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Crepant Blowing-Ups of Canonical Singularities and Its Application to Degenerations of Surfaces

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Let X be a normal algebraic variety over $\mathbb{C}$, and let D be a Weil divisor on it. We would like to know when the sheaf of graded $\mathcal{O}_X$-algebras

$$\mathcal{R}(D) := \oplus_{m \geqq 0} \mathcal{O}_X(mD)$$

is finitely generated, where the $\mathcal{O}_X(mD)$ are reflexive sheaves of rank 1 corresponding to the mD. It is equivalent to saying that there exists a projective morphism $f : X' \longrightarrow X$ which is an isomorphism in codimension 1 and such that the strict transform D' of D on X' is $\mathbb{Q}$-Cartier and f-ample. The problem is trivial in case $\dim X = 2$; f must be an isomorphism and the condition for the finite generatedness is simply that D is $\mathbb{Q}$-Cartier. It is well known that a normal surface singularity X is (analytically) $\mathbb{Q}$-*factorial*, i.e., an arbitrary (analytic) Weil divisor on X is $\mathbb{Q}$-Cartier, if and only if X is a rational singularity. In this paper we announce a partial generalizaton of this fact to 3-dimensional case. (We refer the reader to [KMM] for definitions concerning minimal models.)

THEOREM 1. *Let X be a 3-dimensional normal algebraic* *variety over* $\mathbb{C}$ *which has at most canonical singularities, and let* D *be a Weil divisor on it. Then* $\mathcal{R}(D)$ *is finitely generated.*

We note that a rational Gorenstein singularity is canonical. The theorem is proved in the following way. Let X be as in Theorem 1 and let $\mu : Y \longrightarrow X$ be a desingularization. Then we can write $K_Y = \mu^* K_X + \Sigma_j a_j F_j$ with $a_j \geqq 0$ by definition, where the F_j are exceptional divisors of μ. We define $e(X)$ as the number of divisors F_j for which μ is *crepant*, i.e., $a_j = 0$ (it is easy to see that $e(X)$ does not depend on the choice of μ). For example, $e(X) = 0$ if and only if X has at most terminal singularities. We define also $\sigma(X) := \dim_{\mathbb{Q}} Z_2(X)_{\mathbb{Q}}/\mathrm{Div}(X)_{\mathbb{Q}}$, where $Z_2(X)_{\mathbb{Q}}$ and $\mathrm{Div}(X)_{\mathbb{Q}}$ are groups of $\mathbb{Q}$-divisors and $\mathbb{Q}$-Cartier divisors, respectively (one can prove that $\sigma(X)$ is finite). Thus X is $\mathbb{Q}$-factorial if and only if $\sigma(X) = 0$. Our theorem is proved by induction on $e(X)$ and $\sigma(X)$ in the category consisting of varieties X' with projective birational morphisms $f : X' \longrightarrow X$ which are *crepant*, i.e., $K_{X'} = f^* K_X$; e.g., an isomorphism in codimension 1 is crepant. Theorem 1 in case $e(X) = 0$ is proved by using Brieskorn's flips as in [R]. The termination of log-flips in case $e(X') = n$ produces the existence of the log-flip in case $e(X') = n + 1$ (cf. [KMM]). In the course of the proof, the concept of the *sectional decomposition*, which is a rather trivial generalization of the Zariski decomposition for surfaces (cf.

[F]), plays an essential role. We employ a technique developed in [K] to deal with the difficulty concerning $\mathbb{R}$-divisors which inevitably appear in higher dimensional sectional decompositions (cf. [C]). More precisely, we prove following Lemmas 2 to 5 in our inductive argument.

LEMMA 2. *Let X be a 3-dimensional variety with $\mathbb{Q}$-factorial canonical singularities such that $e(X) \geqq 1$. Then there exists a projective birational morphism $f : X_1 \longrightarrow X$ such that*

(i) X_1 has at most $\mathbb{Q}$-factorial canonical singularities,

(ii) the exceptional locus of f is a prime divisor, and

(iii) f is crepant.

