

A LINEAR OPERATOR AND SOME APPLICATIONS OF FIRST ORDER DIFFERENTIAL SUBORDINATIONS

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1. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$.

For functions $f_j(z) \in A_p$ ($j=1, 2$) defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad ,$$

we define the convolution (or Hadamard product) $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \quad (1.2)$$

With the convolution above, we define

$$\begin{aligned} D^{n+p-1} f(z) &= \frac{z^p}{(1-z)^{n+p}} * f(z) \quad (f(z) \in A_p) \\ &= \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!}, \end{aligned} \quad (1.3)$$

where n is any integer greater than $-p$.

For a function $f(z) \in A_p$, we define the generalized Libera integral operator $J_{v,p}$ by

$$J_{v,p}(f(z)) = \frac{v+p}{z^v} \int_0^z t^{v-1} f(t) dt, \quad v > -p. \quad (1.4)$$

For $p = 1$ and $v \in \mathbb{N}$, the operator $J_{v,1}$ was introduced by Bernardi [1]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some interesting results for the operator $J_{v,p}$ were showed by Saitoh [12] and Saitoh et al. [13].

Now, let the function $\phi_p(a, c)$ be defined by

$$\phi_p(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \quad (z \in \mathbb{U}), \quad (1.5)$$

for $c \neq 0, -1, -2, \dots$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1) \dots (a+n-1), & \text{if } n \in \mathbb{N} \end{cases} \quad (1.6)$$

Also, we define a linear operator $L_p(a, c)$ on A_p by

$$L_p(a, c; z)f(z) = \phi_p(a, c; z) * f(z) \quad (1.7)$$

for $f(z) \in A_p$ and $c \neq 0, -1, -2, \dots$.

The operator $L_1(a, c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

REMARKS. (1) For $f(z) \in A_1 = A$,

$$L_1(n+1, 1; z)f(z) = D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

is Ruscheweyh derivative of $f(z)$ ([8]).

(2) For $f(z) \in A_p$,

$$L_p(n+p, 1; z)f(z) = D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

is Ruscheweyh derivative introduced by Goel and Sohi [3].

(3) For $f(z) \in A_p$,

$$L_p(v+p, v+p+1; z)f(z) = J_{v,p}(f(z))$$

is the generalized Libera integral operator ([12], [13]).

(4) $\phi_1(a, c; z)$ is an incomplete beta function, related to the Gauss hypergeometric functions by

$$\phi_1(a, c; z) = z {}_2F_1(1, a; c; z).$$

It follows from (1.7) that

$$z(L_p(a, c; z)f(z))' = aL_p(a+1, c; z)f(z) - (a-p)L_p(a, c; z)f(z) \quad (1.8)$$

Let the function $f(z)$ and $g(z)$ be analytic in U . Then the $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in U , with $w(0)=0$ and $|w(z)| < 1$ ($z \in U$), such that $f(z) = g(w(z))$ ($z \in U$). We denote this subordination by $f(z) \prec g(z)$.

2. MAIN RESULTS

To establish our main results, we need the following lemmas.

LEMMA 1 [6]. Let $h(z)$ be convex and $B(z)$ be analytic in U with $\operatorname{Re} B(z) \geq 0$. If $p(z)$ is analytic in U and $p(0)=h(0)$, then

$$p(z) + B(z)zp'(z) \prec h(z) \text{ implies } p(z) \prec h(z) \quad (z \in U).$$

LEMMA 2 [9]. Let $\gamma=0$, $\operatorname{Re} \gamma \geq 0$, and let $h(z)$ be convex. If $p(z)$ is analytic in U with $p(0)=h(0)$, then

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z) \text{ implies } p(z) \prec \frac{\gamma}{z} \int_0^z h(t)t^{\gamma-1}dt \quad (z \in U).$$

THEOREM 1. Let λ, a be a real number with $\lambda \geq 0, a > 0$. Let $h(z)$ be a convex function with $h(0)=1, c \neq 0, -1, -2, \dots$ and $g(z) \in A_p$ satisfies

$$\operatorname{Re} \left\{ \frac{L_p(a+1, c; z) g(z)}{L_p(a, c; z) g(z)} \right\} > 0 \quad (2.1)$$

