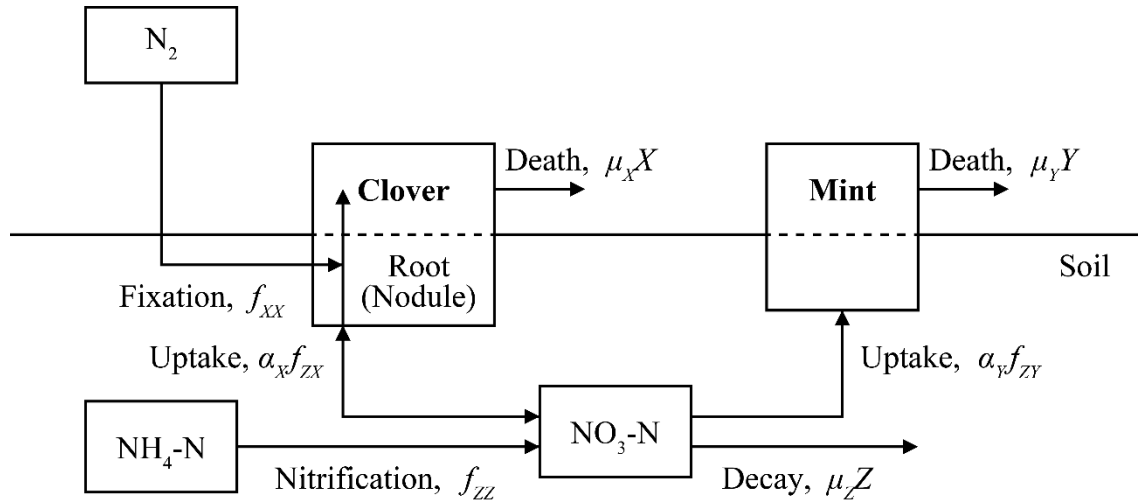


Fig. 8. Structure of the agroecological model.

Remark 3.1 Observing above implies that the Picard–Lindelöf theorem is applicable to guarantee the unique existence of a local solution as the right hand sides of (25) are polynomials.

3.3.1 Physical implication of the model

The interaction functions f_{ZX} , f_{XX} , f_{ZY} , and f_{ZZ} represent the nutrient uptake by clover, the SNF in clover, the nutrient uptake by mint, and the N self-reaction, respectively. The interaction between the plants and the nutrient resource decays if the total abundance of the plants $X + Y$ approaches 1. The quadratic nonlinearity with respect to Z allows the possible presence of the

Allee effect in all the interaction functions. The factors including Z in f_{xx} represent the dependency of the SNF on the N in the environment. The logistic model of f_{zz} assumes the abundance of N stored in the soil as ammonia-N, which is transferred to $\text{NO}_3\text{-N}$ by nitrification.

3.3.2 Mathematical analysis of the model

This subsection discusses the mathematical properties of the model.

Firstly, we consider the unique existence and boundedness of solutions for initial value problems. Because of **Remark 3.1**, we obtain **Propositions 3.1**, **3.2**, and **3.3** below.

Proposition 3.1 There exists a smooth unique solution to (25) with (26) for $t \in [0, \infty)$, satisfying an initial condition

$$X(0) = X_0 > 0, \quad Y(0) = 0, \quad Z(0) = 0. \quad (27)$$

The solution is explicitly given as

$$X(t) = \frac{C_X (K_S \gamma_X \beta_Z - \mu_X)}{\exp((\mu_X - K_S \gamma_X \beta_Z)t) + C_X K_S \gamma_X \beta_Z}, \quad Y(t) = 0, \quad Z(t) = 0 \quad (28)$$

where $C_X = X_0 / (K_S \gamma_X \beta_Z (1 - X_0) - \mu_X)$ if $K_S \gamma_X \beta_Z \neq \mu_X$, and as

$$X(t) = \frac{1}{K_S \gamma_X \beta_Z t + 1/X_0}, \quad Y(t) = 0, \quad Z(t) = 0 \quad (29)$$

if $K_S \gamma_X \beta_Z = \mu_X$.

Proposition 3.2 There exists a smooth unique solution to (25) with (26) for $t \in [0, \infty)$, satisfying an initial condition

$$X(0) = 0, \quad Y(0) = Y_0 > 0, \quad Z(0) = 0. \quad (30)$$

The solution is explicitly given as

$$X(t) = 0, \quad Y(t) = Y_0 \exp(-\mu_Y t), \quad Z(t) = 0. \quad (31)$$

Proposition 3.3 There exists a smooth unique solution to (25) with (26) for $t \in [0, \infty)$, satisfying an initial condition

$$X(0) = 0, \quad Y(0) = 0, \quad Z(0) = Z_0 > 0. \quad (32)$$

The solution is explicitly given as

$$Z(t) = \frac{C_Z (K_Z - \mu_Z)}{\exp((\mu_Z - K_Z)t) + C_Z K_Z}, \quad Y(t) = 0, \quad Z(t) = 0 \quad (33)$$

where $C_Z = Z_0 / (K_Z (1 - Z_0) - \mu_Z)$ if $K_Z \neq \mu_Z$, and as

$$X(t) = 0, \quad Y(t) = 0, \quad Z(t) = \frac{1}{K_Z t + 1/Z_0} \quad (34)$$

if $K_Z = \mu_Z$.

Furthermore, we obtain the following **Theorem 3.1**.

Theorem 3.1 Any continuous solution to (25) with (26) for $t \in [0, \infty)$, satisfying an initial condition

$$\begin{aligned} &X(0) = 0, \quad Y(0) = Y_0 > 0, \quad Z(0) = Z_0 > 0 \\ &\text{or} \\ &X(0) = X_0 > 0, \quad Y(0) = 0, \quad Z(0) = Z_0 > 0 \\ &\text{or} \\ &X(0) = X_0 > 0, \quad Y(0) = Y_0 > 0, \quad Z(0) = 0 \end{aligned} \quad (35)$$

remains

$$\begin{aligned} &X(t) = 0, \quad Y(t) > 0, \quad Z(t) > 0 \\ &\text{or} \\ &X(t) > 0, \quad Y(t) = 0, \quad Z(t) > 0, \\ &\text{or} \\ &X(t) > 0, \quad Y(t) > 0, \quad Z(t) = 0 \end{aligned} \quad (36)$$

respectively.

Proof.

For the first case, let $t_0 \in (0, \infty]$ be the first exit time of $(X, Y, Z)^T$ from

$$\Omega_+^{yz} = \left\{ (x, y, z)^T \mid x = 0, y > 0, z > 0 \right\}. \quad (37)$$

Assuming $t_0 < \infty$ contradicts **Proposition 3.2** or **3.3** because of violating the uniqueness of a solution at $t = t_0$, where $(X, Y, Z)^T = (0, Y, 0)^T$ or $(0, 0, Z)^T$, respectively. We obtain the results for the other two cases in the same manner. $\square$

Finally, the positivity and the (partial) boundedness are stated in **Theorems 3.2** and **3.3** below.

Theorem 3.2 Any continuous solution to (25) with (26) for $t \in [0, \infty)$, satisfying an initial condition

$$X(0) = X_0 > 0, \quad Y(0) = Y_0 > 0, \quad Z(0) = Z_0 > 0 \quad (38)$$

remains

$$X(t) > 0, \quad Y(t) > 0, \quad Z(t) > 0. \quad (39)$$

Proof.

Let $t_0 \in (0, \infty]$ be the first exit time of $(X, Y, Z)^T$ from

$$\mathbb{R}_+^3 = \left\{ (x, y, z)^T \mid x > 0, y > 0, z > 0 \right\}. \quad (40)$$

Assume $t_0 < \infty$. If at least one of $X(t_0)$, $Y(t_0)$, and $Z(t_0)$ is equal to 0, then it contradicts

Propositions 3.1-3 or **Theorem 3.1** because of violating the uniqueness of a solution at $t = t_0$.

$\square$

Theorem 3.3 If $\mu_X \geq 0$, $\mu_Y \geq 0$, and $\mu_X + \mu_Y > 0$, then any continuous solution to (25) with (26) for $t \in [0, \infty)$, satisfying an initial condition

$$X_0 + Y_0 < 1 \quad (41)$$

remains

$$X(t) + Y(t) < 1. \quad (42)$$

Proof.

Let $t_0 \in (0, \infty]$ be the first exit time of $(X, Y, Z)^T$ from

$$\Omega^{xy} = \{(x, y, z)^T \mid x + y < 1\}. \quad (43)$$

Assume $t_0 < \infty$. If $X(t_0) + Y(t_0) = 1$, then

$$\frac{d}{dt}(X + Y) = -\mu_X X - \mu_Y Y < 0 \quad (44)$$

at $t = t_0$, contradicting $X(t) + Y(t) < 1$ for $t < t_0$. $\square$

Next, we identify the equilibria of the model. The agroecological implication of an equilibrium is the balance among nitrification, denitrification, maturation of rhizobia, SNF, uptake of $\text{NO}_3\text{-N}$, and plant growths. Due to the complex structure of the Jacobian matrix (66), we opt for a numerical approach to examine their stability except for the critical cases of $\mu_Y = 0$.

