

LONG-TIME ASYMPTOTICS OF THE DAMPED NONLINEAR KLEIN-GORDON EQUATION WITH A DELTA POTENTIAL

KENJIRO ISHIZUKA

ABSTRACT. We consider the damped nonlinear Klein-Gordon equation with a delta potential:

$$\partial_t^2 u - \partial_x^2 u + 2\alpha \partial_t u + u - \gamma \delta_0 u - |u|^{p-1} u = 0, (t, x) \in \mathbb{R} \times \mathbb{R},$$

where $p > 1$, $\alpha > 0$, $\gamma < 2$, and $\delta_0 = \delta_0(x)$ denotes the Dirac delta with the mass at the origin. When $\gamma = 0$, Côte, Martel and Yuan [8] proved that any global solution either converges to 0 or to the sum of $K \geq 1$ decoupled solitary waves which have alternative signs. In this paper, we first prove that any global solution either converges to 0 or to the sum of $K \geq 1$ decoupled solitary waves. We then construct a single solitary wave solution that moves away from the origin when $\gamma < 0$ and construct an even 2-solitary wave solution when $\gamma \leq -2$. Last we give single solitary wave solutions and even 2-solitary wave solutions an upper bound for the distance between the origin and the solitary wave.

CONTENTS

1. Introduction	2
1.1. Setting of the problem	2
1.2. Main results	3
1.3. Previous results	5
2. Local well-posedness and uniform bound	6
2.1. Linear evolution in the energy space	6
2.2. Proof of Local well-posedness of the Cauchy problem	9
2.3. Uniform bound on global solutions	10
3. Soliton resolution	13
4. Global dynamics below the ground state energy	16
4.1. Minimizing problems and variational structure	16
4.2. Classification of global dynamics	17
4.3. Proof of Theorem 1.2	20
5. Dynamics close to solitary waves	22
5.1. Basic properties of the ground state	22
5.2. Modulation of the center	23
5.3. Decomposition close to the sum of solitary waves	26
5.4. Estimates of the center	27
5.5. Energy estimates	31
5.6. Estimate of the instability term	35
5.7. Proof of Theorem 1.3	37

Key words and phrases. Nonlinear Klein-Gordon equation, Solitons, Soliton resolution, Dirac delta potential.

Acknowledgement	39
References	40

1. INTRODUCTION

1.1. Setting of the problem. We consider the following damped nonlinear Klein-Gordon equation with a delta potential:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + 2\alpha \partial_t u + u - \gamma \delta_0 u - f(u) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ (u(0, x), \partial_t u(0, x)) = (u_0(x), u_1(x)) \in \mathcal{H}, \end{cases} \quad (\text{DNKG})$$

where $\mathcal{H} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $f(u) = |u|^{p-1}u$, $p > 1$, $\alpha > 0$, $\gamma < 2$, and $\delta_0 = \delta_0(x)$ denotes the Dirac delta with the mass at the origin. The singular term in (DNKG) should be interpreted in the sense of distributions. Then, the equation (DNKG) enjoys several invariances:

$$\begin{aligned} u(t, x) &\mapsto -u(t, x), \\ u(t, x) &\mapsto u(t, -x). \end{aligned} \quad (1.1)$$

(DNKG) also rewrites as a first order system for $(u, v) \in \mathcal{H}$

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_x^2 - 1 + \gamma \delta_0 & -2\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(u) \end{pmatrix}.$$

To give the formulation of a formal expression

$$\begin{pmatrix} 0 & 1 \\ \partial_x^2 - 1 + \gamma \delta_0 & -2\alpha \end{pmatrix},$$

we define the operator $\mathcal{A}$ as

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \partial_x^2 - 1 & -2\alpha \end{pmatrix},$$

with domain

$$\mathcal{D} = \{(u, v) \in H^{2,*} \times H^1 : u'(0^+) - u'(0^-) = -\gamma u(0)\} \subset \mathcal{H}, \quad (1.2)$$

where

$$H^{2,*} = H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}).$$

$\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$, which allows us to establish the local well-posedness of (DNKG). We will show it in Section 2. Moreover defining the energy of a solution $\vec{u} = (u, \partial_t u)$ by

$$E_\gamma(\vec{u}(t)) = \frac{1}{2} (\|\vec{u}(t)\|_{\mathcal{H}}^2 - \gamma |u(t, 0)|^2) - \frac{1}{p+1} \|u(t)\|_{L^{p+1}}^{p+1},$$

it holds

$$E_\gamma(\vec{u}(t_2)) - E_\gamma(\vec{u}(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_{L^2}^2 dt. \quad (1.3)$$

When $\gamma = 0$, it is well-known that up to sign and translation, the only stationary solution of (DNKG) is $(Q, 0)$, where Q is the explicit ground state

$$Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}},$$

which solves the equation

$$Q'' - Q + Q^p = 0 \quad \text{on } \mathbb{R}$$

(see [36]). From the expression of Q , we obtain for $|x| > 1$

$$\left| Q(x) - c_Q e^{-|x|} \right| \lesssim e^{-p|x|}, \quad \left| Q'(x) + \frac{x}{|x|} c_Q e^{-|x|} \right| \lesssim e^{-p|x|}, \quad (1.4)$$

where $c_Q = (2p+2)^{\frac{1}{p-1}}$. When $\gamma = 0$ and $p > 2$, it is proved in [8] that any global solution either converges to 0 or to the sum of $K \geq 1$ decoupled solitary waves which have alternative signs, namely for any global solution $\vec{u} \in C([0, \infty), \mathcal{H})$ of (DNKG) there exist $K \geq 0, \sigma = \pm 1$ and functions $z_k : [0, \infty) \rightarrow \mathbb{R}$ for all $k = 1, \dots, K$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\|u(t) - \sigma \sum_{k=1}^K (-1)^k Q(\cdot - z_k(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} \right) &= 0, \\ \lim_{t \rightarrow \infty} (z_{k+1}(t) - z_k(t)) &= \infty \text{ for } k = 1, 2, \dots, K-1. \end{aligned} \quad (1.5)$$

Remark 1.1. If $K = 0$ then the terms $\sum_{k=1}^K$ are 0.

On the other hand, when $\gamma \neq 0$, there may exist a solution of (DNKG) whose asymptotic behavior is different from the $\gamma = 0$ case thanks to the potential-soliton repulsion(or attraction). In fact, when γ satisfies $|\gamma| < 2$ and $\gamma \neq 0$, the positive stationary solution of (DNKG) is $(Q_\gamma, 0)$, where Q_γ is

$$Q_\gamma(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}|x| + \tanh^{-1}\left(\frac{\gamma}{2}\right)\right)} \right)^{\frac{1}{p-1}},$$

which is the solution of

$$Q_\gamma'' - Q_\gamma + \gamma \delta_0 Q_\gamma + Q_\gamma^p = 0. \quad (1.6)$$

Its asymptotic behavior is different from (1.5).

1.2. Main results. We are interested in the long-time asymptotic behavior of global solutions of (DNKG). When $\gamma \neq 0$, it is difficult to describe precisely the asymptotic behavior as $t \rightarrow \infty$ of all global solutions of (DNKG) as in [8] because the potential-soliton repulsion (or attraction) is delicate. However, using the concentration-compactness arguments we obtain the soliton resolution as the following:

Theorem 1.1. *When $\gamma \neq 0$, for any global solution $\vec{u}$ of (DNKG), there exist $K \geq 0$, signs $\sigma_0 = 0, \pm 1$, $\sigma_k = \pm 1$ for any $k \in \{1, 2, \dots, K\}$, and functions*

$z_k : [0, \infty) \rightarrow \mathbb{R}$ for any $k \in \{1, 2, \dots, K\}$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\|u(t) - \sigma_0 Q_\gamma - \sum_{k=1}^K \sigma_k Q(\cdot - z_k(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} \right) &= 0, \\ \lim_{t \rightarrow \infty} (z_{k+1}(t) - z_k(t)) &= \infty \text{ for } k = 1, \dots, K-1, \\ \lim_{t \rightarrow \infty} |z_k(t)| &= \infty \text{ for } k = 1, \dots, K. \end{aligned} \quad (1.7)$$

Remark 1.2. When $\sigma_0 = 0$ and $K = 0$, $\vec{u}$ converges exponentially to 0 in $\mathcal{H}$ as $t \rightarrow \infty$. We explained further in Section 4.2.

Remark 1.3. If $\gamma \leq -2$, we may assume $\sigma_0 = 0$ because Q_γ does not exist.

Theorem 1.1 implies that (1.7) exhausts all solutions in the energy space. On the other hand, it is difficult to determine signs $(\sigma_k)_{0 \leq k \leq K}$ in (1.7). For example, we do not know whether there exists a global solution $\vec{u}$ of (DNKG) satisfying (1.7) with $K = 3$ and $(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (0, 1, 1, -1)$. This is due to the potential-soliton repulsion (attraction). When $\gamma = 0$, we only consider the interaction between solitary waves, and (1.5) exhausts all solutions. On the other hand, when $\gamma \neq 0$ we need to reveal whether the interaction between solitary waves is stronger than the potential-soliton repulsion (attraction). However, the potential-soliton repulsion (attraction) is delicate and prevents the determination of signs of solitary waves.

Remark 1.4. From [8] and (1.2), for any $K \in \mathbb{N}$ and $\sigma = \pm 1$ there exist functions $z_k : [0, \infty) \rightarrow \mathbb{R}$ for $k = 1, 2, \dots, K$ and an odd global solution $\vec{u} \in C([0, \infty), \mathcal{H})$ of (DNKG) such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\|u(t) - \sigma \sum_{k=1}^K \{Q(\cdot - z_k(t)) - Q(\cdot + z_k(t))\}\|_{H^1} + \|\partial_t u(t)\|_{L^2} \right) &= 0, \\ \lim_{t \rightarrow \infty} z_k(t) &= \infty \text{ for } k = 1, 2, \dots, K, \\ \lim_{t \rightarrow \infty} (z_{k+1}(t) - z_k(t)) &= \infty \text{ for } k = 1, 2, \dots, K-1. \end{aligned}$$

As mentioned previously, it is difficult to determine signs $(\sigma_k)_{0 \leq k \leq K}$ of (1.7). On the other hand, when (1.7) with $\sigma_0 = 0$, $K = 1$ and $\gamma < 0$ or (1.7) with $\sigma_0 = 0$, $K = 2$ and $\gamma \leq -2$ we construct a global solution whose asymptotic behavior is different from (1.5). To establish the next result, we define a solution as follows:

Definition 1.1. (1) A solution $\vec{u}$ of (DNKG) is called a *single solitary wave* if there exist a sign $\sigma = \pm 1$ and a function $z : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} (\|u(t) - \sigma Q(\cdot - z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) &= 0, \\ \lim_{t \rightarrow \infty} |z(t)| &= \infty. \end{aligned} \quad (1.8)$$

(2) A solution $\vec{u}$ of (DNKG) is called an *even 2-solitary wave* if $\vec{u}$ is even and there exist a sign $\sigma = \pm 1$ and a function $z : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} (\|u(t) - \sigma(Q(\cdot - z(t)) + Q(\cdot + z(t)))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) &= 0, \\ \lim_{t \rightarrow \infty} z(t) &= \infty. \end{aligned} \quad (1.9)$$

Remark 1.5. When $\gamma = 0$, a solution $\vec{u}$ of (DNKG) satisfying (1.5) with $K = 1$ converges to the stationary solution $\pm Q(\cdot - y)$ for some $y \in \mathbb{R}$ as $t \rightarrow \infty$. However,

when $\gamma < 0$, a solution $\vec{u}$ of (DNKG) is effected the delta potential, and $\pm Q$ cannot stay. Therefore, we define a single solitary wave as described above.

Then, our main result is described as follows.

Theorem 1.2. (1) *If $\gamma < 0$, there exists a single solitary wave of (DNKG).*
 (2) *If $\gamma \leq -2$, there exists an even 2-solitary wave of (DNKG).*

A single solitary wave behaves asymptotically as $t \rightarrow \infty$ as Q moves away from the origin. Furthermore, an even 2-solitary wave converges to the sum of two decoupled solitary waves which have same signs. This implies the potential-soliton repulsion is stronger than the soliton-soliton attraction. Our main result is that if a solution $\vec{u}$ of (DNKG) is a single solitary wave or an even 2-solitary wave, the distance between the center of the solitary wave and the origin has an upper bound.

Theorem 1.3. (1) *Let $\gamma < 0$ and $p \geq 2$. If a global solution $\vec{u}$ is a single solitary wave, then we have*

$$|z(t)| - \frac{1}{2} \log t \lesssim 1.$$

(2) *Let $\gamma < 0$ and $p > 2$. If a global solution $\vec{u}$ is an even 2-solitary wave, then we have*

$$z(t) - \frac{1}{2} \log t \lesssim 1.$$

Remark 1.6. When $-2 < \gamma < 0$, we do not know whether there exists an even 2-solitary wave. However, if there exists an even 2-solitary wave of (DNKG), the distance between two solitons is estimated.

