

## Oscillation Theorems for Quasilinear Elliptic Equations

Hiroyuki USAMI (宇佐美 広介)

Department of Mathematics, Faculty of Integrated Arts &  
Sciences, Hiroshima University

### § 1. Introduction

In this talk we treat quasilinear elliptic equations of the form

$$\operatorname{div}(|Du|^{m-2}Du) + a(x)|u|^{m-2}u = 0 \quad (\text{E})$$

in an exterior domain  $\Omega \subset \mathbb{R}^N$ . Such equations are often called half-linear equations. We always assume that  $N \geq 2$ ,  $m > 1$ , and  $a$  is continuous in  $\Omega$ .

**Definition.** (i) A nontrivial solution  $u$  of (E) (defined near  $\infty$ ) is called *oscillatory* if the set  $\{x \in \Omega \cap \operatorname{dom} u : u(x) = 0\}$  is unbounded.

(ii) Equation (E) is called *oscillatory* if every nontrivial solution (defined near  $\infty$ ) of (E) is oscillatory.

The aim of the talk is to establish sufficient conditions for (E) to be oscillatory. We are interested in especially the case where  $a$  may take on negative values for arbitrarily large  $|x|$ .

## § 2. Reduction to one-dimensional problems

We employ the notation

$$\bar{a}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} a(x) dS \quad \text{for large } r,$$

where  $\omega_N = \int_{|x|=1} dS$ . This function is called the spherical mean of  $a$ .

**Lemma.** *Let  $u$  be a positive solution of (E) defined for  $|x| \geq R$ , sufficiently large.*

(i) *The vector-valued function*

$$w(x) = - \frac{|Du|^{m-2} Du}{u^{m-1}}$$

*satisfies the identity*

$$\operatorname{div} w = a(x) + (m-1) |w|^{m/(m-1)}, \quad |x| \geq R. \quad (1)$$

(ii) *The function*

$$z(r) = \int_{|x|=r} (w(x), \frac{x}{r}) dS, \quad r \geq R,$$

*satisfies the generalized Riccati inequality*

$$z'(r) \geq \frac{m-1}{(\omega_N r^{N-1})^{1/(m-1)}} |z(r)|^{m/(m-1)} + \omega_N r^{N-1} \bar{a}(r), \quad (2)$$

where  $(\cdot, \cdot)$  denotes the usual inner product.

**Proof.** Since the proof of (i) is easy, it is omitted.

Only (ii) will be proved.

Integrating (1) over the sphere  $|x| = r$ , we obtain

$$\int_{|x|=r} \operatorname{div} w \, dS = \omega_N r^{N-1} \bar{a}(r) + (m-1) \int_{|x|=r} |w|^{m/(m-1)} dS. \quad (3)$$

On the other hand, the divergence theorem shows that

$$z'(r) = \frac{d}{dr} \int_{|x|=r} (w(x), \frac{x}{r}) dS = \int_{|x|=r} \operatorname{div} w \, dS, \quad (4)$$

and Holder's inequality shows that

$$\begin{aligned} |z(r)| &\leq \int_{|x|=r} |w| \cdot 1 \, dS \\ &\leq \left( \int_{|x|=r} |w|^{m/(m-1)} dS \right)^{(m-1)/m} \left( \int_{|x|=r} dS \right)^{1/m}. \end{aligned}$$

This is equivalent to

$$\left( \int_{|x|=r} |w|^{m/(m-1)} dS \right)^{(m-1)/m} \geq \left( \omega_N r^{N-1} \right)^{-1/m} |z(r)|. \quad (5)$$

By virtue of (4) and (5), we can verify the validity of (2) from (3). The proof is complete.

Lemma immediately gives the following important result on which our oscillation theory is heavily based.

**Proposition 1.** *Equation (E) is oscillatory if the generalized Riccati inequality (2) has no solutions near  $+\infty$ .*

### § 3. Generalized Riccati Inequalities

By Proposition 1, we find that to establish oscillation criteria for (E), it suffices to analyze inequality (2). But, instead of treating inequality (2) directly, we may well consider the inequality

$$h' \geq \frac{|h|^\alpha}{p(r)} + q(r), \quad (6)$$

which presents more simple form than (2). For this inequality we assume that  $\alpha > 1$ ,  $p$  is a positive continuous function, and  $q$  is a continuous function defined near  $+\infty$ . We emphasize that  $q$  is not assumed to be nonnegative. All infinite integrals appearing in the sequel should be taken in the sense of improper integrals:  $\int^\infty = \lim_{R \rightarrow \infty} \int^R$ .

**Proposition 2.** *Inequality (6) has no solutions defined near  $+\infty$  if there is a positive  $C^1$ -function  $\varphi$  satisfying*

$$\int^\infty \left( \frac{p(r)|\varphi'(r)|^\alpha}{\varphi(r)} \right)^{1/(\alpha-1)} dr < \infty, \quad (7)$$

$$\int^\infty \frac{dr}{p(r)[\varphi(r)]^{\alpha-1}} = \infty,$$

and

$$\int^\infty \varphi(r)q(r)dr = \infty. \quad (8)$$

**Proof.** Suppose to the contrary that (6) admits a solution  $h \in C^1[R, \infty)$ . We may assume that  $\varphi$  is defined for  $r \geq R$ . Multiplying (6) by  $\varphi(r)$ , and integrating the resulting inequality on  $[R, r]$ , we have