LEMMA 3. *There is a function $b : \mathbb{N} \times (\mathbb{N} \cup \{0\}) \longrightarrow \mathbb{N}$ such that $b(r, e)Z_2(X) \subset Div(X)$ for an arbitrary 3-dimensional variety X with at most $\mathbb{Q}$-factorial canonical singularities of index r and $e = e(X)$.*

LEMMA 4. *Let $\varphi : X \longrightarrow Z$ be a projective morphism of 3-dimensional varieties and let D be a Cartier divisor on X. Assume that*

(a) X has at most $\mathbb{Q}$-factorial canonical singularities,

(b) φ is an isomorphism in codimension 1,

(c) *dim* $N^1(X/Z) = 1$, *and*

(d) $(K_X . C) = 0$ *and* $(D.C) < 0$ *for all curves* C *on* X *such that* $\varphi(C)$ *is a point.*

Then there exists a projective morphism $\varphi^+ : X^+ \longrightarrow Z$ *which satisfies the following conditions.*

(i) X^+ *has at most* $\mathbb{Q}$*-factorial canonical singularities,*

(ii) φ^+ *is an isomorphism in codimension 1,*

(iii) *dim* $N^1(X^+/Z) = 1$, *and*

(iv) D^+ *being the strict transform of* D, $(K_{X^+} . C^+) = 0$ *and* $(D^+.C^+) > 0$ *for all curves* C^+ *such that* $\varphi^+(C^+)$ *is a point.*

We call the procedure to obtain φ^+ from φ the *log-flip* with respect to D. Let $f : X \longrightarrow S$ be a projective surjective morphism with connected fibers such that $\dim X = 3$, X has at most $\mathbb{Q}$-factorial canonical singularities, and that $\mathrm{cl}(K_X) = 0$ in $N^1(X/S)$. A Weil divisor D on X is called *f-movable* if $f_*\mathcal{O}_X(D) \neq 0$ and if the cokernel of the natural homomorphism $f^*f_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D)$ has a support of codimension $\geqq 2$. We let $\overline{\mathrm{Apl}}(X/S)$, $\overline{\mathrm{Big}}(X/S)$ and $\overline{\mathrm{Mov}}(X/S)$ denote closed convex cones in $N^1(X/S)$ generated by the numerical classes of f-ample, f-big and f-movable divisors, and $\mathrm{Apl}(X/S)$, $\mathrm{Big}(X/S)$ and $\mathrm{Mov}(X/S)$ their interiors, respectively. The cone $\overline{\mathrm{Apl}}(X/S) \cap \mathrm{Big}(X/S)$ is locally polyhedral in $\mathrm{Big}(X/S)$ by Cone Theorem (cf. [KMM]). A log-flip leaves $\overline{\mathrm{Mov}}(X/S) \cap \mathrm{Big}(X/S)$ stable, while

$\overline{\mathrm{Apl}}(X/S) \cap \mathrm{Big}(X/S)$ is transformed to a neighboring cone $\overline{\mathrm{Apl}}(X^+/S) \cap \mathrm{Big}(X^+/S)$. The *sectional decomposition* of an f-big $\mathbb{R}$-divisor D, which always exists as far as X is $\mathbb{Q}$-factorial, is an expression $D = M + F$ in $Z_2(X)_{\mathbb{R}}$ such that $\mathrm{cl}(M) \in \overline{\mathrm{Mov}}(X/S)$, $F \geqq 0$, and that the natural homomorphisms $f_*\mathcal{O}_X([mM]) \longrightarrow f_*\mathcal{O}_X([mD])$ are bijective for all $m \in \mathbb{N}$.

LEMMA 5. *Let $f : X \longrightarrow S$ be a projective surjective morphism of varieties with connected fibers and let M be an $\mathbb{R}$-divisor on X. Assume that*

(a) $\dim X = 3$ and X has at most $\mathbb{Q}$-factorial canonical singularities of index r and $e = e(X)$,

(b) $cl(K_X) = 0$ in $N^1(X/S)$,

(c) M is f-big and $cl(M) \in \overline{Mov}(X/S)$, and

(d) $D := \ulcorner M \urcorner \in 2b(r, e).Z_2(X)$.

Then there does not exist an infinite sequence of log-flips with respect to the strict transforms of D.