The equilibria (X^*, Y^*, Z^*) are the zeros of

$$\begin{cases} f_X(X^*, Y^*, Z^*) = 0 \\ f_Y(X^*, Y^*, Z^*) = 0 \\ f_Z(X^*, Y^*, Z^*) = 0 \end{cases} \quad (45)$$

For the case of $X^* = 0$, $Y^* \neq 0$, and $Z^* \neq 0$, combining $f_Y = 0$ and $f_Z = 0$ leads to that Z^* necessarily solves the quartic equation

$$-K_Y(\beta_Y - Z^*)\varphi(Z^*)(1 - \varphi(Z^*)) + K_Z(1 - Z^*) - \mu_Z = 0 \quad (46)$$

where

$$\varphi(Z) = \mu_Y / \alpha_Y K_Y Z (\beta_Y - Z), \quad (47)$$

and

$$Y^* = 1 - \varphi(Z^*). \quad (48)$$

For the case of $X^* \neq 0$, $Y^* = 0$, and $Z^* \neq 0$, combining $f_X = 0$ and $f_Z = 0$ leads to that Z^* necessarily solves the quartic equation

$$-K_X(\beta_X - Z^*)\psi(Z^*)(1 - \psi(Z^*)) + K_Z(1 - Z^*) - \mu_Z = 0 \quad (49)$$

where

$$\psi(Z) = \frac{\mu_X}{\alpha_X K_X Z(\beta_X - Z) + K_S(Z + \gamma_X)(\beta_Z - Z)}, \quad (50)$$

and

$$X^* = 1 - \psi(Z^*). \quad (51)$$

For the case of $X^* \neq 0$ and $Y^* \neq 0$, combining $f_X = 0$ and $f_Y = 0$ leads to that Z^* necessarily solves the quadratic equation

$$\begin{aligned} & (\mu_X \alpha_Y K_Y - \mu_Y (\alpha_X K_X + K_S)) Z^{*2} \\ & - (\mu_X \alpha_Y \beta_Y K_Y - \mu_Y (\alpha_X \beta_X K_X + (\beta_Z - \gamma_X) K_S)) Z^* + \mu_Y \beta_Z \gamma_X K_S = 0. \end{aligned} \quad (52)$$

When $\mu_Y \neq 0$, X^* is calculated as

$$X^* = \frac{K_Y(\beta_Y - Z^*)(\varphi(Z^*) - 1) + (K_Z - \mu_Z - K_Z Z^*)/\varphi(Z^*)}{K_X(\beta_X - Z^*) - K_Y(\beta_Y - Z^*)} \quad (53)$$

from a real number Z^* satisfying (46), and then Y^* is calculated as

$$Y^* = 1 - X^* - \varphi(Z^*). \quad (54)$$

When $\mu_Y = 0$, the two solutions of (46) are 0 and β_Y . Any point satisfying

$$X^* + Y^* = 1 - \mu_X/\beta_Z \gamma_X K_S = 1 - \psi(0), \quad Z^* = 0 \quad (55)$$

or

$$X^* = 0, \quad Z^* = 0 \quad (56)$$

is eventually an equilibrium, while $Z^* = \beta_Y$ almost surely does not solve $f_Z = 0$. There are also

equilibriums $(1-\psi(0), 0, 0)^T$ and $(0, 0, 0)^T$ along the X -axis. The three eigenvalues of the Jacobian matrix for the cases of (55) or (56) are 0, $-\beta_Z \gamma_X K_S X^* (\leq 0)$, and

$$\lambda_{XY} = -\psi(0)(\beta_X K_X X^* + \beta_Y K_Y Y^*) + K_Z - \mu_Z. \quad (57)$$

Therefore, a semi-stability condition is given as

$$\beta_X K_X X^* + \beta_Y K_Y Y^* \geq \frac{K_Z - \mu_Z}{\psi(0)}. \quad (58)$$

3.3.3 Identification method for model parameters

The least-squares method is employed to identify the values of model parameters of the model from the observed data series of X , Y , and Z . Here, we use the notation

$$\{\Delta y\} = \sum_i (a_i) \{x_i\} \quad (59)$$

to determine the coefficients a_i from the data series of x_i and y , minimizing the sum of the square errors between $\sum_i a_i x_i$ and Δy , which is the one-step temporal difference of y . There are six implementations of the least-square method, abbreviated as HeZdZ, H3YZdZ, H2XYdZ, H3YZdY, H3YZdY0, and H2XZdX:

HeZdZ to identify K_Z and μ_Z from the data series of Z observed in H1 & H5

$$\{\Delta Z\} = (K_Z - \mu_Z) \{Z\} + (-K_Z) \{Z^2\}; \quad (60)$$

H3YZdZ to identify K_Y , β_Y , K_Z , and μ_Z from the data series of Y and Z observed in H3

$$\{\Delta Z\} = (-K_Y \beta_Y) \{YZ(1-Y)\} + (K_Y) \{YZ^2(1-Y)\} + (K_Z) \{Z(1-Z)\} + (-\mu_Z) \{Z\}; \quad (61)$$

H2XZdZ to identify K_X , β_X , K_Z , and μ_Z from the data series of X and Z observed in H2

$$\{\Delta Z\} = (-K_X \beta_X) \{XZ(1-X)\} + (K_X) \{XZ^2(1-X)\} + (K_Z) \{Z(1-Z)\} + (-\mu_Z) \{Z\}; \quad (62)$$

H3YZdY to identify α_Y and μ_Y from the data series of Y and Z observed in H3 with the results of H3YZdZ

$$\{\Delta Y\} = (\alpha_Y) \{K_Y(1-Y)Z(\beta_Y - Z)Y\} + (-\mu_Y) \{Y\}; \quad (63)$$

H3YZDY0 to identify α_Y from the data series of Y and Z observed in H3 with the results of H3YZdZ, assuming $\mu_Y = 0$

$$\{\Delta Y\} = (\alpha_Y) \{K_Y (1-Y) Z (\beta_Y - Z) Y\}; \quad (64)$$

and H2XZdX to identify α_X , μ_X , K_S , β_Z , and γ_X from the data series of X and Z observed in H2 with the results of H2XZdZ

$$\begin{aligned} \{\Delta X\} = & \{\alpha_X\} \{K_X (1-X) Z (\beta_X - Z) X\} + \{K_S \beta_Z \gamma_X\} \{X (1-X)\} \\ & + \{K_S (\beta_Z - \gamma_X)\} \{X (1-X) Z\} + \{-K_S\} \{X (1-X) Z^2\} + \{-\mu_X\} \{X\}. \end{aligned} \quad (65)$$

There will be three different pairs of K_Z and μ_Z values from HeZdZ, H3YZdZ, and H2XYdZ, and two different pairs of α_Y and μ_Y values from H3YZdY and H3YZdY0.

3.4. Results and discussions

3.4.1 Fate of N and justification of the model structure

The $\delta^{15}\text{N}$ -values of the plant samples from the horizontal plot are summarized in **Table 4**. It is difficult to consistently determine the values of $\delta^{15}\text{N}_{\text{reference}}$ to calculate %Ndfa and %Ntr with (23) and (24), thus, we exclusively use $\delta^{15}\text{N}$ to infer the fate of N derived from air and transferred between the plant species. Each plant sample of Week 9 and Week 12 in the vegetative stage was measured as a whole. In contrast, for separate measurements, the plant samples of Week 17 in the budding stage and Week 20 in the flowering stage were divided into different organs, including root, stolon, stem, leaf, and flower. The $\delta^{15}\text{N}$ -value of the soil sampled immediately before the setup of the experimental plots was 5.217 ‰, referred to as $\delta^{15}\text{N}_{\text{soil}}$. This positive value of $\delta^{15}\text{N}_{\text{soil}}$ is due to the isotope fractionation during the agricultural use of the soil. The $\delta^{15}\text{N}$ -values between $\delta^{15}\text{N}_{\text{air}}$ (= 0.000) and $\delta^{15}\text{N}_{\text{soil}}$ (= 5.217), emphasized in *Italic font* in **Table 4**, imply that the N in the plant samples originates from SNF. However, the converse is not always true because of the isotope fractionation during nitrification and

denitrification in the soil, the transport of N from the nodules or the roots toward the other organs of the plant, and assimilation by the plants (Yoneyama et al., 1986; Yoneyama et al., 1991). All the $\delta^{15}\text{N}$ -values of clover from H2, except for those in the flower and the leaf in Week 20, strongly indicate the activity of SNF and thus the presence of f_{XX} . The $\delta^{15}\text{N}$ -values of mint from H3 support the absence of SNF, but that in the root in Week 17 is $1.340 < \delta^{15}\text{N}_{\text{soil}}$. In significant contrast with the $\delta^{15}\text{N}$ -values from H3, the $\delta^{15}\text{N}$ -values of mint smaller than $\delta^{15}\text{N}_{\text{soil}}$ are more prevalent from H4, leading to the inference that the N symbiotically once fixed in clover was emitted to the soil as negative f_{ZX} and then uptaken by mint as positive f_{ZY} . Indeed, there were stolons of clover invading from H2 to H3 since Week 10 that might have allowed that process to cause $\delta^{15}\text{N} = 1.340$ in Week 17. The isotope fractionation from the budding stage to the flowering is verified with statistical tests. The one-side p -values of the Mann-Whitney U test (Mabaya et al., 2016) for the null hypothesis that the distributions of $\delta^{15}\text{N}$ are identical in Week 17 and Week 20, are 0.0556 in clover from H2, 0.1143 in mint from H3, 0.0357 in clover from H4, and 0.0556 in mint from H4. Large increments of $\delta^{15}\text{N}$ from Week 17 to Week 20 appeared in the roots of clover and mint from H4. The images captured by the root imager indicated that the plant roots developed the most during that period, implicating that the increments are due to the isotope fractionation in biomass, even within the roots.

Table 4. $\delta^{15}\text{N}$ of the plant samples from the horizontal plot.

Week	Part	Clover from H2	Mint from H3	Clover from H4	Mint from H4
9	Whole	3.754	6.606	6.040	*
12	Whole	1.715	6.773	4.168	6.761
17	Leaf	1.573	5.332	2.122	4.800
	Stem	0.684	6.104	1.040	4.591
	Stolon	1.639	*	*	2.077
	Root	1.638	1.340	0.786	0.896
20	Flower	8.336	5.172	5.258	10.615
	Leaf	7.815	25.042	6.443	7.370
	Stem	1.054	16.488	8.262	1.798
	Stolon	3.140	27.378	1.976	5.812
	Root	2.480	**	9.278	8.116

*The sample was too tiny to measure

**The instrument malfunctioned, possibly due to incomplete combustion of the sample

3.4.2 Data of $\text{NO}_3\text{-N}$ in soil pore water and LA for identifying model parameters

During Weeks 0-13, when $\text{NO}_3\text{-N}$ and LA were observed, occasional irrigation and the seasonal climate maintained appropriate soil moisture and thermal environment for plant growth. Therefore, the N cycle is considered the dominant factor of the agroecological system. Determining LA from camera images was suspended after Week 14 because the leaves overlap in mixture areas. The maximum value of $\text{NO}_3\text{-N}$ concentration among the 28 observations from the uncultivated areas of H1 and H5 is 58.0 mg/L, and we estimate the actual maximum at $58.0 \times 28 / (28 - 1) = 60.148$. Then, all the $\text{NO}_3\text{-N}$ values are divided by 60.148 for standardization. **Table 5** shows the observed and standardized data. The time variation in the $\text{NO}_3\text{-N}$ values exhibits a low rate of increase in Weeks 0-4, followed by a high rate of increase in Weeks 5-7, before the decline in Week 8. The peak $\text{NO}_3\text{-N}$ values are in the order of H5 (uncultivated), H3 (mint), H1 (uncultivated), H4 (mixture), and H2 (clover). From the data of **Table 5** alone, it is inferred that the non-leguminous mint improved the survival of the rhizobia even more efficiently than the clover. This may be because the mint, later in maturity, could excrete carbon substrates for a more extended period. Moreover, the high increase of $\text{NO}_3\text{-N}$ in less vegetated strips was likely because of the rhizobia diffusion in the soil pore. The total leaf area increased significantly from Week 5 and finally became stable.