1.3. Previous results. There has been intensive study on global behavior of general (large) solutions for the nonlinear dispersive equations, where the guiding principle is the *soliton resolution conjecture*, which claims that generic global solutions are asymptotic to superposition of solitary waves for large time. In particular, the conjecture has been studied for the energy-critical wave equation. We refer to [15, 31], in which the conjecture is proven for all radial solutions in space dimension $N \geq 3$. Historically we refer to [6, 13, 15, 16], the conjecture has been proved in space dimension $N = 3, 5, 7, \dots$ or $N = 4, 6$ using the method of channels.

In recent years, Jendrej and Lawrie [31] proved the conjecture in space dimension $N \geq 4$ using a novel argument based on the analysis of collision intervals. Similar results regarding the conjecture have been obtained in the case with a potential [33, 39], with a damping term [22], as well as for the wave maps [14, 29], for the heat equation [2], all under some rotational symmetry.

On the other hand, the conjecture is widely open for the nonlinear Klein-Gordon equation, namely the $\alpha = 0$ and $\gamma = 0$ case. Nakanishi and Schlag [43] have proved the conjecture as long as the energy of the solution is not much larger than the ground state energy in space dimension $N = 3$.

For the damped nonlinear Klein-Gordon equation, namely the $\gamma = 0$ case, the energy decay (1.3) makes the analysis simpler than the undamped case. Historically, Keller [34] constructed stable and unstable manifolds in a neighborhood of any stationary solution of (DNKG), and Feireisl [17] proved the conjecture along a time sequence. Burq, Raugel and Schlag [3] proved the conjecture in the full limit $t \rightarrow \infty$ for all radial solutions. Furthermore, Côte, Martel and Yuan [8] proved the conjecture without size restriction and radial assumption. To the best of our

knowledge, this is the first complete proof of the soliton resolution conjecture for a non-integrable equation without any size or symmetry restriction and with moving solitons. Recently, Kim and Kwon [35] proved a similar result to [8] for Calogero-Moser derivative nonlinear Schrödinger equation. Other results for the damped nonlinear Klein-Gordon equation are in [9, 12, 25]. Their arguments are useful for (DNKG).

In addition, several studies have been conducted on multi-solitary waves for the nonlinear dispersive equations. For the nonlinear Klein-Gordon equation, Côte and Muñoz [10] have constructed multi-solitary waves based on the ground state. Furthermore, Côte and Martel [7] have constructed multi-solitary waves based on the stationary solution of the nonlinear Klein-Gordon equation, which extended the work of [10]. Notably, multi-solitary waves usually behave asymptotically as the solitary waves move at a constant speed. More recently, Aryan [1] has constructed 2-solitary waves whose speed converges to 0 as $t \rightarrow \infty$. Strictly in [1] the distance between each solitary wave is $2(1 + O(1)) \log t$ as $t \rightarrow \infty$. As [1], there are several results regarding multi-solitary waves with logarithmic distance; see *e.g.* [20, 40, 41, 42]. Notably, for the damped nonlinear Klein-Gordon equation, multi-solitary waves of (DNKG) are impossible to move at a constant speed by the damping α . [27, 28, 29, 30, 31, 32] are studies on two-bubble solutions.

There are few results about (DNKG) with $\alpha = 0$ because of the singularity of δ_0 . Csobo, Genoud, Ohta and Royer [11] proved the stability of standing waves for (DNKG). On the other hand, there are many results for the nonlinear Schrödinger equation with a delta potential; see *e.g.* [18, 19, 20, 21, 23, 44]. Notably, we have several technical tools in [18, 23].

2. LOCAL WELL-POSEDNESS AND UNIFORM BOUND

In this section we discuss the local well-posedness of the Cauchy problem (DNKG).

To apply the standard theory of operator semigroups (see [4]), we reformulate (DNKG) as a first order system on $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}U(t) - \mathcal{A}U(t) = F(U(t)), \\ U(0) = U_0 \in \mathcal{H}, \end{cases} \quad (2.1)$$

where

$$F(u, v) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}.$$

We demonstrate that the operator $\mathcal{A}$ generates a C^0 -semigroup on $\mathcal{H}$, which will allow us to establish the local well-posedness of (DNKG).

2.1. Linear evolution in the energy space. In this subsection we show that the operator $\mathcal{A}$ generates a strongly continuous group on $\mathcal{H}$. When we consider $\mathcal{A}$ on the domain $\mathcal{D}$, the effect of the delta potential is encoded in the domain of the generator. Furthermore, since $\mathcal{D}$ defined by (1.2) is dense in $\mathcal{H}$ (see [11, Lemma 2.2]) we can consider $\mathcal{A}$ on $\mathcal{H}$. To apply the standard theory of operator semigroups, we discuss the choice of a suitable Hilbert structure on $\mathcal{H}$. For $\mu \geq 0$ we introduce on $\mathcal{H}$ the quadratic form defined by

$$\|(u, v)\|_{\mathcal{H}, \mu, \gamma}^2 = \|u'\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2 - \gamma |u(0)|^2 + \|v\|_{L^2}^2.$$

We denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}, \mu, \gamma}$ the corresponding bilinear form. With $\mu = 1$ we observe that for $U = (u, v) \in \mathcal{D}$,

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}, 1, \gamma} = -2\alpha \|v\|_{L^2}^2 \leq 0.$$

In addition, the following lemma holds.

Lemma 2.1. *For $\gamma < 2$ there exists $C_\gamma > 1$ such that for all $u \in H^1$, we have*

$$C_\gamma^{-1} \|u\|_{H^1}^2 \leq \|u\|_{H^1}^2 - \gamma |u(0)|^2 \leq C_\gamma \|u\|_{H^1}^2. \quad (2.2)$$

Proof. See [11, Lemma 2.3]. $\square$

Proposition 2.2. *The operator $\mathcal{A}$ generates a C^0 -semigroup on $\mathcal{H}$.*

Proof. We assume that $0 < \mu < 1$ is close to 1. For $U = (u, v) \in \mathcal{D}$, we have

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}, \mu, \gamma} = (\mu^2 - 1) \langle u, v \rangle_{L^2} - 2\alpha \|v\|_{L^2}^2,$$

and

$$|\langle \mathcal{A}U, U \rangle_{\mathcal{H}, \mu, \gamma}| \lesssim \|u\|_{L^2}^2 + \|v\|_{L^2}^2.$$

Moreover by Lemma 2.1,

$$\|U\|_{\mathcal{H}, \mu, \gamma}^2 \gtrsim \|u\|_{L^2}^2 + \|v\|_{L^2}^2.$$

Hence, fixing $\beta > 0$ large enough, we have

$$\langle (\pm \mathcal{A} - \beta)U, U \rangle_{\mathcal{H}, \mu, \gamma} \leq 0.$$

Therefore the operators $\pm \mathcal{A} - \beta$ are dissipative. In particular, for $\lambda > \beta$ we have

$$\|(\pm \mathcal{A} - \lambda)U\|_{\mathcal{H}, \mu, \gamma}^2 \geq \|(\pm \mathcal{A} - \beta)U\|_{\mathcal{H}, \mu, \gamma}^2 + (\lambda - \beta)^2 \|U\|_{\mathcal{H}, \mu, \gamma}^2, \quad (2.3)$$

so that $\pm \mathcal{A} - \lambda$ are injective with close range. Next we show that $\mathcal{A} - \lambda$ is surjective. Let $(f_1, f_2) \in \mathcal{H}$. For $U = (u, v) \in \mathcal{D}$, we have

$$(\mathcal{A} - \lambda)U = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \Leftrightarrow \begin{cases} v = \lambda u + f_1, \\ u'' - (\lambda^2 + 2\alpha\lambda + 1)u = f_2 + (\lambda + 2\alpha)f_1. \end{cases}$$

Then by [11, Lemma 2.5], the operator $\mathcal{A} - \lambda$ is surjective. Hence, $\mathcal{A} - \lambda$ has a bounded inverse and by (2.3),

$$\|(\mathcal{A} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{\lambda - \beta}.$$

Therefore by the Hille-Yoshida Theorem, this proves that $\mathcal{A}$ generates a C^0 -semigroup on $\mathcal{H}$. Now the same holds true with $\mathcal{A}$ replaced by $-\mathcal{A}$, and the proof is complete. $\square$

Thus the linear problem is expressed as follows:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + 2\alpha \partial_t u + u - \gamma \delta_0 u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ (u(0, x), \partial_t u(0, x)) = (u_0(x), u_1(x)) \in \mathcal{H}, \end{cases} \quad (2.4)$$

which generates a strongly continuous semigroup of contractions $(S_{\alpha, \gamma}(t))_{t \geq 0}$ in $\mathcal{H}$. Furthermore, we define $(S_{\alpha, \gamma}(t))_{t \geq 0}$ as a one-parameter family on $L^2 \times H^{-1}$ as

$$\begin{pmatrix} \partial_t u(t) \\ \partial_t^2 u(t) \end{pmatrix} = S_{\alpha, \gamma}(t) \begin{pmatrix} \partial_t u(0) \\ \partial_t^2 u(0) \end{pmatrix}.$$

We then have the following estimates:

Proposition 2.3. *There exist $C > 0$ and $\kappa > 0$ such that, for all $t \geq 0$*

$$\|S_{\alpha,\gamma}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{-\kappa t}, \quad \|S_{\alpha,\gamma}(t)\|_{\mathcal{L}(L^2 \times H^{-1})} \leq Ce^{-\kappa t}. \quad (2.5)$$

Proof. We assume that $\vec{u}$ is a solution of (2.4). Let ϵ be sufficiently small, and we define E as

$$E(t) = \frac{1}{2} (\|\partial_t u(t)\|_{L^2}^2 + \|u(t)\|_{H^1}^2 - \gamma|u(t,0)|^2) + \epsilon \left(\int u(t) \partial_t u(t) + \alpha \|u(t)\|_{L^2}^2 \right).$$

Then by direct calculation, E satisfies

$$\begin{aligned} E'(t) &= -2\alpha \|\partial_t u(t)\|_{L^2}^2 + \epsilon (\|\partial_t u(t)\|_{L^2}^2 - \|u(t)\|_{H^1}^2 + \gamma|u(t,0)|^2) \\ &= -\frac{\epsilon}{2} (\|\partial_t u(t)\|_{L^2}^2 + \|u(t)\|_{H^1}^2 - \gamma|u(t,0)|^2) - \frac{\epsilon}{2} (\|u(t)\|_{H^1}^2 - \gamma|u(t,0)|^2) \\ &\quad - (2\alpha - \frac{3}{2}\epsilon) \|\partial_t u(t)\|_{L^2}^2. \end{aligned}$$

By Lemma 2.1 and the smallness of ϵ , we have

$$\begin{aligned} &\epsilon^2 \left(\int u(t) \partial_t u(t) + \alpha \|u(t)\|_{L^2}^2 \right) \\ &\leq (\alpha + \frac{1}{2}) \epsilon^2 \|u(t)\|_{H^1}^2 + \frac{\epsilon^2}{2} \|\partial_t u(t)\|_{L^2}^2 \\ &\leq (\alpha + \frac{1}{2}) C_\gamma^{-1} \epsilon^2 (\|u(t)\|_{H^1}^2 - \gamma|u(t,0)|^2) + \frac{\epsilon^2}{2} \|\partial_t u(t)\|_{L^2}^2 \\ &\leq \frac{\epsilon}{2} (\|u(t)\|_{H^1}^2 - \gamma|u(t,0)|^2) + (2\alpha - \frac{3}{2}\epsilon) \|\partial_t u(t)\|_{L^2}^2. \end{aligned}$$

Therefore we have

$$E'(t) \leq -\epsilon E(t),$$

which implies

$$E(t) \leq e^{-\epsilon t} E(0).$$

Furthermore, we define $\gamma' = \frac{\gamma}{1-\epsilon}$. Then $\gamma < \gamma' < 2$ holds true because ϵ is sufficiently small. Thus by Lemma 2.1, we have

$$\begin{aligned} E(t) &\geq \frac{1}{2} (1-\epsilon) \left(\|u(t)\|_{H^1}^2 - \frac{\gamma}{1-\epsilon} |u(t,0)|^2 \right) + (\frac{1}{2} - \frac{\epsilon}{2}) \|\partial_t u(t)\|_{L^2}^2 \\ &\geq \frac{1}{2} (1-\epsilon) C_{\gamma'}^{-1} \|u(t)\|_{H^1}^2 + (\frac{1}{2} - \frac{\epsilon}{2}) \|\partial_t u(t)\|_{L^2}^2. \end{aligned}$$

Therefore we obtain $E(t) \sim \|\vec{u}(t)\|_{\mathcal{H}}^2$ and

$$\|\vec{u}(t)\|_{\mathcal{H}}^2 \lesssim e^{-\epsilon t} \|\vec{u}(0)\|_{\mathcal{H}}^2.$$

This implies that $\|S_{\alpha,\gamma}(t)\|_{\mathcal{L}(\mathcal{H})} \lesssim e^{-\kappa t}$ for some $\kappa > 0$. Next we estimate $\|S_{\alpha,\gamma}(t)\|_{\mathcal{L}(L^2 \times H^{-1})}$. To estimate $\|S_{\alpha,\gamma}(t)\|_{\mathcal{L}(L^2 \times H^{-1})}$, first we show

$$\|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L^2}^2 \sim \|\partial_t u(t)\|_{L^2}^2 + \|\partial_t^2 u(t)\|_{H^{-1}}^2. \quad (2.6)$$