If $f(z) \in A_p$ satisfies

$$(1-\lambda) \left\{ \frac{L_p(a, c; z) f(z)}{L_p(a, c; z) g(z)} \right\} + \lambda \left\{ \frac{L_p(a+1, c; z) f(z)}{L_p(a+1, c; z) g(z)} \right\} < h(z) \quad (2.2)$$

then we have

$$\frac{L_p(a, c; z) f(z)}{L_p(a, c; z) g(z)} < h(z) \quad (z \in U). \quad (2.3)$$

PROOF. Put

$$H(z) = (1-\lambda) \left\{ \frac{L_p(a, c; z) f(z)}{L_p(a, c; z) g(z)} \right\} + \lambda \left\{ \frac{L_p(a+1, c; z) f(z)}{L_p(a+1, c; z) g(z)} \right\}$$

From assumption, $h(z)$ is convex with $h(0)=1$ and $f(z) \in A_p$ satisfies $H(z) < h(z) \quad (z \in U)$, where $\lambda \geq 0$ and $g(z) \in A_p$ satisfies (2.1).

$$\text{Set } B(z) = \frac{\lambda}{a} \cdot \frac{L_p(a, c; z) g(z)}{L_p(a+1, c; z) g(z)}$$

According to (2.1), we have $\operatorname{Re} B(z) \geq 0$. Define $p(z)$ by

$$p(z) = \frac{L_p(a, c; z) f(z)}{L_p(a, c; z) g(z)} \quad (2.4)$$

We can see that $p(z)$ is analytic in U and $p(0)=1$. Logarithmic differentiating (2.4) and using (1.8), we can get

$$\frac{L_p(a+1, c; z) f(z)}{L_p(a+1, c; z) g(z)} = p(z) + \frac{1}{\lambda} B(z) \cdot z p'(z) \quad (z \in U). \quad (2.5)$$

Then (2.2) can be written

$$p(z) + B(z)zp'(z) \prec h(z).$$

By Lemma 1, we have

$$p(z) \prec h(z).$$

Thus, we have

$$\frac{L_p(a, c; z)f(z)}{L_p(a, c; z)g(z)} \prec h(z).$$

This completes the proof of Theorem 1.

Putting $a = n+p$, $c=1$ in Theorem 1, we have

COROLLARY 1. Let λ be a real number with $\lambda \geq 0$. Let $h(z)$ be a convex function with $h(0)=1$ and $g(z) \in A_p$ satisfies

$$\operatorname{Re} \left\{ \frac{D^{n+p}g(z)}{D^{n+p-1}g(z)} \right\} > 0.$$

If $f(z) \in A_p$ satisfies

$$(1-\lambda) \left\{ \frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \right\} + \lambda \left\{ \frac{D^{n+p}f(z)}{D^{n+p}g(z)} \right\} \prec h(z),$$

then we have

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}g(z)} \prec h(z) \quad (z \in U).$$

Making $a = v+p$, $c = v+p+1$ in Theorem 1, we can show

COROLLARY 2. Let λ be a real number with $\lambda \geq 0$. Let $h(z)$ be a convex function with $h(0)=1$ and $g(z) \in A_p$ satisfies

$$\operatorname{Re} \left\{ \frac{g(z)}{J_{v,p}(g(z))} \right\} > 0.$$

If $f(z) \in A_p$ satisfies

$$(1-\lambda) \left\{ \frac{J_{v,p}(f(z))}{J_{v,p}(g(z))} \right\} + \lambda \frac{f(z)}{g(z)} \prec h(z),$$

then we have

$$\frac{J_{v,p}(f(z))}{J_{v,p}(g(z))} \prec h(z).$$

Next, we prove

THEOREM 2. Let $f(z) \in A_p$ and $h(z)$ be a convex function with $h(0)=1$. Then for any complex number λ with $\operatorname{Re} \lambda \geq 0$ ($\lambda=0$), $a > 0$,

$$(1-\lambda) \left\{ \frac{L_p(a, c; z) f(z)}{z^p} \right\} + \lambda \left\{ \frac{L_p(a+1, c; z) f(z)}{z^p} \right\} < h(z) \quad (z \in U) \quad (2.6)$$

implies

$$\frac{L_p(a, c; z) f(z)}{z^p} < \frac{a}{\lambda z^{a/\lambda}} \int_0^z h(t) t^{a/\lambda - 1} dt < h(z) \quad (z \in U) \quad (2.7)$$

The result is sharp.

PROOF. Choosing $g(z)=z^p$ and $B(z)=\frac{\lambda}{a}$, and use Lemma 2,

Theorem 2 follows from Theorem 1.

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