Table 5. Observed and standardized data of Z (NO₃-N), X (clover LA), and Y (mint LA) for parameter identification (H1, H2, H3, and H5) and model validation (H4)

Week	Z in H1	Z in H2	Z in H3	Z in H5	X in H2	Y in H3	X in H4	Y in H4	Z in H4
0	0.025	0.039	0.044	0.037					0.039
1	0.021	0.020	0.027	0.022	0.005		0.005		0.030
2	0.042	0.020	0.068	0.008	0.009		0.004		0.012
3	0.035	0.039	0.099	0.084	0.010		0.005		0.031
4	0.036	0.040	0.034	0.044	0.017		0.006		0.051
5	0.081	0.078	0.060	0.069	0.025	0.000	0.011		0.075
6	0.550	0.168	0.134	0.389	0.050	0.004	0.020		0.392
7	0.520	0.289	0.362	0.537	0.082	0.015	0.031		0.318
8	0.647	0.168	0.803	0.964	0.138	0.039	0.049		0.299
9	0.494	0.288	0.776	0.712	0.264	0.109	0.087	0.007	0.429
10	0.434	0.175	0.351	0.613	0.351	0.203	0.123	0.007	0.201
11	0.429	0.221	0.196	0.712	0.438	0.304	0.219	0.015	0.163
12	0.181	0.268	0.361	0.421	0.489	0.286	0.222	0.003	0.452
13	0.193	0.080	0.175	0.347	0.534	0.457	0.318	0.012	0.136

The identification method developed in Subsection 3.3 was applied to the observed and standardized data from H1, H2, H3, and H5 to obtain the parameter values shown in **Table 6**. The values of Y from H4 were too small for parameter identification; therefore, the behavior of X , Y , and Z in H4 will be used for validation.

Table 6. Results of parameter identification

Parameter	From HeZdZ	Identified value From H3YZdZ	From H2XYdZ
α_X		1.168	
α_Y		0.128 from H3YZdY, 0.171 from H3YZdY0	
μ_X		0.532	
μ_Y		-0.110 from H3YZdY, 0.000 from H3YZdY0	
μ_Z	0.367	0.108	3.799
K_X		-24.133	
K_S		1.365	
K_Y		3.127	
K_Z	0.849	2.248	5.237
β_X		0.169	
β_Y		3.887	
β_Z		0.234	
γ_X		6.371	

Fig. 9 is a chart to explain the interaction factions representing the Allee effects. The negative part of the red line for f_{ZX} implies N absorption by clover has a strong Allee effect, which involves a population threshold below which the per capita growth rate is negative, causing the population to decline. The green line indicates the dependency of N-fixation on N-concentration in the soil. The blue and orange lines show that N absorption by mint and N self-reaction, respectively, has no Allee effect.

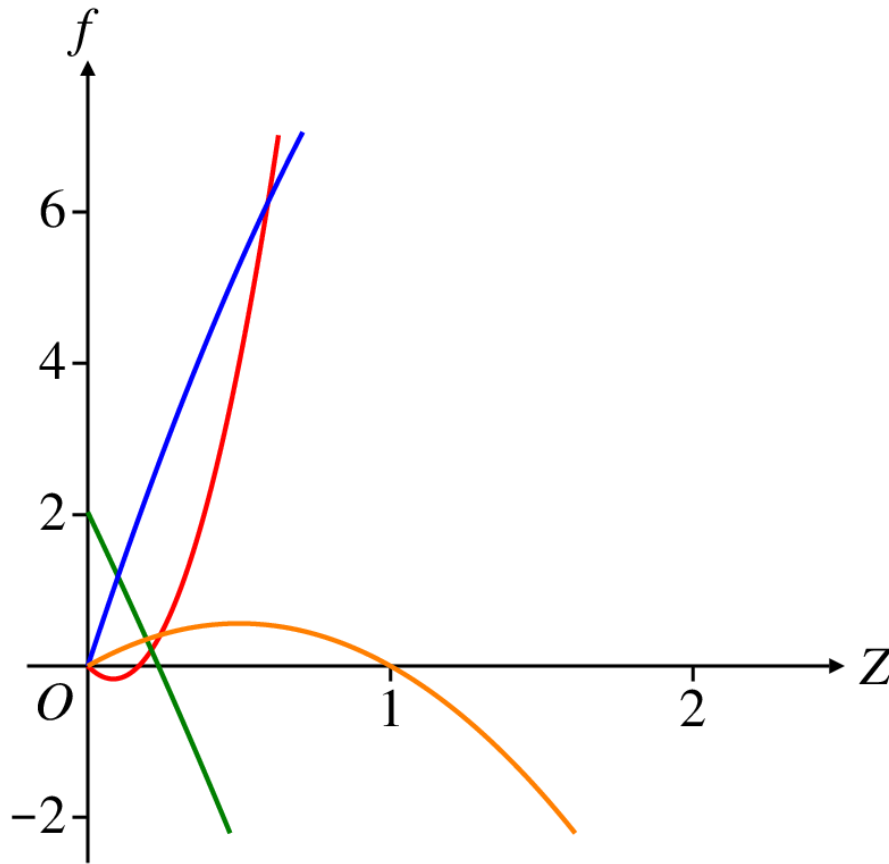


Fig. 9. Plots of $f_{ZX}/(1-X-Y)X$ (red), $f_{XX}/(1-X-Y)X$ (green), $f_{ZY}/(1-X-Y)Y$ (blue), and f_{ZZ} (orange) with the model parameters identified from H3YZdZ

3.4.3 Behavior of the dynamical system

Numerical simulations were implemented for different nine initial values $(X_0, Y_0, Z_0)^T = (0.5 \pm 0.49 k/4, 0.5 \pm 0.49 k/4, 0.5 \pm 0.49 k/4)^T$ ($k = 0, 1, 2, 3, 4$) in each case of parameter values. In all the cases, the numerical solutions to the dynamical system (25) approach to stable

equilibria for any of the nine initial values.

Figs. 10, 11, and 12 show the results when assuming $\mu_Y < 0$ with K_Z and μ_Z identified from HeZdZ, H3YZdZ, and H2XZdZ, respectively. No stable equilibrium exists in Ω^{xy} , but the numerical solutions achieve asymptotic convergence to stable equilibria of $X^* = 0$ and $Y^* > 1$, even if initially $X_0 + Y_0 < 1$, because **Theorem 3.3** is invalid.

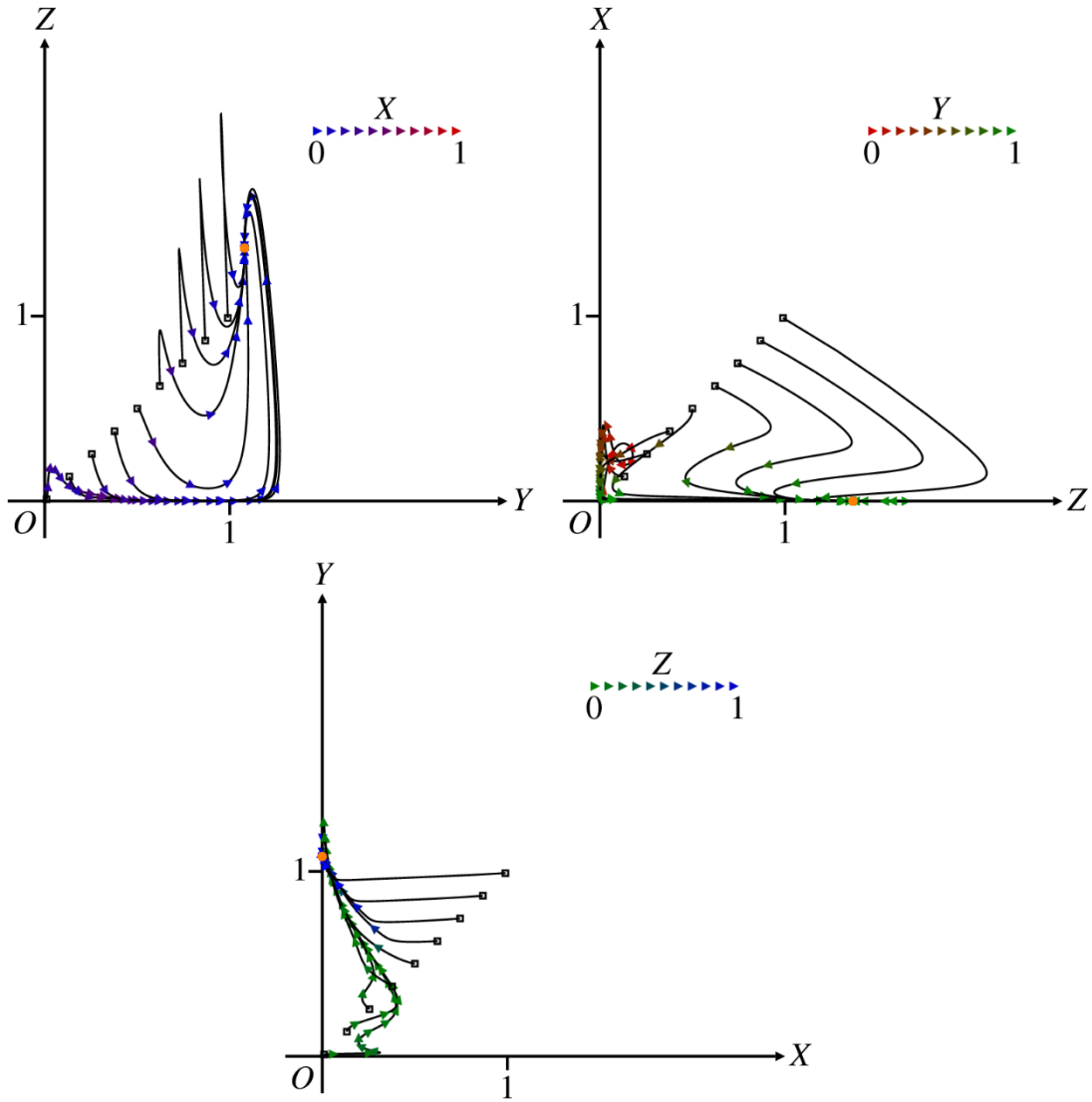


Fig. 10. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from HeZdZ, assuming $\mu_Y < 0$.