By (2.4), for every $\varphi \in H^1(\mathbb{R})$, we have

$$\begin{aligned} |\langle \partial_t^2 u(t), \varphi \rangle| &= | - \langle -\partial_x^2 u(t) + u(t), \varphi \rangle + \gamma \langle \delta_0 u(t), \varphi \rangle - 2\alpha \langle \partial_t u(t), \varphi \rangle | \\ &\lesssim | \langle -\partial_x^2 u(t) + u(t), \varphi \rangle | + | \langle \delta_0 u(t), \varphi \rangle | + | \langle \partial_t u(t), \varphi \rangle | \\ &\lesssim \|\vec{u}(t)\|_{\mathcal{H}} \|\varphi\|_{H^1}. \end{aligned}$$

Thus $\|\partial_x^2 u(t)\|_{H^{-1}} \lesssim \|\vec{u}(t)\|_{\mathcal{H}}$ holds. Next, we consider the equation:

$$\langle -\partial_x^2 u(t) + u(t) - \gamma \delta_0 u(t), u(t) \rangle = -\langle \partial_t^2 u(t), u(t) \rangle - 2\alpha \langle \partial_t u(t), u(t) \rangle. \quad (2.7)$$

Then, by Lemma 2.1, we have

$$\langle -\partial_x^2 u(t) + u(t) - \gamma \delta_0 u(t), u(t) \rangle = \|u(t)\|_{H^1}^2 - \gamma |u(t, 0)|^2 \gtrsim \|u(t)\|_{H^1}^2. \quad (2.8)$$

By (2.7), (2.8), $|\langle \partial_t^2 u(t), u(t) \rangle| \lesssim \|\partial_t^2 u(t)\|_{H^{-1}} \|u(t)\|_{H^1}$ and $|\langle \partial_t u(t), u(t) \rangle| \lesssim \|\partial_t u(t)\|_{L^2} \|u(t)\|_{H^1}$, we have

$$\|u(t)\|_{H^1}^2 \lesssim (\|\partial_t^2 u(t)\|_{H^{-1}} + \|\partial_t u(t)\|_{L^2}) \|u(t)\|_{H^1},$$

which implies (2.6) is valid. Therefore by (2.6), we have

$$\|(\partial_t u(t), \partial_t^2 u(t))\|_{L^2 \times H^{-1}} \lesssim \|\vec{u}(t)\|_{\mathcal{H}} \lesssim e^{-\kappa t} \|\vec{u}(0)\|_{\mathcal{H}} \lesssim e^{-\kappa t} \|(\partial_t u(0), \partial_t^2 u(0))\|_{L^2 \times H^{-1}},$$

and complete the proof. $\square$

2.2. Proof of Local well-posedness of the Cauchy problem. We now consider the equation (2.1). We show that the equation (2.1) is locally well-posed in $\mathcal{H}$ using the standard arguments in [4, Chapter 4].

Proposition 2.4. *The following properties hold.*

- (1) *For any $U_0 \in \mathcal{H}$, the equation (2.1) has a unique maximal solution $U \in C([0, T(U_0)), \mathcal{H})$, with initial data U_0 .*
- (2) *If $T(U_0) < \infty$, then $\lim_{t \rightarrow T(U_0)-0} \|U(t)\|_{\mathcal{H}} = \infty$.*
- (3) *$T : \mathcal{H} \rightarrow (0, \infty]$ is lower semicontinuous: given the initial conditions $U_0, U_{0,n} \in \mathcal{H}$ such that $U_{0,n}$ converges to U_0 in $\mathcal{H}$, we have that*

$$T(U_0) \leq \liminf_{n \rightarrow \infty} T(U_{0,n}).$$

Moreover if $U_{0,n} \rightarrow U_0$ and if $T < T(U_0)$, then $U_n \rightarrow U$ in $C([0, T], \mathcal{H})$, where U_n and U are the solutions of (2.1) corresponding to the initial data $U_{0,n}$ and U_0 .

We prove Proposition 2.4 using the same argument as [11]. To show the same as [11], we only need to observe the nonlinearity F and the continuous semigroup $(S_{\alpha, \gamma}(t))_{t \geq 0}$.

Proof. To apply the arguments in [11, Section 2], we estimate the nonlinearity F and the continuous semigroup $(S_{\alpha, \gamma}(t))_{t \geq 0}$.

First, we estimate F . We note that

$$|f(u) - f(v)| \lesssim (1 + |u|^{p-1} + |v|^{p-1})|u - v|. \quad (2.9)$$

Therefore, for any $R > 0$, there exists $C_R > 0$ such that, for any $U, V \in \mathcal{H}$ with $\|U\|_{\mathcal{H}} < R$ and $\|V\|_{\mathcal{H}} < R$, we have

$$\|F(U) - F(V)\|_{\mathcal{H}} \leq C_R \|U - V\|_{\mathcal{H}}. \quad (2.10)$$

Next we estimate $(S_{\alpha, \gamma}(t))_{t \geq 0}$. This has been estimated by (2.5). Thus by (2.5) and (2.10) we complete the proof to apply [11, Section 2]. $\square$

By the well-posedness, for a solution $\vec{u}$ of (2.1), by [4, Chapter 4], we have

$$\vec{u}(t) = S_{\alpha,\gamma}(t)\vec{u}(0) + \int_0^t S_{\alpha,\gamma}(t-s)F(\vec{u}(s))ds. \quad (2.11)$$

Furthermore, by differentiating (2.11) we have

$$\begin{pmatrix} \partial_t u(t) \\ \partial_t^2 u(t) \end{pmatrix} = S_{\alpha,\gamma}(t) \begin{pmatrix} \partial_t u(0) \\ \partial_t^2 u(0) \end{pmatrix} + \int_0^t S_{\alpha,\gamma}(t-s) \begin{pmatrix} 0 \\ p|u(s)|^{p-1}\partial_t u(s) \end{pmatrix} ds. \quad (2.12)$$

By the well-posedness, we also have the time invariance:

$$u(t, x) \mapsto u(t+s, x), \quad (2.13)$$

where $s > 0$.

2.3. Uniform bound on global solutions. By subsection 2.2, we obtain the local well-posedness of (DNKG). In this subsection, we obtain a uniform bound for global solutions of (DNKG). This proof is essentially identical to that in [8, Theorem 2.2]. To prove the following theorem, we only need to consider the effect of a delta potential.

Theorem 2.5. *There exists a function $\mathcal{F} : [0, \infty) \rightarrow [0, \infty)$ such that $\mathcal{F}(0) = 0$ and for any global solution $\vec{u}$ of (DNKG), we have*

$$\sup_{t \in [0, \infty)} \|\vec{u}(t)\|_{\mathcal{H}} \leq \mathcal{F}(\|\vec{u}(0)\|_{\mathcal{H}}).$$

Proof. We assume that $\vec{u}$ is a global solution of (DNKG). We define $E_\gamma, M, W : [0, \infty) \rightarrow \mathbb{R}$ as:

$$\begin{aligned} E_\gamma(t) &= E_\gamma(\vec{u}(t)), \\ M(t) &= \frac{1}{2}\|u(t)\|_{L^2}^2 + \alpha \int_0^t \|u(s)\|_{L^2}^2 ds, \\ W(t) &= \frac{1}{2}\|\vec{u}(t)\|_{\mathcal{H}}^2 - \frac{\gamma}{2}|u(t, 0)|^2. \end{aligned}$$

By Lemma 2.1, we have

$$\|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L^2}^2 \sim \|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L^2}^2 - \gamma|u(t, 0)|^2. \quad (2.14)$$

Then, by direct computations, we have

$$M'(t) = \int u(t)\partial_t u(t)dx + \alpha\|u(t)\|_{L^2}^2 \quad (2.15)$$

$$= \int u(t)\partial_t u(t)dx + 2\alpha \int_0^t \int u(s)\partial_t u(s)dxds + \alpha\|u(0)\|_{L^2}^2, \quad (2.16)$$

$$M''(t) = \|\partial_t u(t)\|_{L^2}^2 - (\|u(t)\|_{H^1}^2 - \gamma|u(t, 0)|^2) + \|u(t)\|_{L^{p+1}}^{p+1} \quad (2.17)$$

$$= \frac{p+3}{2}\|\partial_t u(t)\|_{L^2}^2 + \frac{p-1}{2}(\|u(t)\|_{H^1}^2 - \gamma|u(t, 0)|^2) - (p+1)E_\gamma(t), \quad (2.18)$$

$$W'(t) = -2\alpha\|\partial_t u(t)\|_{L^2}^2 + \int f(u(t))\partial_t u(t)dx. \quad (2.19)$$

Furthermore by (2.14), (2.15), and the Cauchy-Schwarz inequality, there exists $C_1 > 0$ such that for all $t \geq 0$

$$|M'(t)| \leq C_1 W(t). \quad (2.20)$$

Moreover by (1.3) and (2.18), there exists $C_2 > 0$ such that for all $t \geq 0$

$$M''(t) \geq C_2 W(t) - (p+1)E_\gamma(0). \quad (2.21)$$

In addition, by (2.16) and the Cauchy-Schwarz inequality,

$$\begin{aligned} |M'(t)| &\leq \|u(t)\|_{L^2} \|\partial_t u(t)\|_{L^2} \\ &\quad + 2\alpha \left(\int_0^t \|u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\ &\quad + \alpha \|u(0)\|_{L^2}^2. \end{aligned}$$

Let $\epsilon > 0$ to be chosen later, we estimate

$$\begin{aligned} |M'|^2 &\leq (1+\epsilon) \left[\|u\|_{L^2} \|\partial_t u\|_{L^2} + 2\alpha \left(\int_0^t \|u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \right]^2 \\ &\quad + \left(1 + \frac{1}{\epsilon} \right) \alpha^2 \|u(0)\|_{L^2}^4, \end{aligned}$$

because for $x > 0, y > 0, \epsilon > 0$ the inequality $x + y \leq \sqrt{(1+\epsilon)x^2 + (1+\frac{1}{\epsilon})y^2}$ is valid. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |M'(t)|^2 &\leq (1+\epsilon)M(t) \left[2\|\partial_t u(t)\|_{L^2}^2 + 4\alpha \int_0^t \|\partial_t u(s)\|_{L^2}^2 ds \right] \\ &\quad + \left(1 + \frac{1}{\epsilon} \right) \alpha^2 \|u(0)\|_{L^2}^4. \end{aligned} \quad (2.22)$$

Furthermore by (1.3) and (2.18), we have

$$\begin{aligned} M''(t) &= 2\|\partial_t u(t)\|_{L^2}^2 + (p-1)W(t) \\ &\quad + 2\alpha(p+1) \int_0^t \|\partial_t u(s)\|_{L^2}^2 ds - (p+1)E_\gamma(0). \end{aligned}$$

Now we fix ϵ such that

$$0 < \epsilon < \left(\frac{p+7}{8} \right)^{\frac{1}{3}} - 1. \quad (2.23)$$

Then M'' satisfies

$$\begin{aligned} M''(t) &\geq (1+\epsilon)^3 \left[2\|\partial_t u(t)\|_{L^2}^2 + 4\alpha \int_0^t \|\partial_t u(s)\|_{L^2}^2 ds \right] \\ &\quad + \frac{p-1}{2} W(t) - (p+1)E_\gamma(0). \end{aligned} \quad (2.24)$$

We assume $\lim_{t \rightarrow \infty} M'(t) = \infty$. Then by (2.22) and (2.24), we have for t large enough,

$$(1+\epsilon)|M'(t)|^2 < M''(t)M(t). \quad (2.25)$$

This inequality (2.25) implies $\frac{d^2}{dt^2} [M^{-\epsilon}(t)] < 0$, and there exists $t_1 > 0$ such that $\frac{d}{dt} [M^{-\epsilon}(t_1)] < 0$ and for all $t \geq t_1$,

$$0 \leq M^{-\epsilon}(t) \leq M^{-\epsilon}(t_1) + (t - t_1) \frac{d}{dt} [M^{-\epsilon}(t_1)], \quad (2.26)$$

which is contradictory. Therefore, we obtain

$$\liminf_{t \rightarrow \infty} M'(t) < \infty. \quad (2.27)$$

Next we prove that there exists $C_3 > 0$ such that

$$\sup_{t \in [0, \infty)} |M'(t)| \leq C_3 (|E_\gamma(0)| + |M'(0)|). \quad (2.28)$$

By (2.20) and (2.21), we obtain

$$M''(t) \geq C_4 |M'(t)| - (p+1)E_\gamma(0), \quad (2.29)$$

where $C_4 = \frac{C_2}{C_1}$. Thus, the following two inequalities hold.

$$\frac{d}{dt} (C_4 M'(t) - (p+1)E_\gamma(0)) \geq C_4 (C_4 M'(t) - (p+1)E_\gamma(0)), \quad (2.30)$$

$$\frac{d}{dt} (-C_4 M'(t) + (p+1)E_\gamma(0)) \leq -C_4 (-C_4 M'(t) + (p+1)E_\gamma(0)). \quad (2.31)$$

We assume that there exists $t' \geq 0$ such that $M'(t') > \frac{p+1}{C_4} E_\gamma(0)$. Then by (2.30), we have for $t \geq t'$

$$C_4 M'(t) - (p+1)E_\gamma(0) \geq e^{C_4(t-t')} (C_4 M'(t') - (p+1)E_\gamma(0)).$$

Therefore $\lim_{t \rightarrow \infty} M'(t) = \infty$ holds, which contradicts (2.27). Thus, we have for $t \geq 0$

$$M'(t) \leq \frac{p+1}{C_4} E_\gamma(0) \leq \frac{p+1}{C_4} (|E_\gamma(0)| + |M'(0)|). \quad (2.32)$$

Next, by (2.31), we have

$$-C_4 M'(t) + (p+1)E_\gamma(0) \leq e^{-C_4 t} (-C_4 M'(0) + (p+1)E_\gamma(0)).$$

Therefore, we obtain

$$M'(t) \geq -\frac{p+1}{C_4} |E_\gamma(0)| - |M'(0)|. \quad (2.33)$$

By (2.32) and (2.33), we obtain (2.28) by defining $C_3 = \frac{2(p+1)}{C_4} + 1$.