$$h\varphi \geq c_1 + \int_R^r h\varphi' ds + \int_R^r \frac{\varphi|h|^\alpha}{p} ds + \int_R^r \varphi q ds \quad (9)$$

for  $r \geq R$ , where  $c_1$  is a constant. By Hölder's inequality we have

$$\int_R^r |h\varphi'| ds = \int_R^r \left( \frac{\varphi}{p} \right)^{1/\alpha} |h| \cdot \left( \frac{p}{\varphi} \right)^{1/\alpha} |\varphi'| ds$$

$$\leq c_2 \left( \int_R^r \frac{\varphi |h|^\alpha}{p} ds \right)^{1/\alpha} \equiv c_2 [H(r)]^{1/\alpha}$$

for  $r \geq R$ , where

$$c_2 = \left( \int_R^\infty \left( \frac{p}{\varphi} \right)^{1/(\alpha-1)} |\varphi'|^{\alpha/(\alpha-1)} ds \right)^{(\alpha-1)/\alpha}.$$

Hence we find from (9) that

$$\begin{aligned} h(r)\varphi(r) &\geq c_1 - c_2 [H(r)]^{1/\alpha} + \frac{1}{2}H(r) \\ &\quad + \frac{1}{2} \int_R^r \frac{\varphi |h|^\alpha}{p} ds + \int_R^r \varphi q ds, \quad r \geq R. \end{aligned} \quad (10)$$

Since the function  $-c_2 \xi^{1/\alpha} + \xi/2$  is bounded from below on  $[0, \infty)$  by the fact  $\alpha > 1$ , assumption (8) shows that the right hand side of (10) tends to  $+\infty$  as  $r \rightarrow +\infty$ . It follows therefore that  $h(r) > 0$ ,  $r \geq r_0 \geq R$  for some sufficiently large  $r_0$ . Again from (10) we have

$$h(r)\varphi(r) \geq \frac{1}{2} \int_R^r \frac{\varphi h^\alpha}{p} ds, \quad r \geq r_1 \geq r_0 \quad (11)$$

for some sufficiently large  $r_1 \geq r_0$ . Differentiating  $H$ , we obtain by (11)

$$H'(r) = \frac{[h(r)\varphi(r)]^\alpha}{p(r)[\varphi(r)]^{\alpha-1}} \geq \frac{2^{-\alpha}[H(r)]^\alpha}{p(r)[\varphi(r)]^{\alpha-1}}, \quad r \geq r_1.$$

Dividing the both sides by  $[H(r)]^\alpha$  and integrating, we have

$$\frac{1}{\alpha-1} [H(r_1)]^{1-\alpha} \geq 2^{-\alpha} \int_{r_1}^r \frac{ds}{p\varphi^{\alpha-1}}, \quad r \geq r_1,$$

which contradicts (7). The proof is complete.

#### § 4. Results

We are now in a position to state our main results.

**Theorem.** *Eq. (E) is oscillatory if there exists a positive  $C^1$ -function  $\rho$  satisfying*

$$\int^{\infty} \frac{r^{N-1} |\rho'(r)|^m}{[\rho(r)]^{m-1}} dr < \infty, \quad \int^{\infty} \frac{dr}{[r^{N-1} \rho(r)]^{1/(m-1)}} = \infty; \text{ and}$$

$$\int^{\infty} r^{N-1} \rho(r) \bar{a}(r) dr = \infty.$$

**Corollary.** (i) *Eq. (E) is oscillatory if*

$$\int^{\infty} r^{m-1-\varepsilon} \bar{a}(r) dr = \infty \text{ for some } \varepsilon > 0.$$

(ii) *Let  $N \geq m$ . Then, Eq. (E) is oscillatory if*

$$\int^{\infty} r^{N-1} \bar{a}(r) dr = \infty.$$

(iii) *Let  $N = m$ . Then, Eq. (E) is oscillatory if*

$$\int^{\infty} r^{m-1} (\log r)^{m-1-\varepsilon} \bar{a}(r) dr = \infty \text{ for some } \varepsilon > 0.$$

Since these results can be easily proved by combining Propositions 1 and 2, the proofs are left to the readers.

**Remark 1.** Generally, the assumption  $\varepsilon > 0$  in the statement of Corollary can not be weakened to  $\varepsilon \geq 0$ . In fact, if  $N + 1 - 2m > 0$ , then the equation

$$\operatorname{div}(|Du|^{m-2} Du) + (N+1-2m) |x|^{-m} |u|^{m-2} u = 0,$$

has a nonoscillatory solution  $u(x) = |x|^{-1}$ , and for this equation, obviously  $\int^{\infty} r^{m-1} \bar{a}(r) dr = \infty$ .

**Remark 2.** Let us consider the case where  $a$  has radial symmetry:  $a(x) \equiv a_0(|x|)$ . In this case, obviously  $\bar{a} \equiv a_0$ . Suppose moreover that  $a_0(r) \geq 0$  near  $+\infty$ . Then, it has been shown [1, 2] that:

(i) Eq. (E) has positive radial solutions defined near  $\infty$  if

$$\int^{\infty} r^{m-1} a_0(r) dr < \infty.$$

(ii) Let  $N = m$ . Eq. (E) has positive radial solutions defined near  $\infty$  if

$$\int^{\infty} r^{m-1} (\log r)^{m-1} a_0(r) dr < \infty.$$

Comparing Remarks 1 and 2 with Corollary, we find that our results are optimal in some sense.

### References

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