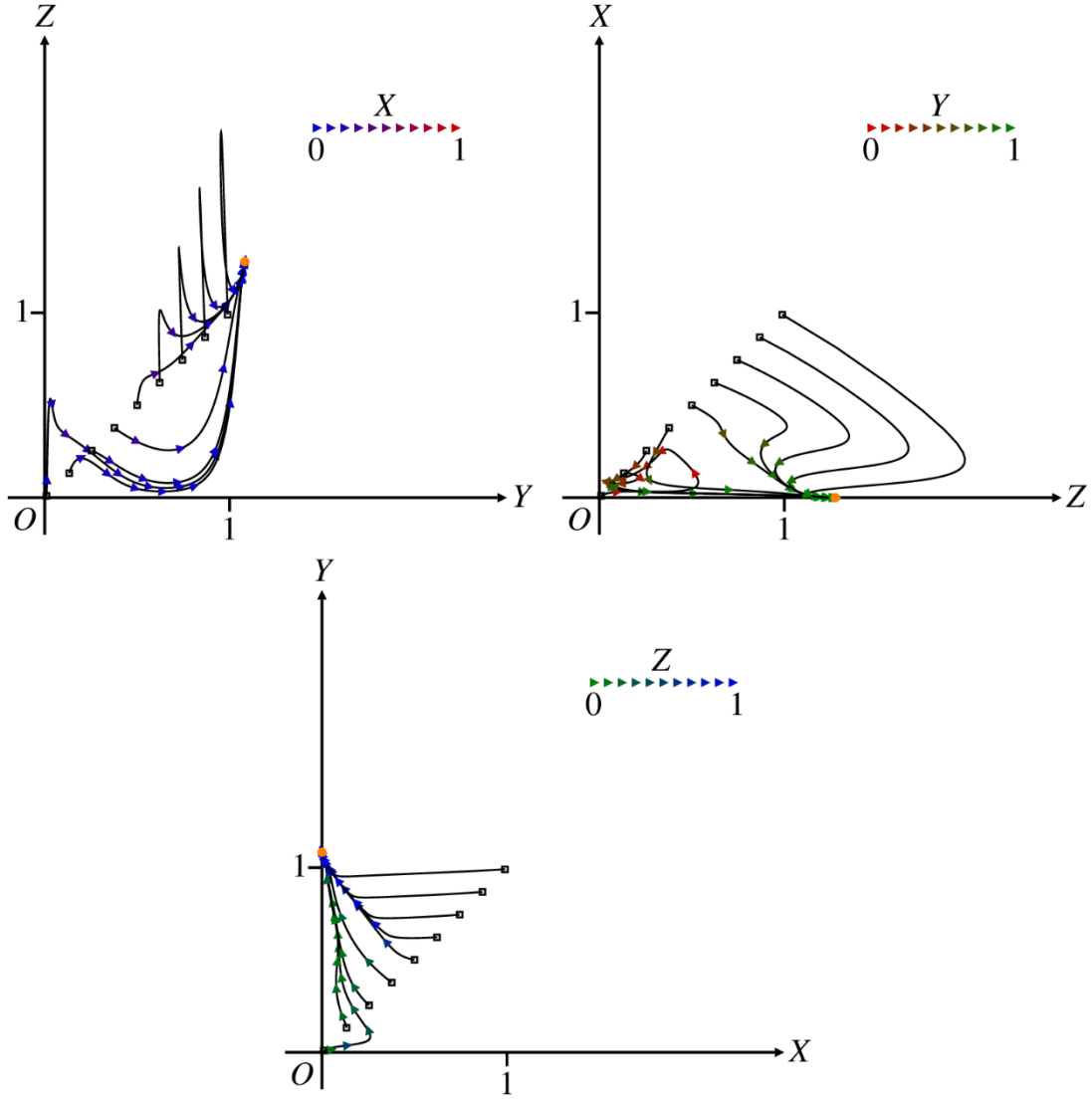


Fig. 11. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from H3YZdZ, assuming $\mu_Y < 0$.

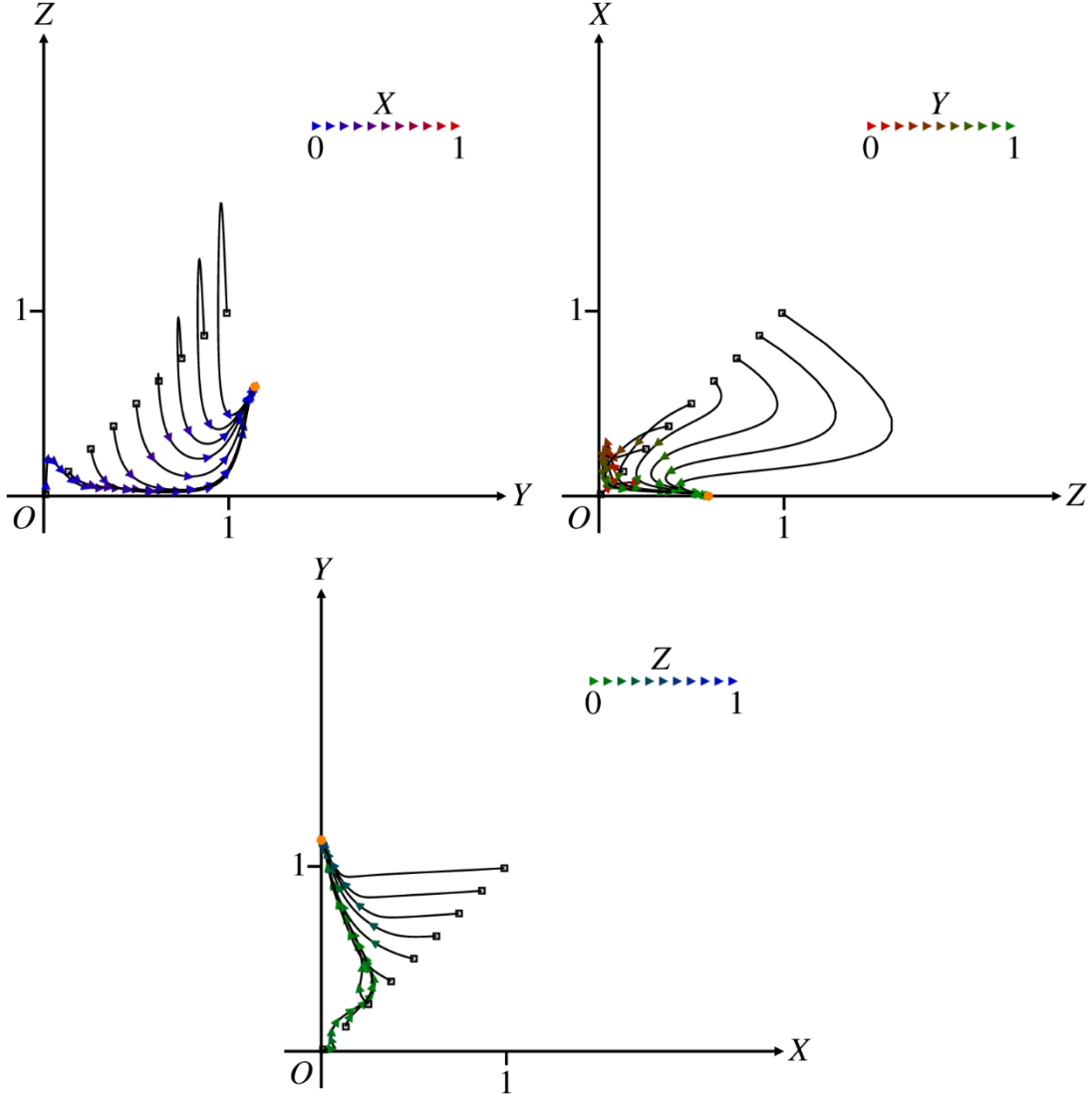


Fig. 12. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from $H2XZdZ$, assuming $\mu_Y < 0$.

Figs. 13, 14, and 15 show the results when assuming $\mu_Y = 0$ with K_Z and μ_Z identified from $HeZdZ$, $H3YZdZ$, and $H2XZdZ$, respectively. For this critical case of $\mu_Y = 0$ and thus $\varphi(Z^*) = 0$, we observe bifurcations occurring when the values of K_Z and μ_Z change. **Fig. 16** is a generic bifurcation diagram of asymptotically approached Y^* with respect to μ_Z , fixing K_Z at 1. It consists of a segment of isolated stable equilibria, achieving (46) and (48), and a region satisfying (55) and (58), with **Theorems 3.2** and **3.3** valid. The region implies that the

total abundance of the plants $X^* + Y^*$ in equilibrium is equal to 1, with the nutrient Z tending to zero after the peak occurs at an earlier stage.