Lastly, we prove the global bound

$$\sup_{t \in [0, \infty)} W(t) < \infty. \quad (2.34)$$

For $t \geq 0$ and $\tau \in (0, 1]$, integrating (2.21) on $(t, t+\tau)$, we have

$$C_2 \int_t^{t+\tau} W(s) ds - \tau(p+1)E_\gamma(0) \leq M'(t+\tau) - M'(t).$$

Furthermore by (2.28), we obtain

$$\int_t^{t+\tau} W(s) ds \leq \frac{p+1}{C_2} |E_\gamma(0)| + \frac{2C_3}{C_2} (|E_\gamma(0)| + |M'(0)|).$$

Thus, by defining $C_5 = \frac{p+1+2C_3}{C_2}$, for all $t \geq 0$ and $0 < \tau \leq 1$ we have

$$\int_t^{t+\tau} W(s) ds \leq C_5 (|E_\gamma(0)| + |M'(0)|). \quad (2.35)$$

We note that C_5 is independent of τ . By (2.19), there exists $C_6 > 0$ such that for all $t \geq 0$

$$W'(t) \leq -2\alpha \|\partial_t u(t)\|_{L^2}^2 + \int |u(t)|^p |\partial_t u(t)| dx \leq C_6 \|u(t)\|_{L^{2p}}^{2p}.$$

For $\tau \in (0, 1)$ and $t \geq \tau$, integrating on $(t - \tau, t)$, we have

$$W(t) \leq W(t - \tau) + C_6 \left(\int_{t-\tau}^t \|u(s)\|_{L^{2p}}^{2p} ds \right).$$

By using (2.14), (2.35), and the Sobolev inequality, there exists $C_7 > 0$ such that for all $0 < \tau \leq 1$ and $t \geq 1$

$$W(t) \leq W(t - \tau) + C_7 \left(\int_{t-\tau}^t W(s) ds \right)^p. \quad (2.36)$$

If $0 \leq t \leq 1$, by observing (2.36) with $t = \tau$, we have

$$W(t) \leq W(0) + C_7 \left(\int_0^t W(s) ds \right)^p \leq W(0) + C_7 \{C_5(|E_\gamma(0)| + |M'(0)|)\}^p. \quad (2.37)$$

If $t \geq 1$, by integrating (2.36) on $\tau \in (0, 1)$, we have

$$\begin{aligned} W(t) &\leq \int_0^1 W(t - \tau) d\tau + C_7 \{C_5(|E_\gamma(0)| + |M'(0)|)\}^p \\ &\leq C_5(|E_\gamma(0)| + |M'(0)|) + C_7 \{C_5(|E_\gamma(0)| + |M'(0)|)\}^p. \end{aligned} \quad (2.38)$$

Thus we obtain (2.34). In particular, by (2.37) and (2.38) W has an upper bound that depends only on $\|\vec{u}(0)\|_{\mathcal{H}}$, and by (2.14) we complete the proof. $\square$

3. SOLITON RESOLUTION

In this section, we prove Theorem 1.1. Before proving Theorem 1.1, we introduce the result of the concentration-compactness argument [18, Lemma 2.5].

Proposition 3.1. *Let $\gamma \neq 0$. We assume that $(u_n)_n \subset H^1$ is a bounded sequence satisfying*

$$\lim_{n \rightarrow \infty} \left\| -\partial_x^2 u_n + u_n - \gamma \delta_0(x) u_n - |u_n|^{p-1} u_n \right\|_{H^{-1}} = 0. \quad (3.1)$$

Then there exist a subsequence $(v_n)_n \subset (u_n)_n$, $K \geq 0$, signs $\sigma_0 = 0, \pm 1, \sigma_k = \pm 1$ for any $k \in \{1, 2, \dots, K\}$, and a sequence $(\xi_{k,n})_{k \in \{1, 2, \dots, K\}} \subset \mathbb{R}^K$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| v_n - \sigma_0 Q_\gamma - \sum_{k=1}^K \sigma_k Q(\cdot - \xi_{k,n}) \right\|_{H^1} &= 0, \\ \lim_{n \rightarrow \infty} (\xi_{k+1,n} - \xi_{k,n}) &= \infty, \text{ for } k = 1, \dots, K-1, \\ \lim_{n \rightarrow \infty} |\xi_{k,n}| &= \infty \text{ for } k = 1, \dots, K. \end{aligned} \quad (3.2)$$

Remark 3.1. When $\gamma = 0$, (1.6) allows the solution to translate. Thus, Proposition 3.1 also allows v_n in (3.2) to translate. For further details, please refer to [37, Appendix A] and [38, Theorem III.4].

Remark 3.2. In [18, Lemma 2.5], Fukuizumi and Jeanjean prove Proposition 3.1 using [26, Theorem 5.1].

Here, we prove Theorem 1.1. First, we define

$$E_{0,R} = \mathbb{R},$$

$$E_{K,R} = \{z \in \mathbb{R}^K : |z_k| > R \text{ for } 1 \leq k \leq K \text{ and } z_{k+1} - z_k > R \text{ for } 1 \leq k \leq K-1\},$$

where $K \geq 1$ and $R > 0$. Furthermore for $K \geq 0$ we denote $\vec{\sigma}^{K+1} = (\sigma_0, \sigma_1, \dots, \sigma_K) \in \{0, \pm 1\} \times \{\pm 1\}^K$. Then for $K \geq 0$, $\vec{\sigma}^{K+1}$, $R > 0$, and $\epsilon > 0$ we define

$$d_{K,\vec{\sigma}^{K+1},R}(u) = \inf_{z \in E_{K,R}} \|u - \sigma_0 Q_\gamma - \sum_{k=1}^K \sigma_k Q(\cdot - z_k)\|_{H^1},$$

$$\mathcal{N}_{K,\vec{\sigma}^{K+1},R,\epsilon} = \{u \in H^1(\mathbb{R}) : d_{K,\vec{\sigma}^{K+1},R}(u) < \epsilon\}.$$

Notably, when $R > 0$ is sufficiently large and $\epsilon > 0$ is sufficiently small, the family of sets $(\mathcal{N}_{K,\vec{\sigma}^{K+1},R,\epsilon})$ is disjoint.

Proof of Theorem 1.1. We assume that $\vec{u}$ is a global solution of (DNKG). First, we prove

$$\lim_{t \rightarrow \infty} \|\partial_t u(t)\|_{L^2} = 0, \quad (3.3)$$

$$\lim_{t \rightarrow \infty} \|\partial_x^2 u(t) + u(t) - \gamma \delta_0 u(t) - |u(t)|^{p-1} u(t)\|_{H^{-1}} = 0. \quad (3.4)$$

We define

$$\vec{v}(t) = \begin{pmatrix} v(t) \\ \partial_t v(t) \end{pmatrix} = \begin{pmatrix} \partial_t u(t) \\ \partial_t^2 u(t) \end{pmatrix}.$$

By (2.12), we obtain

$$\vec{v}(t) = S_{\alpha,\gamma}(t) \vec{v}(0) + \int_0^t S_{\alpha,\gamma}(t-s) \begin{pmatrix} 0 \\ p|u(s)|^{p-1} \partial_t u(s) \end{pmatrix} ds. \quad (3.5)$$

Furthermore by Proposition 2.3, for all $t \geq 0$,

$$\|S_{\alpha,\gamma}(t) \vec{v}(0)\|_{L^2 \times H^{-1}} \lesssim e^{-\kappa t}.$$

By (1.3), we have $v = \partial_t u \in L^2([0, \infty) \times \mathbb{R})$. Furthermore, by Theorem 2.5, we have $u \in L^\infty([0, \infty), H^1)$. Hence, we have

$$\begin{aligned} \left\| \int_0^{\frac{t}{2}} S_{\alpha,\gamma}(t-s) \vec{w}(s) ds \right\|_{L^2 \times H^{-1}} &\lesssim \int_0^{\frac{t}{2}} e^{-\kappa(t-s)} \|u\|_{L^\infty([0,\infty), H^1)}^{p-1} \|v(s)\|_{L^2} ds \\ &\lesssim e^{-\frac{\kappa t}{2}} \|u\|_{L^\infty([0,\infty), H^1)}^{p-1} \|v\|_{L^2([0,\infty) \times \mathbb{R})}, \\ \left\| \int_{\frac{t}{2}}^t S_{\alpha,\gamma}(t-s) \vec{w}(s) ds \right\|_{L^2 \times H^{-1}} &\lesssim \int_{\frac{t}{2}}^t e^{-\kappa(t-s)} \|u\|_{L^\infty([0,\infty), H^1)}^{p-1} \|v(s)\|_{L^2} ds \\ &\lesssim \|u\|_{L^\infty([0,\infty), H^1)}^{p-1} \|v\|_{L^2((\frac{t}{2}, \infty) \times \mathbb{R})}, \end{aligned}$$

where

$$\vec{w}(t) = \begin{pmatrix} 0 \\ p|u(t)|^{p-1} \partial_t u(t) \end{pmatrix}.$$

Thus, we have $\lim_{t \rightarrow \infty} \|\vec{v}(t)\|_{L^2 \times H^{-1}} = 0$, and therewe obtain (3.3) and (3.4).

Subsequently, $(u(n))_n \subset H^1(\mathbb{R})$ is bounded and satisfies (3.1). Therefore, by Proposition 3.1, there exist $\tilde{K} \geq 0$, $\vec{\sigma}^{\tilde{K}+1} \in \{0, \pm 1\} \times \{\pm 1\}^{\tilde{K}}$, $(\tilde{t}_n)_n$ such that for any $R > 0$,

$$\lim_{n \rightarrow \infty} d_{\tilde{K}, \vec{\sigma}^{\tilde{K}+1}, R}(u(\tilde{t}_n)) = 0.$$

Next we prove that for any sequence $t_n \rightarrow \infty$, u satisfies

$$\lim_{n \rightarrow \infty} d_{\tilde{K}, \vec{\sigma}^{\tilde{K}+1}, R}(u(t_n)) = 0. \quad (3.6)$$

We assume that there exist a sequence $t_n \rightarrow \infty$ and $R > 0$ such that

$$\limsup_{n \rightarrow \infty} d_{\tilde{K}, \vec{\sigma}^{\tilde{K}+1}, R}(u(t_n)) > 0.$$

Then there exist a subsequence $(t'_n)_n \subset (t_n)_n$ and a sufficiently small $\epsilon > 0$ such that for all $n \in \mathbb{N}$

$$u(t'_n) \notin \mathcal{N}_{\tilde{K}, \vec{\sigma}^{\tilde{K}+1}, R, \epsilon}. \quad (3.7)$$

Given that $(u(t'_n))_n$ is bounded and satisfies (3.1), by Proposition 3.1 there exist a subsequence $(t''_n)_n \subset (t'_n)_n$, $\hat{K}$, and $\vec{\sigma}^{\hat{K}+1}$ such that

$$\lim_{n \rightarrow \infty} d_{\hat{K}, \vec{\sigma}^{\hat{K}+1}, R}(u(t''_n)) = 0.$$

Notably, by (3.7), we have $(\hat{K}, \vec{\sigma}^{\hat{K}+1}) \neq (\tilde{K}, \vec{\sigma}^{\tilde{K}+1})$. Therefore, when $n, m \in \mathbb{N}$ satisfy $u(t_n) \in \mathcal{N}_{\tilde{K}, \vec{\sigma}^{\tilde{K}+1}, R, \epsilon}$ and $u(t'_m) \in \mathcal{N}_{\hat{K}, \vec{\sigma}^{\hat{K}+1}, R, \epsilon}$, there exists $s \in \mathbb{R}$ between t_n and t'_m such that

$$u(s) \notin \bigcup_{K \geq 0, \vec{\sigma}^{K+1} \in \{0, \pm 1\} \times \{\pm 1\}^K} \mathcal{N}_{K, \vec{\sigma}^{K+1}, R, \epsilon}.$$

Based on this argument, there exists a sequence $s_n \rightarrow \infty$ such that for all $n \in \mathbb{N}$,

$$u(s_n) \notin \bigcup_{K \geq 0, \vec{\sigma}^{K+1} \in \{0, \pm 1\} \times \{\pm 1\}^K} \mathcal{N}_{K, \vec{\sigma}^{K+1}, R, \epsilon}. \quad (3.8)$$