The following theorems are immediate applications of Theorem 1.

THEOREM 6. *Let $f : X \longrightarrow S$ be a projective surjective morphism with connected fibers and let D be a Weil divisor on X. Assume that $\dim X = 3$, X has at most canonical singularities, $cl(K_X) = 0$ in $N^1(X/S)$, and that D is f-big. Then the sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{R}(X/S, D) :=$*

$\oplus_{m\geqq 0}\, f_*\mathcal{O}_X(mD)$ *is finitely generated.*

THEOREM 7. *Let* X_1 *and* X_2 *be two* $\mathbb{Q}$*-factorial terminal good minimal models of dimension 3 which are birationally equivalent. Then they are joined by a sequence of log-flips.*

A singularity of a 3-dimensional normal variety Z is called *flipping* if it comes from a flipping contraction $\varphi : X \longrightarrow Z$ from a variety X with terminal singularities (cf. [KMM]). The existence of minimal models for algebraic 3-folds follows if the existence of the flips are proved, and the latter is equivalent to the finite generatedness of $\mathcal{R}(K_Z)$ for flipping singularities of dimension 3. Theorem 1 gives a sufficient condition for this to hold ; we construct a double covering $\pi : \tilde{Z} \longrightarrow Z$ by using a section s of $\mathcal{O}_Z(-2K_Z)$. If $\tilde{Z}$ is a canonical singularity, then the finite generatedness of $\mathcal{R}(\pi^*K_Z)$ implies that of $\mathcal{R}(K_Z)$. In this way we obtain the following corollaries to Theorem 1.

COROLLARY 8. *Let* $\varphi : X \longrightarrow Z$ *be a flipping contraction from a 3-folds with terminal singularities and let* $\mu : Y \longrightarrow X$ *be a desingularization whose exceptional locus is a simple normal crossing divisor* $\Sigma_j F_j$ *. Let* S *be a Weil divisor on* X *which is the zeroes of a section in* $H^0(X, \mathcal{O}_X(-2K_X)) \simeq H^0(Z, \mathcal{O}_Z(-2K_Z))$*. Write* $K_Y = \mu^*K_X + \Sigma_j a_jF_j$ *and* $\mu^*S = S' + \Sigma_j s_jF_j$ *, where* S' *is the strict transform of* S*. Assume that* $2a_j + 1 \geqq s_j$ *for all* j*. Then* $\mathcal{R}(K_Z)$ *is* *finitely generated.*

COROLLARY 9. *Let Z be a flipping singularity of dimension 3 and let S be a Weil divisor on Z which corresponds to a section of $\mathcal{O}_Z(-K_Z)$. Assume that S has at most rational singularities. Then $\mathcal{R}(K_Z)$ is finitely generated.*

COROLLARY 10. *Let Z be as in Corollary 9 and let H be an effective Cartier divisor on Z which contains $Sing(Z)$. Assume that H is a normal surface and let $\tilde{H} \longrightarrow H$ be a double covering constructed by using the restriction of a section of $\mathcal{O}_Z(-2K_Z)$ to H. If $\tilde{H}$ has at most elliptic singularities, then $\mathcal{R}(K_Z)$ is finitely generated.*

Finally, by using the criteria in Corollaries 9 and 10, we obtain an alternative proof of the following theorem of Tsunoda [T] (Shokurov and Mori also announced to have their proofs in private letters).

THEOREM 11. *Let $f : X \longrightarrow S$ be a projective surjective morphism of smooth varieties with connected fibers such that $\dim X = 3$ and $\dim S = 1$. Assume that singular fibers of f are reduced and simple normal crossing while smooth fibers have non-negative Kodaira dimension. Then there exists a minimal model $f' : X' \longrightarrow S$ of f, i.e., f' is a projective surjective morphism which is birationally equivalent to f, X' has at most $\mathbb{Q}$-factorial terminal* *singularities, and that* $K_{X'}$ *is* f'*-nef. In particular, smooth fibers of* f' *are minimal models of corresponding fibers of* f.

By applying Nakayama's theory [N], we can extend our results to the case where the base space is a complex analytic space ; X may be a germ of an analytic space in Theorem 1 and S a disc in Theorem 11.

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