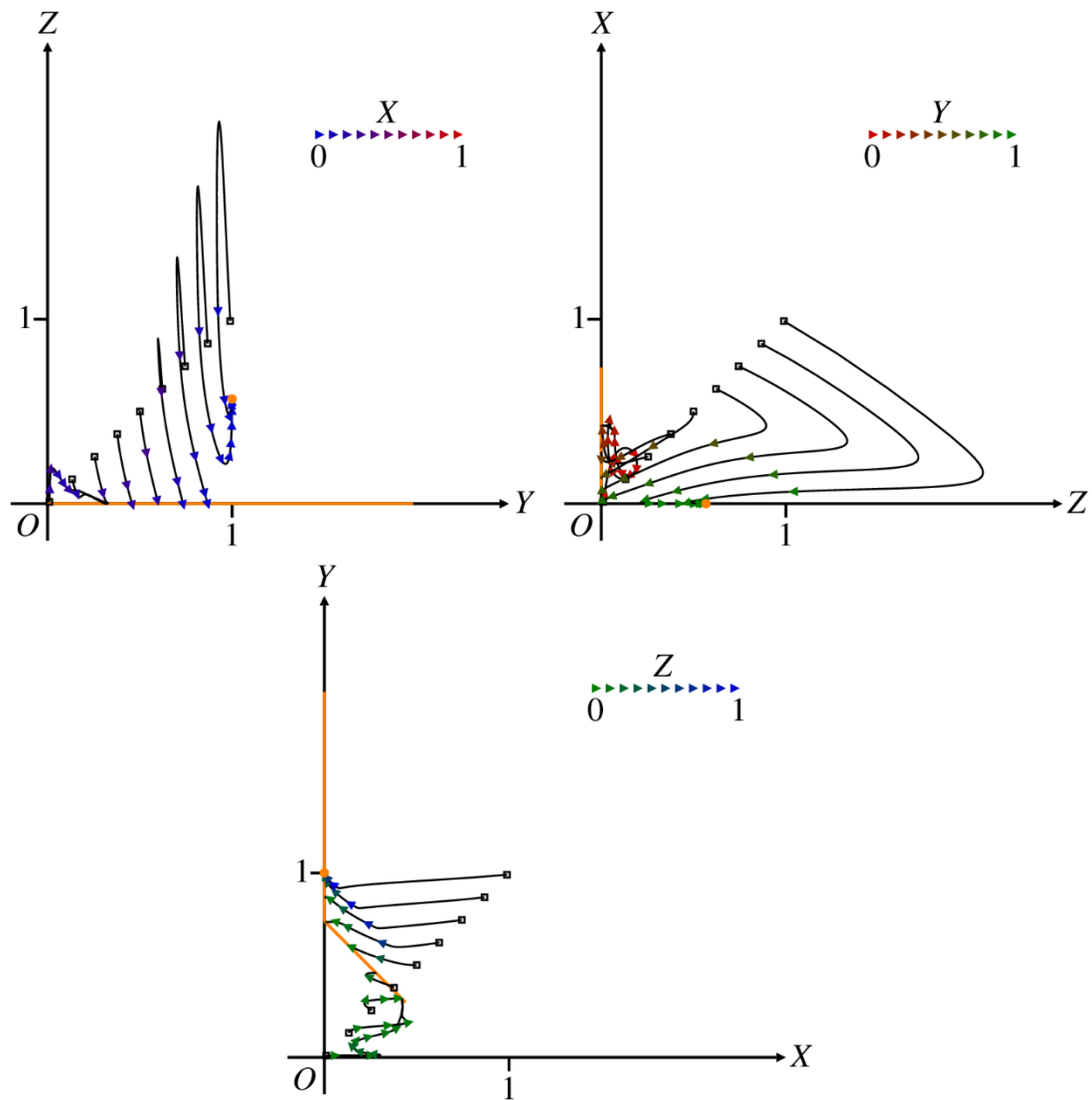


Fig. 13. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from HeZdZ, assuming $\mu_Y = 0$.

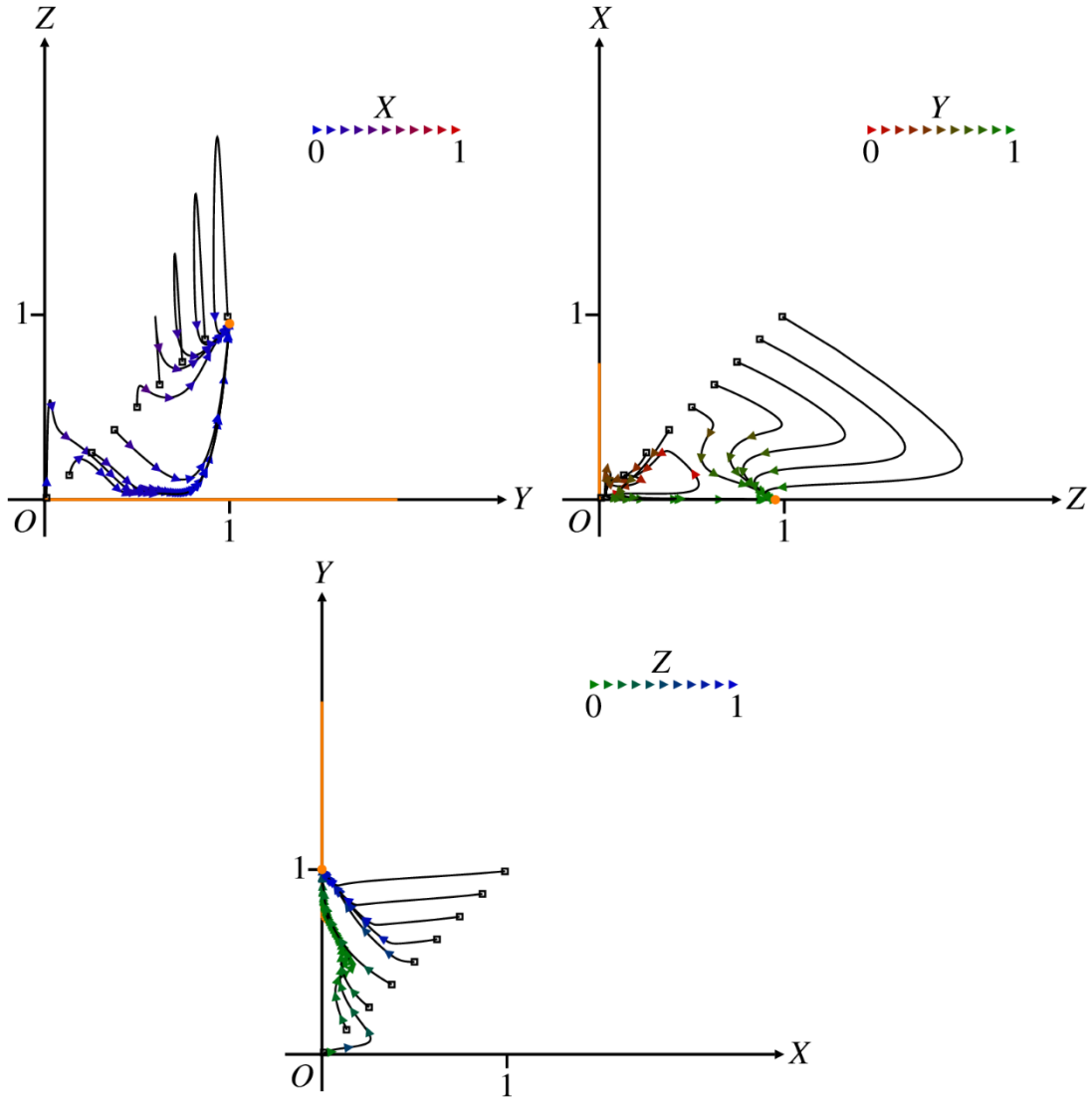


Fig. 14. Asymptotic behaviour of the dynamical system with K_Z and μ_Z identified from H3YZdZ, assuming $\mu_Y = 0$.

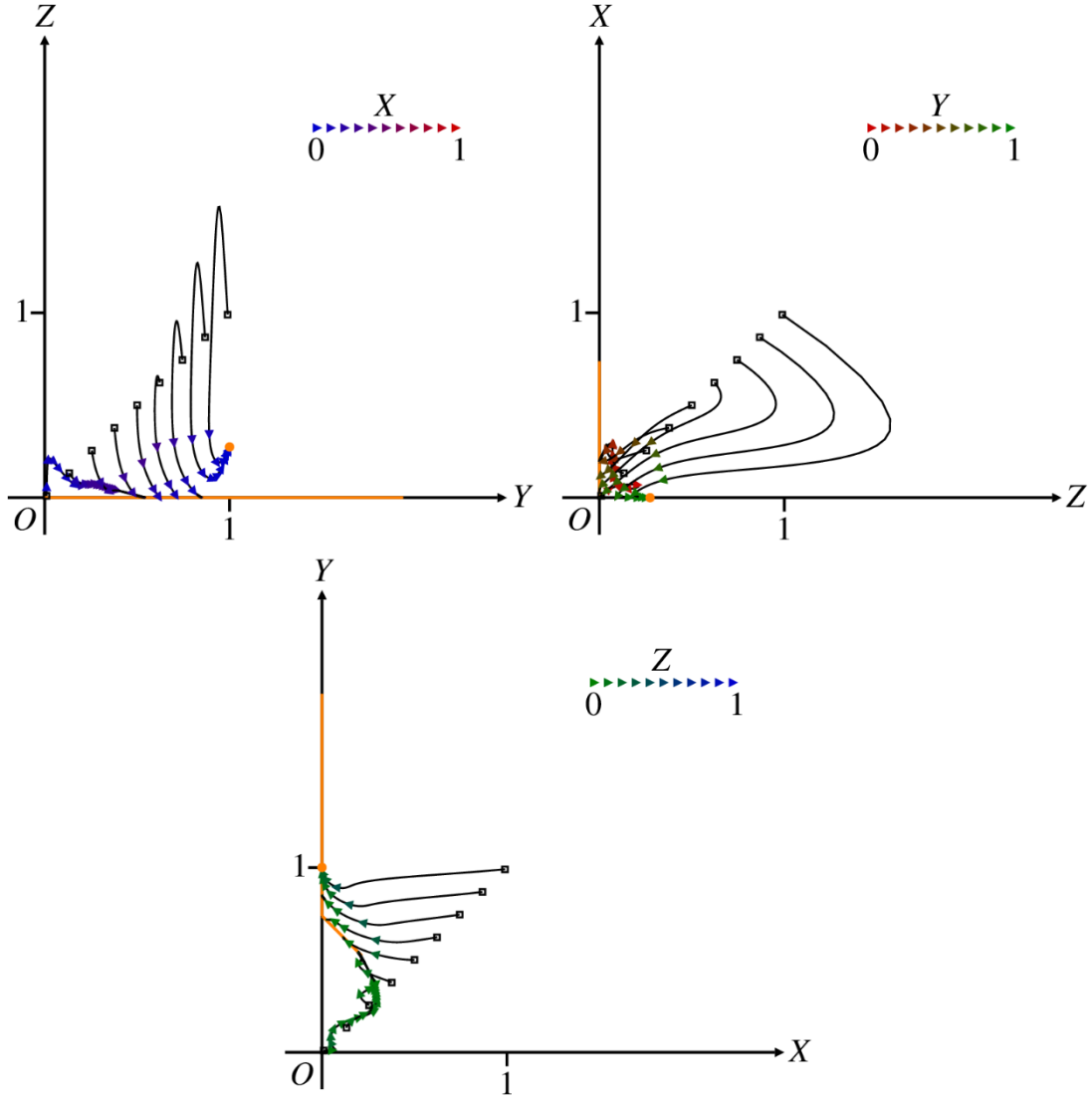


Fig. 15. Asymptotic behaviour of the dynamical system with K_z and μ_z identified from $H2XZdZ$, assuming $\mu_y = 0$.