Then, $(u(s_n))_n$ is bounded and $u(s_n)$ satisfies (3.1). Therefore by Proposition 3.1, there exist a subsequence $(s'_n)_n \subset (s_n)_n$, $\dot{K}$, $\vec{\sigma}^{\dot{K}+1}$ such that

$$\lim_{n \rightarrow \infty} d_{\dot{K}, \vec{\sigma}^{\dot{K}+1}, R}(u(s'_n)) = 0,$$

which contradicts (3.8). Therefore for any $t_n \rightarrow \infty$, u satisfies (3.6) and we have

$$\lim_{t \rightarrow \infty} d_{\tilde{K}, \vec{\sigma}^{\tilde{K}+1}, R}(u(t)) = 0.$$

We then prove Theorem 1.1. We define $z(t) = (z_k(t)) \in \mathbb{R}^{\tilde{K}}$ as

$$\|u(t) - \tilde{\sigma}_0 Q_\gamma - \sum_{k=1}^{\tilde{K}} \tilde{\sigma}_k Q(\cdot - z_k(t))\|_{H^1} = \inf_{z \in \mathbb{R}^{\tilde{K}}} \|u(t) - \tilde{\sigma}_0 Q_\gamma - \sum_{k=1}^{\tilde{K}} \tilde{\sigma}_k Q(\cdot - z_k)\|_{H^1}.$$

Notably, for any $R > 0$, the minimum is attained for $z(t)$ satisfying $z(t) \in E_{R, \tilde{K}}$ when t is sufficiently large. Therefore, we obtain (1.7) with $K = \tilde{K}$, and $\sigma_k = \tilde{\sigma}_k$ for $0 \leq k \leq \tilde{K}$, and complete the proof. $\square$

4. GLOBAL DYNAMICS BELOW THE GROUND STATE ENERGY

4.1. Minimizing problems and variational structure. First, we collect the properties of the ground state energy. First we define $K_\gamma, J_\gamma : H^1(\mathbb{R}) \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} K_\gamma(\varphi) &= \|\varphi\|_{H^1}^2 - \gamma|\varphi(0)|^2 - \|\varphi\|_{L^{p+1}}^{p+1}, \\ J_\gamma(\varphi) &= \frac{1}{2} (\|\varphi\|_{H^1}^2 - \gamma|\varphi(0)|^2) - \frac{1}{p+1} \|\varphi\|_{L^{p+1}}^{p+1}, \end{aligned}$$

and consider the following minimizing problems:

$$\begin{aligned} n_\gamma &= \inf\{J_\gamma(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\}, K_\gamma(\varphi) = 0\}, \\ r_\gamma &= \inf\{J_\gamma(\varphi) : \varphi \in H_{rad}^1(\mathbb{R}) \setminus \{0\}, K_\gamma(\varphi) = 0\}, \end{aligned}$$

where $H_{rad}^1(\mathbb{R}) := \{\varphi \in H^1(\mathbb{R}) : \varphi(x) = \varphi(-x)\}$. In [18, 23], we obtain the following statements regarding n_γ and r_γ :

Proposition 4.1. *The following properties hold.*

- (1) if $0 \leq \gamma < 2$, Q_γ is a minimizer of n_γ and r_γ .
- (2) if $-2 < \gamma < 0$, Q_γ is a minimizer of r_γ . Furthermore $n_\gamma = J_0(Q)$ has no minimizer.
- (3) if $\gamma \leq -2$, $n_\gamma = J_0(Q)$ has no minimizer and $r_\gamma = 2J_0(Q)$ has no minimizer.

Remark 4.1. In [23], the radial case of Proposition 4.1 is restricted to $p > 5$. However, the essence of the proof of Proposition 4.1 is whether the nontrivial solution of (1.6) exists. Thus, Proposition 4.1 is valid in $p > 1$.

Furthermore, we obtain a uniform bound on K_γ .

Proposition 4.2. (1) For any $\varphi \in H^1(\mathbb{R})$ with $J_\gamma(\varphi) < n_\gamma$, we have

$$K_\gamma(\varphi) \gtrsim \min\{n_\gamma - J_\gamma(\varphi), \|\varphi\|_{H^1}^2\} \text{ or } K_\gamma(\varphi) \lesssim -(n_\gamma - J_\gamma(\varphi)). \quad (4.1)$$

(2) For any $\varphi \in H_{rad}^1(\mathbb{R})$ with $J_\gamma(\varphi) < r_\gamma$, we have

$$K_\gamma(\varphi) \gtrsim \min\{r_\gamma - J_\gamma(\varphi), \|\varphi\|_{H^1}^2\} \text{ or } K_\gamma(\varphi) \lesssim -(r_\gamma - J_\gamma(\varphi)). \quad (4.2)$$

This uniform bound is useful to classify the global behavior of solutions whose energy is less than ground state energy for the nonlinear dispersive equations. To prove Proposition 4.2, we refer to [23, Proposition 2.18] and [24, Lemma 2.12].

Proof of Proposition 4.2. First, we prove (1). We may assume $\varphi \neq 0$. We define $j(\lambda) = J_\gamma(e^\lambda \varphi)$. Then we have

$$j(\lambda) = \frac{e^{2\lambda}}{2} (\|\varphi\|_{H^1}^2 - \gamma|\varphi(0)|^2) - \frac{e^{(p+1)\lambda}}{p+1} \|\varphi\|_{L^{p+1}}^{p+1}, \quad (4.3)$$

$$j'(\lambda) = e^{2\lambda} (\|\varphi\|_{H^1}^2 - \gamma|\varphi(0)|^2) - e^{(p+1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1}, \quad (4.4)$$

$$j''(\lambda) = 2e^{2\lambda} (\|\varphi\|_{H^1}^2 - \gamma|\varphi(0)|^2) - (p+1)e^{(p+1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1}. \quad (4.5)$$

By (4.4) and (4.5), we obtain $j'(0) = K_\gamma(\varphi)$ and

$$j''(\lambda) \leq 2j'(\lambda). \quad (4.6)$$

First we consider $K_\gamma(\varphi) < 0$. Then we have $j'(0) < 0$ and $j'(\lambda) > 0$ for sufficiently small $\lambda < 0$. Therefore there exists $\lambda_0 < 0$ such that $j'(\lambda_0) = 0$. By integrating (4.6) on $[\lambda_0, 0]$, we have

$$j'(0) - j'(\lambda_0) \leq 2(j(0) - j(\lambda_0)). \quad (4.7)$$

By (4.7), $j(\lambda_0) \geq n_\gamma$, and $j'(\lambda_0) = 0$, we obtain

$$K_\gamma(\varphi) = j'(0) \leq -2(n_\gamma - J_\gamma(\varphi)).$$

Second we consider $K_\gamma(\varphi) \geq 0$. Since $\varphi \neq 0$, we consider $K_\gamma(\varphi) > 0$. If

$$K_\gamma(\varphi) \geq \frac{p-1}{3} \|\varphi\|_{L^{p+1}}^{p+1},$$

we have

$$(1 + \frac{p-1}{3})K_\gamma(\varphi) \geq \frac{p-1}{3} (\|\varphi\|_{H^1}^2 - \gamma|\varphi(0)|^2).$$

By Lemma 2.1, we obtain $K_\gamma(\varphi) \gtrsim \|\varphi\|_{H^1}^2$. If

$$K_\gamma(\varphi) < \frac{p-1}{3} \|\varphi\|_{L^{p+1}}^{p+1},$$

there exists $\lambda_1 > 0$ such that

$$0 < j'(\lambda) < \frac{p-1}{3} e^{(p+1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1} \quad (4.8)$$

for $0 < \lambda < \lambda_1$. Then by direct calculation and (4.8), we have for $0 < \lambda < \lambda_1$

$$\begin{aligned} j''(\lambda) &= 2j'(\lambda) - (p-1)e^{(p+1)\lambda} \|\varphi\|_{L^{p+1}}^{p+1} \\ &< -j'(\lambda). \end{aligned}$$

In particular, as long as (4.8) holds, j' decreases. Furthermore $\lim_{\lambda \rightarrow \infty} j'(\lambda) = -\infty$ holds and we may assume $j'(\lambda_1) = 0$. Then by integrating $j'' < -j'$ on $[0, \lambda_1]$, we have

$$j'(\lambda_1) - j'(0) < -(j(\lambda_1) - j(0)).$$

Since $j(\lambda_1) \geq n_\gamma$, we obtain

$$K_\gamma(\varphi) > n_\gamma - J_\gamma(\varphi).$$

Gathering these arguments, we obtain (4.1).

Notably, we only use the properties of minimizing problems and variational structure to obtain (4.1). Therefore, we obtain (4.2) by the same argument. $\square$

4.2. Classification of global dynamics. In this subsection, we describe the classification of the global behavior of solutions whose energy is less than the ground state energy. To prove the following proposition, we refer to [25].

Proposition 4.3. *Let $\vec{u}$ be a solution of (DNKG).*

- (1) *If $\vec{u}$ satisfies $E_\gamma(\vec{u}(0)) < n_\gamma$ and $K_\gamma(u(0)) \geq 0$, then $\vec{u}$ is global and converges exponentially to 0 in $\mathcal{H}$ as $t \rightarrow \infty$.*
- (2) *If $\vec{u}$ satisfies $E_\gamma(\vec{u}(0)) < n_\gamma$ and $K_\gamma(u(0)) < 0$, then $\vec{u}$ blows up in finite time.*
- (3) *If $\vec{u}$ is even and satisfies $E_\gamma(\vec{u}(0)) < r_\gamma$ and $K_\gamma(u(0)) \geq 0$, then $\vec{u}$ is global and converges exponentially to 0 in $\mathcal{H}$ as $t \rightarrow \infty$.*

(4) If $\vec{u}$ is even and satisfies $E_\gamma(\vec{u}(0)) < r_\gamma$ and $K_\gamma(u(0)) < 0$, then $\vec{u}$ blows up in finite time.

Proof. We define $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ as

$$\mathcal{P}(\varphi_1, \varphi_2) = \int \varphi_1 \varphi_2 + \alpha \|\varphi_1\|_{L^2}^2.$$

Then, we have

$$|\mathcal{P}(\varphi_1, \varphi_2)| \leq C_1 \|(\varphi_1, \varphi_2)\|_{\mathcal{H}}, \quad (4.9)$$

where $C_1 = \alpha + \frac{1}{2}$. First we prove (1). Since $J_\gamma(u(0)) < n_\gamma$ and $K_\gamma(u(0)) \geq 0$, as long as $\vec{u}$ exists, $K_\gamma(u(t)) \geq 0$ holds and by (1.3), $n_\gamma - E_\gamma(\vec{u}(t)) \geq n_\gamma - E_\gamma(\vec{u}(0))$ holds. Then, we have

$$\begin{aligned} E_\gamma(\vec{u}(t)) &= \frac{1}{p+1} K_\gamma(u(t)) + \frac{1}{2} \|\partial_t u(t)\|_{L^2}^2 + \frac{p-1}{2(p+1)} (\|u(t)\|_{H^1}^2 - \gamma |u(t,0)|^2) \\ &\geq \frac{1}{2} \|\partial_t u(t)\|_{L^2}^2 + \frac{p-1}{2(p+1)} (\|u(t)\|_{H^1}^2 - \gamma |u(t,0)|^2). \end{aligned}$$

Therefore there exist $C_2, C_3 > 0$ such that

$$C_2 \|\vec{u}(t)\|_{\mathcal{H}}^2 \leq E_\gamma(\vec{u}(t)) \leq C_3 \|\vec{u}(t)\|_{\mathcal{H}}^2. \quad (4.10)$$

In particular, as long as $\vec{u}$ exists, $\|\vec{u}(t)\|_{\mathcal{H}}$ is uniformly bounded in $\mathcal{H}$. Therefore, $\vec{u}$ is global. Furthermore, by Proposition 4.2, there exists $C_4 > 0$ such that

$$K_\gamma(u(t)) \geq C_4 \min \{n_\gamma - E_\gamma(\vec{u}(0)), \|u(t)\|_{H^1}^2\}.$$

If $\|u(t)\|_{H^1}^2 \leq n_\gamma - E_\gamma(\vec{u}(0))$, we have

$$K_\gamma(u(t)) \geq C_4 \|u(t)\|_{H^1}^2. \quad (4.11)$$

If $\|u(t)\|_{H^1}^2 > n_\gamma - E_\gamma(\vec{u}(0))$, by (1.3), (4.10), and $E_\gamma(\vec{u}(t)) \leq E_\gamma(\vec{u}(0)) < n_\gamma$, we have

$$K_\gamma(u(t)) \geq C_4 (n_\gamma - E_\gamma(\vec{u}(0))) \geq \frac{C_2 C_4 (n_\gamma - E_\gamma(\vec{u}(0)))}{n_\gamma} \|u(t)\|_{H^1}^2. \quad (4.12)$$

By (4.11) and (4.12), there exists $C_5 > 0$ such that

$$K_\gamma(u(t)) \geq C_5 \|u(t)\|_{H^1}^2. \quad (4.13)$$

Here, we assume that $C_6 > 0$ is a large constant such that

$$C_2 C_6 - C_1 > 1, \quad 2\alpha C_6 - 1 > \max(C_5, C_3 + \frac{C_1}{C_6}) \quad (4.14)$$

hold. Then by (4.9), (4.10), (4.14), we obtain

$$(C_2 C_6 - C_1) \|\vec{u}(t)\|_{\mathcal{H}}^2 \leq \mathcal{P}(\vec{u}(t)) + C_6 E_\gamma(\vec{u}(t)) \leq (C_1 + C_3 C_6) \|\vec{u}(t)\|_{\mathcal{H}}^2. \quad (4.15)$$

Furthermore, by (4.13), (4.14), and (4.15), we have

$$\begin{aligned} \partial_t [\mathcal{P}(\vec{u}) + C_6 E_\gamma(\vec{u})] &= (1 - 2\alpha C_6) \|\partial_t u\|_{L^2}^2 - K_\gamma(u) \\ &\leq (1 - 2\alpha C_6) \|\vec{u}\|_{\mathcal{H}}^2 \\ &\leq -\frac{C_3 C_6 + C_1}{C_6} \|\vec{u}\|_{\mathcal{H}}^2 \\ &\leq -\frac{1}{C_6} [\mathcal{P}(\vec{u}) + C_6 E_\gamma(\vec{u})]. \end{aligned}$$

Thus, we have

$$\|\vec{u}(t)\|_{\mathcal{H}} \lesssim e^{-\frac{t}{C_6}} \|\vec{u}(0)\|_{\mathcal{H}},$$

and complete the proof of (1).