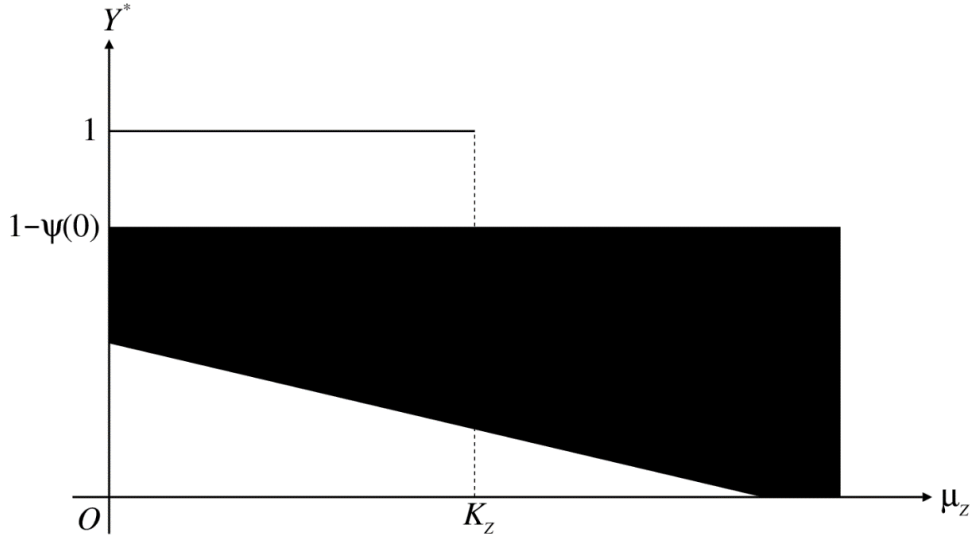


Fig. 16. Bifurcation diagram of asymptotically approached Y^* with respect to μ_z as the bifurcation parameter when $\mu_y = 0$ and $K_z = 1$.

Figs. 17, 18, and 19 show the results when assuming $\mu_y = 0.110 > 0$ (μ_y from H3YZdY multiplied by -1) with K_z and μ_z identified from HeZdZ, H3YZdZ, and H2XZdZ, respectively. Linearly extrapolating the values of α_y and μ_y from H3YZdY and H3YZdY0 determines $\alpha_y = 0.213$ to be consistent with $\mu_y = 0.110$. The numerical solutions initially exhibit oscillatory behaviors before asymptotically converging to the equilibria, which are in $\mathbb{R}_+^3 \cap \Omega^{xy}$ and unique under the assumptions of **Theorem 3.3**. However, as shown in **Fig. 18** with K_z and μ_z identified from H3YZdZ, the numerical solutions approach to another stable equilibrium of $X^* = 0$ if the initial values are of $X_0 + Y_0 > 1$.

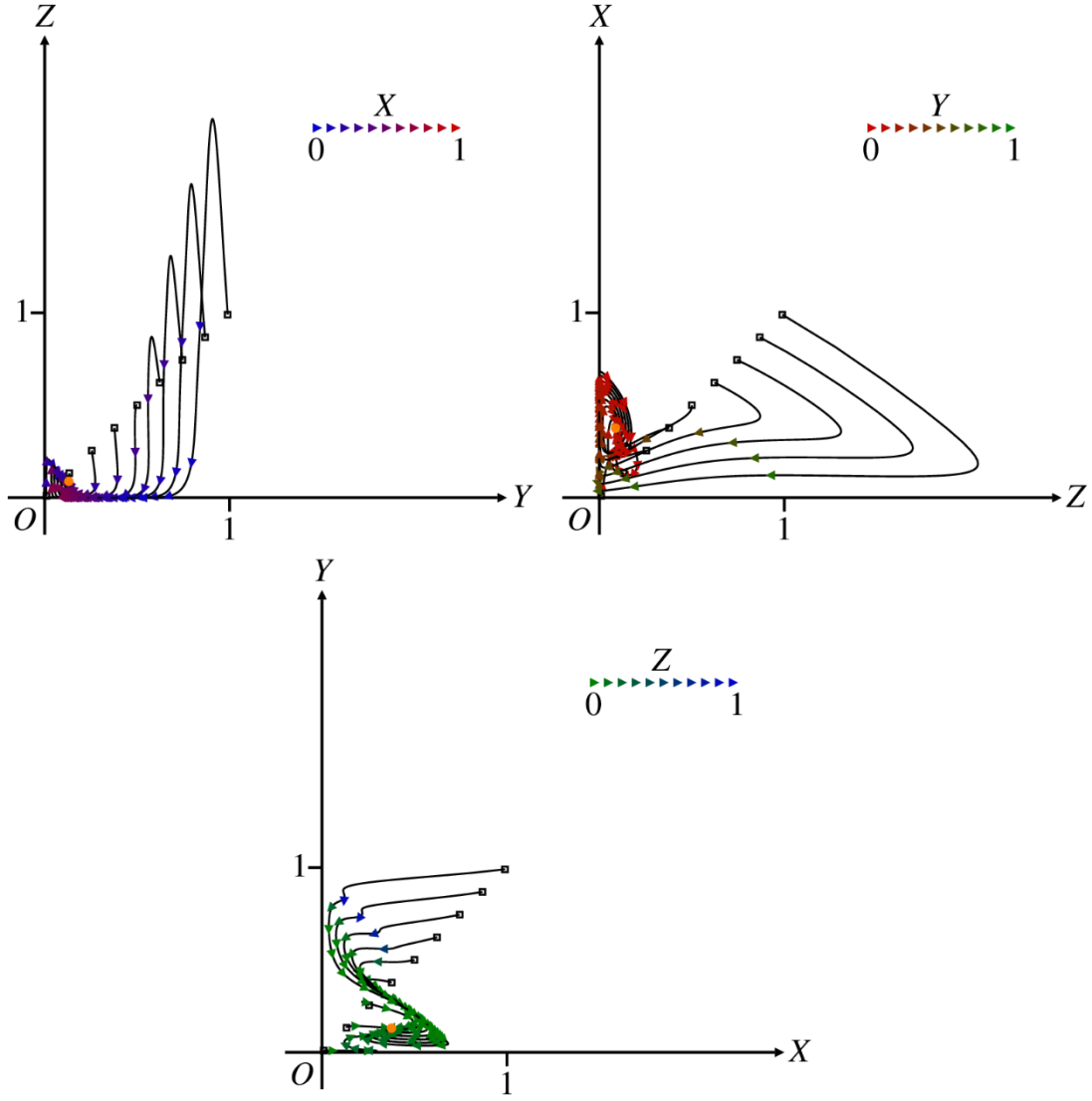


Fig. 17. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from HeZdZ, assuming $\mu_Y > 0$.