Next, we prove (2). We assume that $\vec{u}$ is global. By (1.3) and Proposition 4.2, we have for all $t \geq 0$, $K_\gamma(u(t)) \lesssim -(n_\gamma - E_\gamma(\vec{u}(0)))$. Thus, there exists $C_7 > 0$ such that

$$\begin{aligned} \partial_t \mathcal{P}(\vec{u}) &= \|\partial_t u\|_{L^2}^2 - K_\gamma(u) \\ &\geq C_7(n_\gamma - J_\gamma(u)) \\ &\geq C_7(n_\gamma - E_\gamma(\vec{u})) \\ &\geq C_7(n_\gamma - E_\gamma(\vec{u}(0))). \end{aligned}$$

By the above estimate, $\lim_{t \rightarrow \infty} \mathcal{P}(\vec{u}(t)) = \infty$ holds. Therefore $\lim_{t \rightarrow \infty} \|\vec{u}(t)\|_{\mathcal{H}} = \infty$ holds, which contradicts Theorem 2.5. Thus, we have proved (2).

Finally, we prove (3) and (4). Let $\vec{u}$ be even. Then, by Proposition 4.2, we prove them using the same arguments as in (1) and (2). Therefore, we complete the proof. $\square$

Remark 4.2. We assume that a solution $\vec{u}$ of (DNKG) satisfies $\lim_{t \rightarrow \infty} \|\vec{u}(t)\|_{\mathcal{H}} = 0$. Then, $\vec{u}$ satisfies $\|\vec{u}(t)\|_{\mathcal{H}} \ll 1$ for some $t \in \mathbb{R}$, and therefore $\vec{u}(t)$ satisfies $E_\gamma(\vec{u}(t)) < n_\gamma$ and $K_\gamma(u(t)) \geq 0$. Therefore by Proposition 4.3, $\vec{u}$ converges exponentially to 0 as $t \rightarrow \infty$. Based on this argument, when a global solution $\vec{u}$ satisfies (1.7) with $\sigma = 0$ and $K = 0$, $\vec{u}$ converges exponentially to 0 as $t \rightarrow \infty$.

We define $\mathcal{H}_{rad}$ as

$$\mathcal{H}_{rad} = \{(\varphi_1, \varphi_2) \in \mathcal{H} : \varphi_1(x) = \varphi_1(-x), \varphi_2(x) = \varphi_2(-x)\}.$$

Furthermore, we define $\mathcal{B}$, $\mathcal{B}_{rad}$, $\mathcal{V}$, $\mathcal{V}_{rad}$ by

$$\begin{aligned} \mathcal{B} &= \{\vec{\varphi} \in \mathcal{H} : \vec{u} \text{ blows up in finite time}\}, \\ \mathcal{B}_{rad} &= \{\vec{\varphi} \in \mathcal{H}_{rad} : \vec{u} \text{ blows up in finite time}\}, \\ \mathcal{V} &= \{\vec{\varphi} \in \mathcal{H} : \vec{u} \text{ is global and } \lim_{t \rightarrow \infty} \|\vec{u}(t)\|_{\mathcal{H}} = 0\}, \\ \mathcal{V}_{rad} &= \{\vec{\varphi} \in \mathcal{H}_{rad} : \vec{u} \text{ is global and } \lim_{t \rightarrow \infty} \|\vec{u}(t)\|_{\mathcal{H}} = 0\}, \end{aligned}$$

where $\vec{u}$ is the solution of (DNKG) with initial data $\vec{\varphi}$. Then, $\mathcal{B}, \mathcal{B}_{rad}, \mathcal{V}, \mathcal{V}_{rad}$ have the following topological properties.

Lemma 4.4. $\mathcal{B}$ and $\mathcal{V}$ are open in $\mathcal{H}$. Furthermore $\mathcal{B}_{rad}$ and $\mathcal{V}_{rad}$ are open in $\mathcal{H}_{rad}$.

Proof. Let $\vec{u}$ be a solution of (DNKG). If $\vec{u}(0) \in \mathcal{V}$, then there exists $t > 0$ such that $\|\vec{u}(t)\|_{\mathcal{H}} \ll 1$. Then, $E_\gamma(\vec{u}(t)) < n_\gamma$ and $K_\gamma(u(t)) \geq 0$ are valid. Therefore, by Proposition 2.4 and Proposition 4.2, $\mathcal{V}$ is open.

Next, we consider $\mathcal{B}$. Let $(U_{n,0})_n \subset \mathcal{H} \setminus \mathcal{B}$ and $U_0 \in \mathcal{H}$ satisfy

$$U_{n,0} \rightarrow U_0 \text{ in } \mathcal{H},$$

as $n \rightarrow \infty$. Then $(U_{n,0})$ is bounded in $\mathcal{H}$. We assume that $\vec{u}_n$ is the solution of (DNKG) with initial data $U_{n,0}$ and $\vec{u}$ is the solution of (DNKG) with initial data

U_0 . Furthermore, we assume that $U_0 \in \mathcal{B}$. Then, by (2) of Proposition 2.4, there exists $t > 0$ such that

$$\|\vec{u}(t)\|_{\mathcal{H}} > \left(\sup_{n \in \mathbb{N}} \mathcal{F}(\|U_{n,0}\|_{\mathcal{H}}) \right) + 1, \quad (4.16)$$

where $\mathcal{F}$ is the function that emerges from Theorem 2.5. On the other hand, by (3) of Proposition 2.4, there exists $m \in \mathbb{N}$ such that

$$\|\vec{u}(t)\|_{\mathcal{H}} < \|\vec{u}_m(t)\|_{\mathcal{H}} + 1. \quad (4.17)$$

By (4.16) and (4.17), we have

$$\mathcal{F}(\|U_{m,0}\|_{\mathcal{H}}) + 1 \leq \left(\sup_{n \in \mathbb{N}} \mathcal{F}(\|U_{n,0}\|_{\mathcal{H}}) \right) + 1 < \|\vec{u}(t)\|_{\mathcal{H}} < \|\vec{u}_m(t)\|_{\mathcal{H}} + 1,$$

and therefore we have

$$\|\vec{u}_m(t)\|_{\mathcal{H}} > \mathcal{F}(\|U_{m,0}\|_{\mathcal{H}}),$$

which contradicts Theorem 2.5. Therefore $U_0 \in \mathcal{H} \setminus \mathcal{B}$ holds. Thus, $\mathcal{H} \setminus \mathcal{B}$ is closed, which implies that $\mathcal{B}$ is open.

The same holds true for $\mathcal{B}, \mathcal{V}$ replaced with $\mathcal{B}_{rad}, \mathcal{V}_{rad}$, and the proof is complete. $\square$

4.3. Proof of Theorem 1.2. In this subsection, we prove Theorem 1.2. Before we prove Theorem 1.2, we collect the properties of Q . Since Q is a minimizer of n_0 , we have

$$K_0(Q) = \|Q\|_{H^1}^2 - \|Q\|_{L^{p+1}}^{p+1} = 0. \quad (4.18)$$

Notably, by Proposition 4.1, if $\gamma < 0$, $n_\gamma = n_0$ holds, and if $\gamma \leq -2$, $r_\gamma = 2n_0$ holds. Then, we define j as

$$j(\lambda) = J_0(e^\lambda Q) = \frac{e^{2\lambda}}{2} \|Q\|_{H^1}^2 - \frac{e^{(p+1)\lambda}}{p+1} \|Q\|_{L^{p+1}}^{p+1}.$$

Then, we have

$$\begin{aligned} j'(\lambda) &= e^{2\lambda} \|Q\|_{H^1}^2 - e^{(p+1)\lambda} \|Q\|_{L^{p+1}}^{p+1}, \\ j''(\lambda) &= 2e^{2\lambda} \|Q\|_{H^1}^2 - (p+1)e^{(p+1)\lambda} \|Q\|_{L^{p+1}}^{p+1}. \end{aligned}$$

We define $c_j = \min(\frac{1}{2} \log 2, \frac{1}{p+1} \log \frac{2(p+1)}{p+3})$. Then by (4.18), for $|\lambda| < c_j$ we have

$$j''(\lambda) < -\frac{p-1}{4} \|Q\|_{H^1}^2. \quad (4.19)$$

We assume that $z > 0$ is sufficiently large. For $\sigma = 0, 1$, we define

$$\begin{aligned} \mathcal{Q}_\sigma &= Q(\cdot - z) + \sigma Q(\cdot + z), \\ \mathcal{Q}_{\lambda,\sigma} &= e^\lambda \mathcal{Q}_\sigma, \\ \vec{\mathcal{Q}}_{\lambda,\sigma} &= (\mathcal{Q}_{\lambda,\sigma}, 0), \\ \mathcal{A}_\sigma &= \{\vec{\mathcal{Q}}_{\lambda,\sigma} : -c_j \leq \lambda \leq c_j\}. \end{aligned}$$

Furthermore, we define

$$j_\sigma(\lambda) = E_\gamma(\vec{\mathcal{Q}}_{\lambda,\sigma}) = \frac{e^{2\lambda}}{2} \|\mathcal{Q}_\sigma\|_{H^1}^2 - \frac{e^{(p+1)\lambda}}{p+1} \|\mathcal{Q}_\sigma\|_{L^{p+1}}^{p+1} - \frac{\gamma e^{2\lambda}}{2} |\mathcal{Q}_\sigma(0)|^2.$$

Then by direct calculation, we have

$$\begin{aligned} j'_\sigma(\lambda) &= e^{2\lambda} \|\mathcal{Q}_\sigma\|_{H^1}^2 - e^{(p+1)\lambda} \|\mathcal{Q}_\sigma\|_{L^{p+1}}^{p+1} - \gamma e^{2\lambda} |\mathcal{Q}_\sigma(0)|^2, \\ j''_\sigma(\lambda) &= 2e^{2\lambda} \|\mathcal{Q}_\sigma\|_{H^1}^2 - (p+1)e^{(p+1)\lambda} \|\mathcal{Q}_\sigma\|_{L^{p+1}}^{p+1} - 2\gamma e^{2\lambda} |\mathcal{Q}_\sigma(0)|^2. \end{aligned}$$

Since $z > 0$ is sufficiently large, using the Taylor expansion, we have

$$|\|\mathcal{Q}_\sigma\|_{H^1}^2 - (\sigma+1)\|Q\|_{H^1}^2| \lesssim e^{-2z}, \quad |\|\mathcal{Q}_\sigma\|_{L^{p+1}}^{p+1} - (1+\sigma)\|Q\|_{L^{p+1}}^{p+1}| \lesssim e^{-(p+1)z}.$$

Thus, for $-c_j < \lambda < c_j$, we have

$$|j_\sigma(\lambda) - (1+\sigma)j(\lambda)| + |j'_\sigma(\lambda) - (1+\sigma)j'(\lambda)| + |j''_\sigma(\lambda) - (1+\sigma)j''(\lambda)| \leq e^{-z}, \quad (4.20)$$

and we start to prove Theorem 1.2.