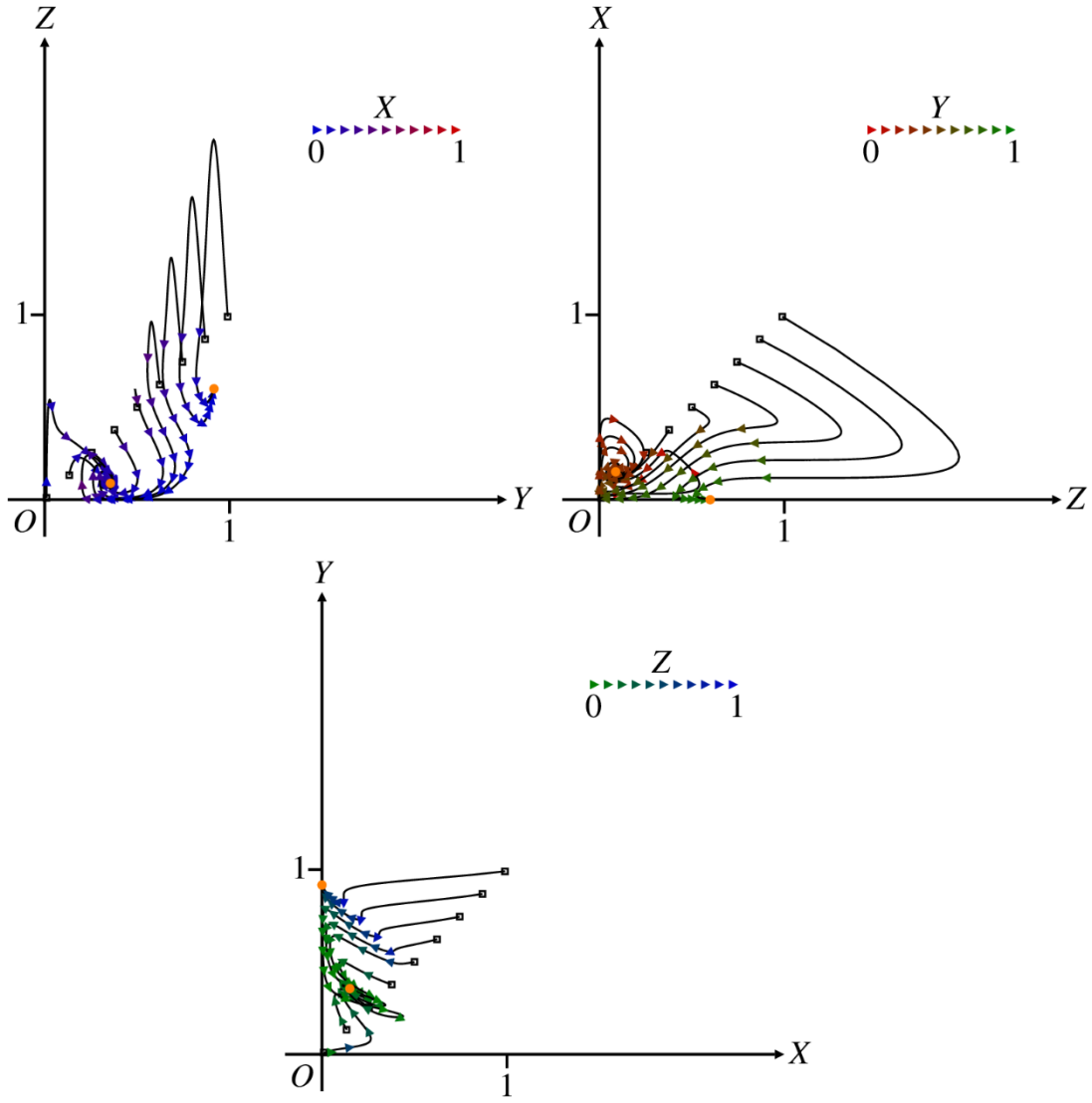


Fig. 18. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from H3YZdZ, assuming $\mu_Y > 0$.

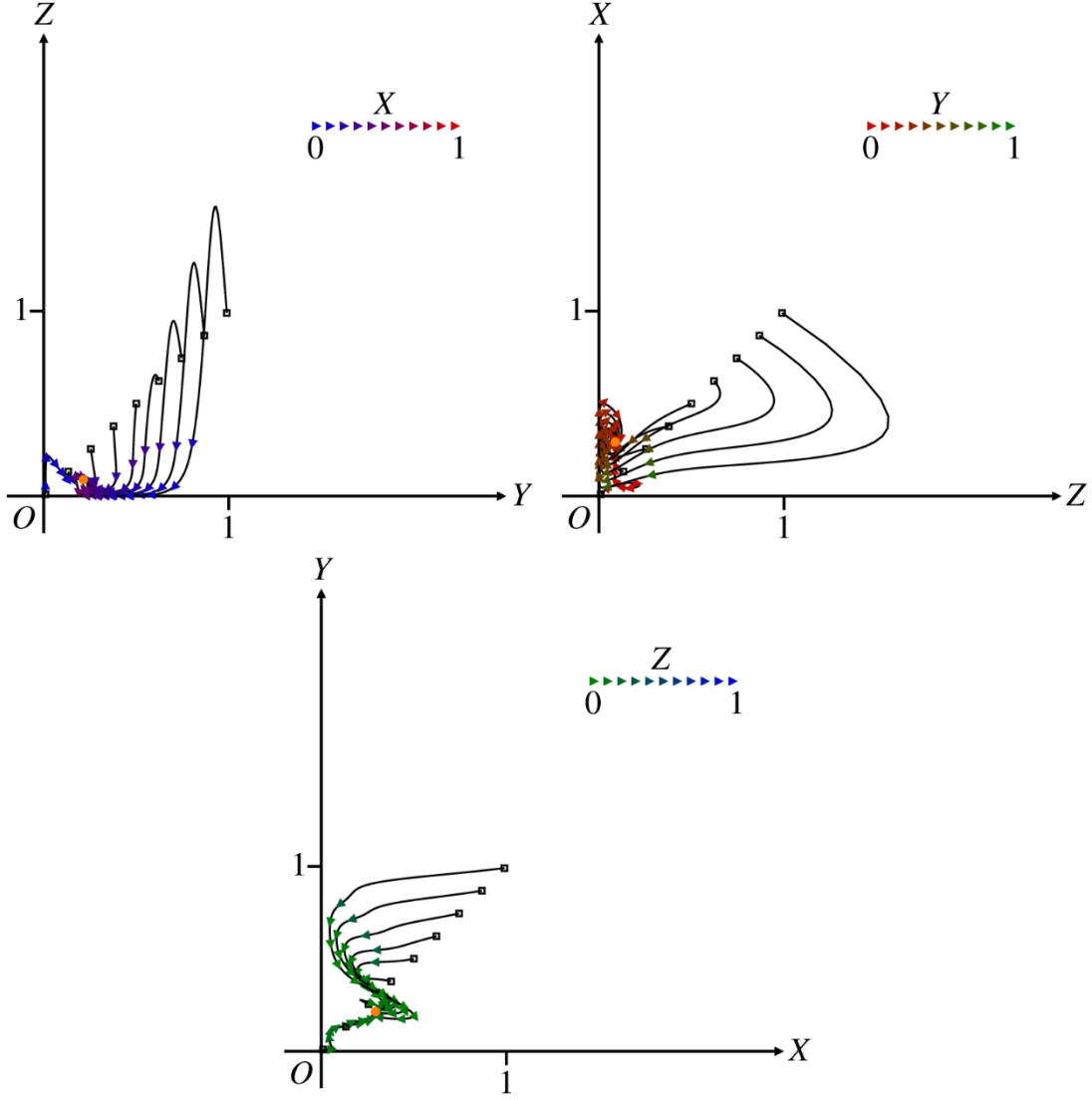


Fig. 19. Asymptotic behavior of the dynamical system with K_Z and μ_Z identified from H2XZdZ, assuming $\mu_Y > 0$.

Table. 7, where $\Omega_+^{xz} = \{(x, y, z)^T \mid x > 0, y = 0, z > 0\}$, summarizes the computational results to demonstrate how equilibria bifurcate with respect to μ_X when $K_X = -3.01$. The equilibria and the eigenvalues of the Jacobian matrix (66) at them were calculated for each μ_X ranging between -1.000 and 0.100 with an increment of 0.001 . When μ_X is small enough, a stable equilibrium does not exist, so the dynamical system becomes oscillatory in a limit cycle. The asymptotical behavior is sensitive to the initial value if at least one stable equilibrium exists. The most significant case is when $\mu_X = -0.070$, and the computational

results are presented as the bifurcation diagram with respect to the initial values $(X_0, Y_0, Z_0)^T = (p, p, p)^T$ ($0.0 < p < 1.5$), shown in **Fig. 20**. In this **Fig. 20**, the Hopf bifurcation can be seen: there are three Zones (I), (II), and (III), where the system dynamics sharply shifts between asymptotical convergence to the different two stable equilibriums and oscillation in a limit cycle.

Table 7 Bifurcation of equilibriums with respect to μ_x when $K_x = -3.01$

Range of μ_x	Number of equilibriums in $\mathbb{R}_+^3$	Number of equilibriums in Ω_+^{yz}	Number of equilibriums in Ω_+^{xz}
(Figures in parentheses indicate the numbers of stable equilibriums)			
$\mu_x \leq -0.265$	0 (0)	3 (0)	2 (0)
$-0.264 \leq \mu_x \leq -0.246$	1 (0)	3 (0)	2 (0)
$-0.245 \leq \mu_x \leq -0.217$	1 (1)	3 (0)	2 (0)
$-0.216 \leq \mu_x \leq -0.178$	0 (0)	3 (1)	2 (0)
$-0.177 \leq \mu_x \leq -0.071$	1 (0)	3 (1)	2 (1)
$-0.070 \leq \mu_x, \mu_x \neq 0.000$	2 (0)	3 (1)	2 (1)
$\mu_x = 0.000$	2 (0)	3 (1)	5 (1)

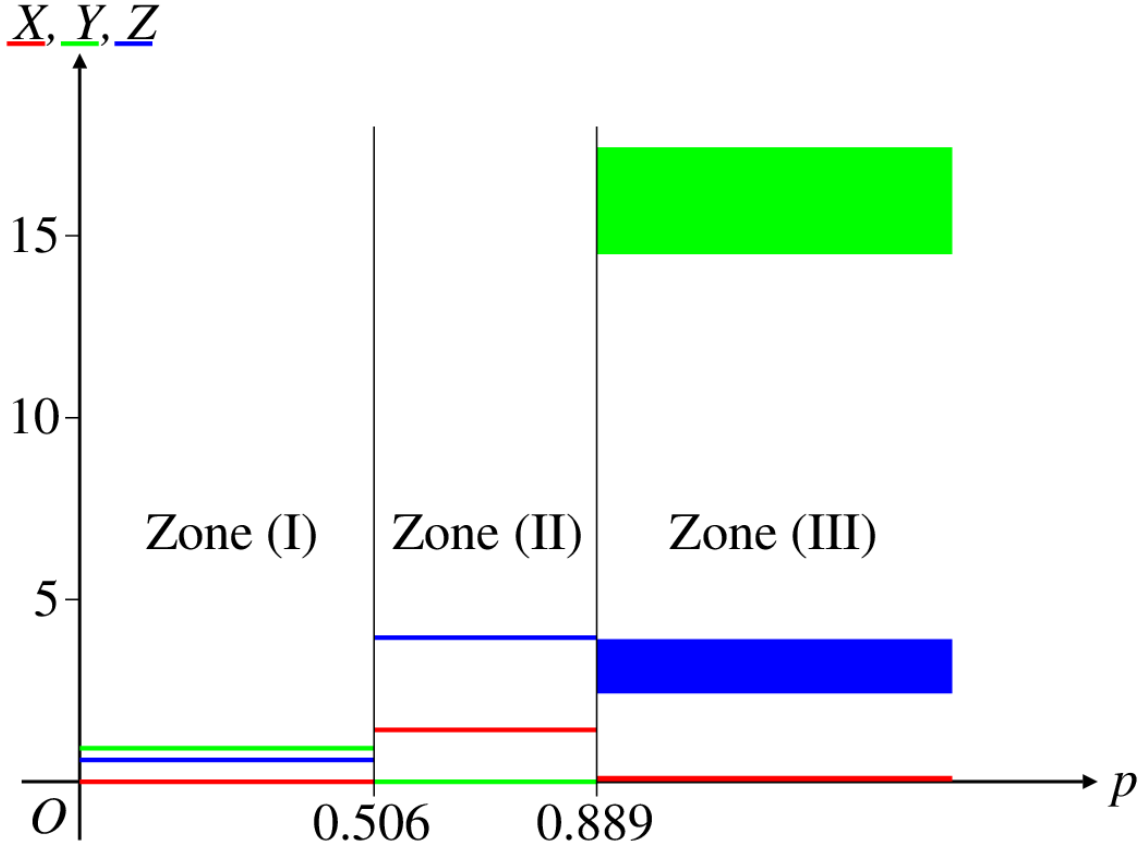


Fig. 20. Bifurcation diagram of asymptotically approached X (in red color), Y (in green color), and Z (in blue color) with respect to the initial value as the bifurcation parameter when $\mu_X = -0.070$ and $K_X = -0.301$. Zone (I): asymptotical convergence to the stable equilibrium in Ω_+^{yz} ; Zone (II): asymptotical convergence to the stable equilibrium in Ω_+^{xz} ; Zone (III) oscillation in a limit cycle in $\mathbb{R}_+^3$.