Proof of Theorem 1.2. Given that z is large, for $-c_j < \lambda < c_j$, we have

$$j''_\sigma(\lambda) < -\frac{p-1}{8} \|Q\|_{H^1}^2.$$

Thus, for $-c_j < \lambda_1 \leq \lambda_2 < c_j$, we have

$$j'_\sigma(\lambda_1) - j'_\sigma(\lambda_2) \geq C(\lambda_2 - \lambda_1), \quad (4.21)$$

where $C = \frac{p-1}{8} \|Q\|_{H^1}^2$. Furthermore, by (4.20) $|j'_\sigma(0)| \leq e^{-z}$ holds true and by (4.21), we have $j'_\sigma(-e^{-\frac{1}{2}z}) > 0$, $j'_\sigma(e^{-\frac{1}{2}z}) < 0$. Thus there exists $-e^{-\frac{1}{2}z} < \lambda' < e^{-\frac{1}{2}z}$ such that $j'_\sigma(\lambda') = 0$. By (4.21), for $-c_j < \lambda < c_j$, we have

$$\begin{aligned} j_\sigma(\lambda') - j_\sigma(\lambda) &= \int_\lambda^{\lambda'} j'_\sigma(s) - j'_\sigma(\lambda') ds \\ &= \left| \int_\lambda^{\lambda'} |j'_\sigma(s) - j'_\sigma(\lambda')| ds \right| \\ &\geq C \left| \int_{\lambda'}^\lambda |s - \lambda'| ds \right| \\ &= \frac{C}{2} |\lambda - \lambda'|^2. \end{aligned}$$

By $j(0) = n_0$ and $|\lambda'| \leq e^{-\frac{1}{2}z}$, we have

$$|j_\sigma(\lambda') - (1+\sigma)n_0| \lesssim e^{-\frac{1}{2}z}. \quad (4.22)$$

Based on these arguments, we have

$$\begin{aligned} j'_\sigma(-c_j) &> 0, \quad j'_\sigma(c_j) < 0, \\ j_\sigma(-c_j) &< (1+\sigma)n_0, \quad j_\sigma(c_j) < (1+\sigma)n_0. \end{aligned}$$

Now we assume $\gamma < 0$. Then $n_\gamma = n_0$ holds and by Proposition 4.3, $\vec{\mathcal{Q}}_{-c_j,0} \in \mathcal{V}$ and $\vec{\mathcal{Q}}_{c_j,0} \in \mathcal{B}$. Since $\mathcal{A}_0$ is connected and by Lemma 4.4, there exists $-c_j \leq \lambda^* \leq c_j$ such that $\vec{\mathcal{Q}}_{\lambda^*,0} \notin \mathcal{B}$ and $\vec{\mathcal{Q}}_{\lambda^*,0} \notin \mathcal{V}$. Moreover by (4.22), we have

$$E_\gamma(\vec{\mathcal{Q}}_{\lambda^*,0}) - n_0 \lesssim e^{-\frac{1}{2}z}. \quad (4.23)$$

Since z is sufficiently large, by Theorem 1.1 and (4.23), we know from the energy of the solution that the solution $\vec{u}_{*,0}$ of (DNKG) with initial data $\vec{\mathcal{Q}}_{\lambda^*,0}$ satisfies

$$\lim_{t \rightarrow \infty} (\|u_{*,0}(t) - \sigma' Q(\cdot - z(t))\|_{H^1} + \|\partial_t u_{*,0}(t)\|_{L^2}) = 0,$$

where $\sigma' = \pm 1$ and $\lim_{t \rightarrow \infty} |z(t)| = \infty$. This completes the proof of (1) of Theorem 1.2, since (1.1) holds.

Next we prove (2) of Theorem 1.2. When $\gamma \leq -2$ holds true, we have $r_\gamma = 2n_0$. Thus, considering $\sigma = 1$, we can prove (2) as in the proof of (1) of Theorem 1.2. $\square$

5. DYNAMICS CLOSE TO SOLITARY WAVES

In this section, we prove Theorem 1.3.

5.1. Basic properties of the ground state. In this subsection, we collect some basic properties. First, to prove Theorem 1.3, we need to consider the spectral decomposition around the ground state Q . The linearized operator $\mathcal{L}$ around Q of the stationary equation is denoted by

$$\mathcal{L} = -\partial_x^2 + 1 - pQ^{p-1}. \quad (5.1)$$

We introduce standard properties of the operator $\mathcal{L}$. For further details, please refer to [5, Theorem 3.1].

Lemma 5.1. (1) *The unbounded operator $\mathcal{L}$ on L^2 with domain H^2 is self-adjoint, its continuous spectrum is $[1, \infty)$, its kernel is $\text{span}\{Q'\}$ and it has a unique negative eigenvalue $-\nu_0^2$, with corresponding smooth normalized even eigenfunction ϕ ($\|\phi\|_{L^2} = 1$). Moreover, on $\mathbb{R}$, for $m \in \mathbb{N}$,*

$$|\partial_x^m \phi(x)| \lesssim e^{-\sqrt{1+\nu_0^2}|x|}.$$

(2) *There exists $c > 0$ such that, for all $\varphi \in H^1(\mathbb{R})$,*

$$\langle \mathcal{L}\varphi, \varphi \rangle \geq c\|\varphi\|_{H^1}^2 - \frac{1}{c} (\langle \varphi, \phi \rangle^2 + \langle \varphi, Q' \rangle^2).$$

Remark 5.1. In general, Lemma 5.1 holds for any spatial dimension. On the other hand, when we consider the one-dimensional case, ν_0 and ϕ are explicitly expressed as

$$\begin{aligned} \nu_0 &= \sqrt{\frac{(p-1)(p+3)}{4}}, \\ \phi(x) &= \left(\frac{1}{\cosh(\frac{p-1}{2}x)} \right)^{\frac{p+1}{p-1}}. \end{aligned}$$

Furthermore, we have

$$\phi = \left(\frac{2}{p+1} \right)^{\frac{p+1}{2(p-1)}} Q^{\frac{p+1}{2}}.$$

When $\gamma = 0$, linearizing equation (DNKG) around the static solution Q is given by

$$\frac{d}{dt}v = \begin{pmatrix} 0 & 1 \\ -\mathcal{L} & -2\alpha \end{pmatrix} v. \quad (5.2)$$

We define $\nu^\pm$ and $Y^\pm$ as follows:

$$\begin{aligned} \nu^\pm &= -\alpha \pm \sqrt{\alpha^2 + \nu_0^2}, \\ Y^\pm &= \begin{pmatrix} 1 \\ \nu^\pm \end{pmatrix} \phi. \end{aligned}$$

Given that $\nu^+ > 0$, the solution $v = e^{\nu^+ t} Y^+$ of (5.2) exhibits the exponential instability of $(Q, 0)$. In particular, we see that the presence of the damping $\alpha > 0$ in the equation does not remove the exponential instability.

In (DNKG), when the center of the ground state is far from the origin, the ground state is less affected by the delta potential. Therefore when we consider a single solitary wave or an even 2-solitary wave, we ignore the delta potential and consider the spectral decomposition as in the $\gamma = 0$ case.

Second We recall properties of soliton interactions. For further details, you can refer to [9, Lemma 2.1].

Lemma 5.2. *We assume that $z_1, z_2 \in \mathbb{R}$ satisfy $|z_1 - z_2| \gg 1$. Then, by defining*

$$Q_1 = Q(\cdot - z_1), \quad Q_2 = Q(\cdot - z_2),$$

the following properties hold.

(1) *The following equation holds:*

$$\int Q^p e^{-x} = 2c_Q. \quad (5.3)$$

(2) *For any $0 < m' < m$,*

$$0 < \int Q_1^m Q_2^m \lesssim e^{-m'|z_1 - z_2|}, \quad (5.4)$$

$$0 < \int Q_1^m Q_2^{m'} \lesssim e^{-m'|z_1 - z_2|}. \quad (5.5)$$

(3) *The following estimate holds:*

$$\left| \langle f(Q_1 + Q_2) - f(Q_1) - f(Q_2), \partial_x Q_1 \rangle - \frac{z_1 - z_2}{|z_1 - z_2|} \cdot 2c_Q^2 e^{-|z_1 - z_2|} \right| \lesssim e^{-\min(p-1, 2)|z_1 - z_2|}.$$

Proof. First, we prove (1). Let $R > 0$. By integration by parts, we obtain that as $R \rightarrow \infty$,

$$\begin{aligned} \int_{-R}^R Q^p e^{-x} &= \int_{-R}^R (-Q'' + Q) e^{-x} \\ &= c_Q + \int_{-R}^R (-Q' + Q) e^{-x} + O(e^{-2R}) \\ &= 2c_Q + O(e^{-2R}). \end{aligned}$$

Therefore we obtain (5.3). (2) and (3) are proven by [8, Lemma 3.2], [9, Lemma 2.1], and (5.3). Thus, we complete the proof. $\square$

5.2. Modulation of the center. Here, we assume that a solution $\vec{u}$ of (DNKG) is a single solitary wave or an even 2-solitary wave. We have some freedom for the choice of the center of the solitary wave. So we choose the center of the solitary wave.

Lemma 5.3. (1) *Let $\sigma = \pm 1$. There exists $\delta > 0$ small enough such that for any $\varphi = (\varphi_1, \varphi_2) \in \mathcal{H}$ and $z \in \mathbb{R}$ satisfying*

$$\|\varphi_1 - \sigma Q(\cdot - z)\|_{H^1} + \|\varphi_2\|_{L^2} \leq \delta, \quad (5.6)$$

there exists a unique $\tilde{z} \in \mathbb{R}$ such that

$$\begin{aligned} \int \{\varphi_2 + 2\alpha(\varphi_1 - \sigma Q(\cdot - \tilde{z}))\} Q'(\cdot - \tilde{z}) &= 0, \\ |z - \tilde{z}| &\lesssim \|\varphi_1 - \sigma Q(\cdot - z)\|_{H^1} + \|\varphi_2\|_{L^2}. \end{aligned} \quad (5.7)$$

Moreover, $\tilde{z}$ is C^1 with respect to (φ, z) .

- (2) Let $\sigma = \pm 1$. There exists $\delta > 0$ small enough such that for any $\varphi = (\varphi_1, \varphi_2) \in \mathcal{H}_{rad}$ and $z \in \mathbb{R}$ satisfying

$$z > \frac{1}{\delta}, \quad \|\varphi_1 - \sigma(Q(\cdot - z) + Q(\cdot + z))\|_{H^1} + \|\varphi_2\|_{L^2} \leq \delta, \quad (5.8)$$

there exists a unique $\tilde{z} \in \mathbb{R}$ such that

$$\begin{aligned} \int \{\varphi_2 + 2\alpha(\varphi_1 - \sigma(Q(\cdot - \tilde{z}) + Q(\cdot + \tilde{z})))\} Q'(\cdot - \tilde{z}) &= 0, \\ |z - \tilde{z}| &\lesssim \|\varphi_1 - \sigma(Q(\cdot - z) + Q(\cdot + z))\|_{H^1} + \|\varphi_2\|_{L^2}. \end{aligned} \quad (5.9)$$

Moreover, $\tilde{z}$ is C^1 with respect to (φ, z) .

This lemma is based on [25, Lemma 4.1]. However, [25, Lemma 4.1] has complicated notations and does not consider the even case. Thus, we provide the proof.

Proof of Lemma 5.3. First, we prove (1). For $\varphi \in \mathcal{H}$ satisfying (5.6) for some $z \in \mathbb{R}, \sigma = \pm 1$, we define $\Phi_{(1)}[\hat{z}] : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\Phi_{(1)}[\hat{z}] := \int \{\varphi_2 + 2\alpha(\varphi_1 - \sigma Q(\cdot - \hat{z}))\} Q'(\cdot - \hat{z}).$$

Then, $|\Phi_{(1)}[z]| \lesssim \delta$ holds and we have

$$\partial_{\hat{z}} \Phi_{(1)}[\hat{z}] = 2\alpha\sigma \|Q'\|_{L^2}^2 - \int \{\varphi_2 + 2\alpha(\varphi_1 - \sigma Q(\cdot - \hat{z}))\} Q''(\cdot - \hat{z}).$$

By (5.6), we fix $\delta \ll 1$, and obtain $\partial_{\hat{z}} \Phi_{(1)}[z] \neq 0$. Thus by the inverse function theorem, there exists $\tilde{z} \in \mathbb{R}$ such that (5.7).

Next we prove (2). For $\varphi \in \mathcal{H}_{rad}$ satisfying (5.8) for some $z \in \mathbb{R}, \sigma = \pm 1$, we define $\Phi_{(2)}[\hat{z}] : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\Phi_{(2)}[\hat{z}] := \int \{\varphi_2 + 2\alpha(\varphi_1 - \sigma(Q(\cdot - \hat{z}) + Q(\cdot + \hat{z})))\} Q'(\cdot - \hat{z}).$$

Then $|\Phi_{(2)}[z]| \lesssim \delta$ holds and we have

$$\begin{aligned} \partial_{\hat{z}} \Phi_{(2)}[\hat{z}] &= 2\alpha\sigma \left(\|Q'\|_{L^2}^2 - \int Q'(\cdot - \hat{z}) Q'(\cdot + \hat{z}) \right) \\ &\quad - \int \{\varphi_2 + 2\alpha(\varphi_1 - \sigma(Q(\cdot - \hat{z}) + Q(\cdot + \hat{z})))\} Q''(\cdot - \hat{z}). \end{aligned}$$

By Lemma 5.2 and (5.8), by the inverse function theorem, there exists $\tilde{z} \in \mathbb{R}$ such that (5.9). $\square$

By Lemma 5.3, we can rewrite the assumption of (1.8) and (1.9).