3.4.4 Validation with the real-world experiment

As shown in the rightest three columns of **Table 5**, clover (X) grew rapidly in Weeks 0-13, along with $\text{NO}_3\text{-N}$ (Z), showing fluctuations from Weeks 6, while the growth rate and maturity of mint (Y) were slow. This dynamical behavior in the experiment is the most consistent with that of the model with $\mu_Y (< 0)$, K_Z , and μ_Z identified from H3YZdY and H3YZdZ , as shown in **Fig. 11**, in the early stages. However, contrary to the asymptotic behavior indicated in **Fig. 11**, the coexistence of clover and mint was achieved in H4 of the experiment, as shown in the photo of **Fig. 21**, implicating the mortality μ_Y of mint turned from negative to positive. Significantly high weekly maximum soil temperatures over 35°C during the hottest Weeks 9-

21 explain the shift of the mortality. **Fig. 21** also indicates horizontal growth of the plants, which is more in clover, invalidating the model's hypotheses. The growth trends of the plants were faster in the horizontal plot than in the inclined plot facing to the south, most probably due to the significant difference in the maximum temperatures during Weeks 9-21: 35.4 °C on average in H2 and H3 and 38.7 °C on average in I2 and I3. The soil moisture in the inclined plot was less than the appropriate level. There were eyewitnesses of wetted zones propagating downward in the soil surfaces of the inclined plot during irrigation events. The resulting advection of NO₃-N invalidated the assumptions of the parameter identification method proposed in Subsection 3.3. Indeed, applying that method to the inclined plot yielded irrational negative values of the N absorption coefficient α_x by clover and the N fixation coefficient K_s .



Fig. 21. Photo of half of the horizontal plot in Week 20 covering a part of H3, the entire H4, and the entire H5.

According to these observations of the real-world experiment, the model (25) needs amendments to take the horizontal growths of the plants and the transport of NO₃-N into

account. A system of partial differential equations will be a reasonable extension of the model governing the spatially distributed concentrations of X , Y , and Z , whose fluxes are determined by the tropisms of the plants and the fluid mechanics in the soil.

3.5. Conclusions

This study investigated the dynamical relationships among the leguminous plant (clover), the non-leguminous plant (mint), and the nutrient resource ($\text{NO}_3\text{-N}$) with the lumped dynamical system model and the field experiment, primarily considering the setup of the horizontal plot. The two plant species were assumed to compete over the nutrient resource, but the involvement of SNF based on the mutualism between clover and rhizobia made the dynamical relationships asymmetric. Through mathematical analysis, we confirmed that continuous solutions remain positive for positive initial conditions and clarified the range of parameters μ_X and μ_Y to guarantee the boundedness of positive continuous solutions.

The values of $\delta^{15}\text{N}$ determined for the plant samples in different growth stages semi-quantified the fate of N, such as SNF and transfer between the plant species, well supporting the presence of f_{XX} representing SNF, possibly negative f_{ZX} , and positive f_{ZY} in the model. Moreover, the results of stable isotope analysis for different organs with the observation using the root imager implicated the isotope fractionation in the biomass.

The repetitive use of the least square method identified the values of the model parameters from the observed and standardized data of LA and $\text{NO}_3\text{-N}$ during the early growth stages of the plants. Then, we analyzed the quadratic nonlinearity of f_{ZX} with respect to Z , indicating the strong Allee effect that involves a negative growth rate of clover and transfer of N from clover to the soil. The identified mortality μ_Y of mint was negative.

Then, based on those values of the model parameters obtained from the field experiment, we analytically calculated the several types of equilibria and numerically examined their stability, implying one or more of the three variables may asymptotically become extinct.

Indeed, the model comprehensively encompasses various cases regarding the dynamical system's asymptotic behaviors. The sign of μ_Y was found essential to determine which types of stable equilibria eventually appear. Bifurcation analysis was conducted for the critical case of $\mu_Y = 0$, delineating asymptotically approached Y^* with respect to μ_Z .

Finally, we conducted model validation against the measurements and observations of the real-world experiment. A contradiction exists in the asymptotic behaviors between the model and the experiment during the late growth stages. However, allowing for variable signs of the mortality μ_Y of mint from negative to positive due to the hot summer environment well explains the phenomena in the real world. Reasonable values of the model parameters were not obtained from the inclined plot, where advection by the irrigation water was dominant in transporting $\text{NO}_3\text{-N}$. The suggested alternative model considering the horizontal growths of the plants and the transport of $\text{NO}_3\text{-N}$ is a system of partial differential equations, including the partial derivatives of first and second orders to represent the mechanisms of tropisms and diffusions. We expect to reproduce complex spatio-temporal vegetation patterns, including extinction, coexistence, and segregation, from such a model in future studies.

Appendix A. Jacobian matrix of (25)

Each entry of the Jacobian matrix of (25) is expressed as

$$\left\{ \begin{array}{l} \frac{\partial f_x}{\partial X} = \alpha_x K_x (1 - 2X - Y)Z(\beta_x - Z) + K_s (1 - 2X - Y)(Z + \gamma_x)(\beta_z - Z) - \mu_x \\ \frac{\partial f_x}{\partial Y} = -\alpha_x K_x Z(\beta_x - Z)X - K_s (Z + \gamma_x)(\beta_z - Z)X \\ \frac{\partial f_x}{\partial Z} = \alpha_x K_x (1 - X - Y)(\beta_x - 2Z)X + K_s (1 - X - Y)(-2Z + \beta_z - \gamma_x)X \\ \frac{\partial f_y}{\partial X} = -\alpha_y K_y Z(\beta_y - Z)Y \\ \frac{\partial f_y}{\partial Y} = \alpha_y K_y (1 - X - 2Y)Z(\beta_y - Z) - \mu_y \\ \frac{\partial f_y}{\partial Z} = \alpha_y K_y (1 - X - Y)(\beta_y - 2Z)Y \\ \frac{\partial f_z}{\partial X} = -K_x (1 - 2X - Y)Z(\beta_x - Z) + K_y Z(\beta_y - Z)Y \\ \frac{\partial f_z}{\partial Y} = K_x Z(\beta_x - Z)X - K_y (1 - X - 2Y)Z(\beta_y - Z) \\ \frac{\partial f_z}{\partial Z} = -K_x (1 - X - Y)(\beta_x - 2Z)X - K_y (1 - X - Y)(\beta_y - 2Z)Y + K_z (1 - 2Z) - \mu_z \end{array} \right. \quad (66)$$

CHAPTER 4: Concluding remarks

The significance of developing the two models is summarized as follows.

The RBM with Duffing oscillator was to provide a novel dynamical model for analyzing the dynamics of synchronized biennial-bearing of olives, involving environmental, socio-political and biological factors to comprehend yield fluctuations and the underlying causes in the Levant, specifically in Syria. The model expounds on the factors affecting olive production by including socio-political factors and yield anomalies. The instability that occurs during the pollination on-season is illustrated in the results and challenges the assumption that anemophily is the cause of synchronization. The model's behavior illustrates the emergence of bounded chaos and its potential for instability or periodic oscillation, diversifying our understanding of olive yield dynamics. Policy making and agricultural practices in olive production could be improved through implementation of the findings of this research in the region.

The plant-nutrient interactions model advanced our understanding of various population dynamics phenomena, such as extinction, coexistence, and segregation, having practical applications, notably in augmenting the efficiency of mixed cropping systems, particularly between leguminous and non-leguminous plants, to achieve higher yields. The complex dynamics of the symbiotic relationship between leguminous and non-leguminous plants are captured effectively using the plant-nutrient interactions model. Through meticulous mathematical analysis, stability, boundedness and existence of the solutions were established, while identifying the equilibrium of the system. The model's predictions were supported by the field experiments of clover and mint, validation of the model behavior and assumptions was done using photometer and stable isotope analysis. The Allee effect was observed in the relationship between clover and the soil environment, highlighting nonlinear response to varying ecological conditions. In addition, the model has several asymptotic behaviors and possible bifurcations, illustrating its robustness in representing plant-nutrient dynamics.

Research on nonlinear plant population dynamics is significant because it aids in understanding how plant populations interact, grow, and adapt to changing environments. The ecosystem response to disturbances can be predicted considering density-dependent factors, complex feedback mechanisms, and spatial-temporal patterns, which this thesis addressed the least. These issues are significant in numerous parts of the world where political unrest and climate change affect the ecosystems.

In Syria, on which Chapter 2 focused, new opportunities and challenges have been introduced by the regime shift in December 2024. Land use, conservation efforts, and agricultural practices would experience positive or negative impacts due to changes in governance and political instability. This political regime shift could lead to ecological regime shifts in plant population dynamics, such as an increase in invasive species or a decrease in native flora. In addition, changes in the new government's policies and priorities could advance or limit research efforts.

From this perspective, understanding nonlinear plant population dynamics could help develop strategies to improve food security, restore ecosystems, and adapt to socio-political changes. This perspective would also focus on integrating political and ecological systems, thus showing the need to address the challenges using interdisciplinary approaches.

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