Lemma 5.4. (1) When a solution $\vec{u}$ satisfies (1.8) for some $\sigma = \pm 1$ and $z(t)$, there exist a C^1 function $\tilde{z}$ and $T > 0$ such that

$$\lim_{t \rightarrow \infty} (\|u(t) - \sigma Q(\cdot - \tilde{z}(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) = 0,$$

$$\lim_{t \rightarrow \infty} |\tilde{z}(t)| = \infty,$$

$$\int \{\partial_t u(t) + 2\alpha(u(t) - \sigma Q(\cdot - \tilde{z}(t)))\} Q'(\cdot - \tilde{z}(t)) = 0 \text{ for } t > T.$$

Furthermore, we have

$$\lim_{t \rightarrow \infty} |z(t) - \tilde{z}(t)| = 0. \quad (5.10)$$

(2) When an even solution $\vec{u}$ satisfies (1.9) for some $\sigma = \pm 1$ and $z(t)$, there exist a C^1 function $\tilde{z}$ and $T > 0$ such that

$$\lim_{t \rightarrow \infty} (\|u(t) - \sigma(Q(\cdot - \tilde{z}(t)) + Q(\cdot + \tilde{z}(t)))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) = 0,$$

$$\lim_{t \rightarrow \infty} \tilde{z}(t) = \infty,$$

$$\int \{\partial_t u(t) + 2\alpha(u(t) - \sigma(Q(\cdot - \tilde{z}(t)) + Q(\cdot + \tilde{z}(t))))\} Q'(\cdot - \tilde{z}(t)) = 0 \text{ for } t > T.$$

Furthermore, we have

$$\lim_{t \rightarrow \infty} |z(t) - \tilde{z}(t)| = 0. \quad (5.11)$$

Proof. We prove (1). By (1.8), for $\delta > 0$ there exists $T > 0$ such that for $t > T$

$$\|u(t) - \sigma Q(\cdot - z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} < \delta.$$

Then, by Lemma 5.3, there exists the C^1 function $\tilde{z}$ such that for $t > T$

$$\int \{\partial_t u(t) + 2\alpha(u(t) - \sigma Q(\cdot - \tilde{z}(t)))\} Q'(\cdot - \tilde{z}(t)) = 0. \quad (5.12)$$

Furthermore, we have

$$|z(t) - \tilde{z}(t)| \lesssim \|u(t) - \sigma Q(\cdot - z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}.$$

By (1.8), we obtain (5.11), and have

$$\lim_{t \rightarrow \infty} |\tilde{z}(t)| = \infty. \quad (5.13)$$

We also have

$$\|u(t) - \sigma Q(\cdot - \tilde{z}(t))\|_{H^1} \leq \|u(t) - \sigma Q(\cdot - z(t))\|_{H^1} + \|Q(\cdot - \tilde{z}(t)) - Q(\cdot - z(t))\|_{H^1}.$$

We note that since

$$\|Q(\cdot - \tilde{z}(t)) - Q(\cdot - z(t))\|_{H^1} \lesssim |z(t) - \tilde{z}(t)|$$

holds true, we have

$$\|u(t) - \sigma Q(\cdot - \tilde{z}(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} \lesssim \|u(t) - \sigma Q(\cdot - z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} + |z(t) - \tilde{z}(t)|.$$

By (1.8) and (5.11), the right term converges to zero as $t \rightarrow \infty$. Therefore, we have

$$\lim_{t \rightarrow \infty} \|u(t) - \sigma Q(\cdot - \tilde{z}(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} = 0. \quad (5.14)$$

By (5.12), (5.13), (5.14), we complete the proof. we prove (2) using the same argument. $\square$

Furthermore by (1.1) and (2.13), we only need the following proposition to prove Theorem 1.3.

Proposition 5.5. (1) *Let $\gamma < 0$ and $p \geq 2$. If a global solution $\vec{u}$ satisfies*

$$\begin{cases} \lim_{t \rightarrow \infty} (\|u(t) - Q(\cdot - z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) = 0, \\ \lim_{t \rightarrow \infty} z(t) = \infty, \\ \|u(t) - Q(\cdot - z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} \ll 1, \quad z(t) \gg 1, \\ \int \{\partial_t u(t) + 2\alpha(u(t) - Q(\cdot - z(t)))\} Q'(\cdot - z(t)) = 0, \end{cases} \quad (5.15)$$

then we have $z(t) - \frac{1}{2} \log t \lesssim 1$.

(2) *Let $\gamma < 0$ and $p > 2$. If an even global solution $\vec{u}$ satisfies*

$$\begin{cases} \lim_{t \rightarrow \infty} (\|u(t) - Q(\cdot - z(t)) - Q(\cdot + z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) = 0, \\ \lim_{t \rightarrow \infty} z(t) = \infty, \\ \|u(t) - Q(\cdot - z(t)) - Q(\cdot + z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}, \quad z(t) \gg 1, \\ \int \{\partial_t u(t) + 2\alpha(u(t) - Q(\cdot - z(t)) - Q(\cdot + z(t)))\} Q'(\cdot - z(t)) = 0, \end{cases} \quad (5.16)$$

then we have $z(t) - \frac{1}{2} \log t \lesssim 1$.

5.3. Decomposition close to the sum of solitary waves. In this subsection, to prove (1) and (2) of Proposition 5.5 together, we consider certain conditions. First we consider the condition of $\vec{u}$ of (DNKG) such that

$$\begin{cases} \lim_{t \rightarrow \infty} (\|u(t) - Q(\cdot - z(t)) - \sigma Q(\cdot + z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) = 0, \\ \lim_{t \rightarrow \infty} z(t) = \infty, \\ \|u(t) - Q(\cdot - z(t)) - \sigma Q(\cdot + z(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2} \ll 1, \quad z(t) \gg 1, \\ \int \{\partial_t u(t) + 2\alpha(u(t) - Q(\cdot - z(t)) - \sigma Q(\cdot + z(t)))\} Q'(\cdot - z(t)) = 0, \end{cases} \quad (5.17)$$

where $\sigma = 0, 1$. Here, we define certain notations in order to investigate the dynamics of a single solitary wave or an even 2-solitary wave. For any $z \in C^1([0, \infty); \mathbb{R})$, we define $Q_{\pm}$, $\vec{Q}_{\pm}$, and $\phi_{\pm}$ as

$$Q_{\pm}(x) = Q(x \mp z(t)), \quad \vec{Q}_{\pm} = \begin{pmatrix} Q_{\pm} \\ 0 \end{pmatrix}, \quad \phi_{\pm}(x) = \phi(x \mp z(t)).$$

Furthermore, we define $\vec{\varepsilon} = (\varepsilon, \eta)$ as

$$\vec{\varepsilon} = \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \vec{u} - \vec{Q}_+ - \sigma \vec{Q}_-.$$

Then, (DNKG) can be rewritten as

$$\begin{cases} \partial_t \varepsilon = \eta + z'(\partial_x Q_+ - \sigma \partial_x Q_-), \\ \partial_t \eta = - \left\{ -\partial_x^2 \varepsilon + \varepsilon - (pQ_+^{p-1} + \sigma pQ_-^{p-1})\varepsilon \right\} - 2\alpha\eta \\ \quad + \gamma \delta_0 u + (f(u) - f(Q_+) - f(\sigma Q_-) - f'(Q_+)\varepsilon - f'(\sigma Q_-)\varepsilon). \end{cases} \quad (5.18)$$

In addition, we define $\mathcal{L}_t$ and R as

$$\begin{aligned}\mathcal{L}_t &= -\partial_x^2 + 1 - pQ_+^{p-1} - \sigma pQ_-^{p-1}, \\ R &= f(u) - f(Q_+) - f(\sigma Q_-) - f'(Q_+)\varepsilon - f'(\sigma Q_-)\varepsilon.\end{aligned}$$

Subsequently, we rewrite (5.17) and (5.18) as

$$\begin{cases} \lim_{t \rightarrow \infty} \|\vec{\varepsilon}(t)\|_{\mathcal{H}} = 0, \quad \lim_{t \rightarrow \infty} z(t) = \infty, \quad \|\vec{\varepsilon}(t)\|_{\mathcal{H}} \ll 1, \quad z(t) \gg 1, \\ \int (\eta + 2\alpha\varepsilon) \partial_x Q_+ = 0, \end{cases} \quad (5.19)$$

and we have

$$\frac{d}{dt} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathcal{L}_t & -2\alpha \end{pmatrix} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} + \begin{pmatrix} z'(\partial_x Q_+ - \sigma \partial_x Q_-) \\ \gamma \delta_0 u + R \end{pmatrix}. \quad (5.20)$$

Before estimating $\vec{\varepsilon}$, we define θ_1, θ_2 as follows:

$$\theta_1 = \min(p-1, \frac{3}{2}), \quad (5.21)$$

$$\theta_2 = \min(p, 2). \quad (5.22)$$

Furthermore, we define θ_3 as

$$\theta_3 = \begin{cases} \theta_2 & \text{if } 1 < p \leq 2, \\ \frac{1}{1 - \min(p-2, \frac{1}{3})} & \text{if } p > 2. \end{cases} \quad (5.23)$$

Then, we have for $p > 1$,

$$1 < \theta_3 \leq \theta_2. \quad (5.24)$$

These are useful constants to estimate $\vec{\varepsilon}$ and z . Furthermore, we set the condition as

$$\sigma = 0 \quad \text{or} \quad \sigma = 1, \quad p > 2, \quad (5.25)$$

in order to consider a single solitary wave and an even 2-solitary wave.

Remark 5.2. Theorem 1.3 imposes the restriction $p > 2$ and $p \geq 2$. Both restrictions ensure the smoothness of the nonlinear term. On the other hand, when we consider a single solitary wave, the restriction on p is relaxed because we do not consider soliton interactions. More details are provided later, and we will keep the restriction on p as weak as possible when $\sigma = 0$.

5.4. Estimates of the center. In this subsection we start to estimate $\vec{\varepsilon}$. First, we estimate the remainder term R . We define R_1 as

$$R_1 = R - p\sigma Q_+^{p-1} Q_-.$$

Then, we have

$$R = p\sigma Q_+^{p-1} Q_- + R_1. \quad (5.26)$$

Here, we estimate R and R_1 .

Lemma 5.6. *Let $\vec{u}$ be a solution of (DNKG) satisfying (5.19) and (5.25) for some $z(t)$. Then, the following estimates hold:*

$$\left| \int R_1 \partial_x Q_+ \right| \lesssim e^{-2\theta_1 z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2}, \quad (5.27)$$

$$\left| \int R_1 \phi_+ \right| \lesssim e^{-2\theta_1 z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2}, \quad (5.28)$$

$$\left| \int R \partial_x Q_+ \right| \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2}, \quad (5.29)$$

$$\left| \int R \phi_+ \right| \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2}, \quad (5.30)$$

$$\|R\|_{L^2} \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3}. \quad (5.31)$$

Proof. We note that $p\sigma Q_+^{p-1}Q_- = \sigma f'(Q_+)Q_-$. First, R_1 satisfies

$$\begin{aligned} R_1 &= f(u) - f(Q_+) - f(\sigma Q_-) - f'(Q_+)\varepsilon - f'(\sigma Q_-)\varepsilon - \sigma f'(Q_+)Q_- \\ &= f(Q_+ + \sigma Q_- + \varepsilon) - f(Q_+ + \sigma Q_-) - f'(Q_+ + \sigma Q_-)\varepsilon \\ &\quad + (f'(Q_+ + \sigma Q_-) - f'(Q_+))\varepsilon \\ &\quad + f(Q_+ + \sigma Q_-) - f(Q_+) - f(\sigma Q_-) - f'(Q_+)(\sigma Q_-) \\ &\quad - f'(\sigma Q_-)\varepsilon. \end{aligned}$$

In the above term, by Sobolev embedding, we have $\|\varepsilon\|_{L^\infty} \ll 1$ and

$$|f(Q_+ + \sigma Q_- + \varepsilon) - f(Q_+ + \sigma Q_-) - f'(Q_+ + \sigma Q_-)\varepsilon| \lesssim |\varepsilon|^{\theta_2}. \quad (5.32)$$

In addition, when $\sigma = 1$ and $p > 2$, by direct computation we obtain

$$\begin{aligned} |(f'(Q_+ + Q_-) - f'(Q_+))\varepsilon| &\lesssim (|Q_+|^{p-2}|Q_-| + |Q_-|^{p-1})|\varepsilon|, \\ |f(Q_+ + Q_-) - f(Q_+) - f(Q_-) - f'(Q_+)(Q_-)| &\lesssim |Q_+|^{p-2}|Q_-|^2 + |Q_+||Q_-|^{p-1}, \\ |f'(Q_-)\varepsilon| &\lesssim |Q_-|^{p-1}|\varepsilon|. \end{aligned}$$

Therefore by Lemma 5.2 and $e^{-2z}\|\vec{\varepsilon}\|_{\mathcal{H}} \lesssim e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^3$, we obtain (5.27) and (5.28) in the case of $\sigma = 1, p > 2$. Notably, when $\sigma = 0$, $|R_1|$ becomes the left side of (5.32). Therefore (5.27) and (5.28) also hold true. Next, we estimate (5.29) and (5.30). By Lemma 5.2, we have

$$\begin{aligned} \left| \int Q_+^{p-1}Q_- \partial_x Q_+ \right| &\lesssim e^{-2z}, \\ \left| \int Q_+^{p-1}Q_- \phi_+ \right| &\lesssim e^{-2z}. \end{aligned}$$

By (5.26), (5.27), and (5.28), we obtain (5.29) and (5.30).

Finally, we estimate (5.31). When $\sigma = 0$, R becomes $f(u) - f(Q_+) - f'(Q_+)\varepsilon$. Thus, by (5.24) and (5.32), we have (5.31). We assume $\sigma = 1, p > 2$. Then, R satisfies

$$\begin{aligned} R &= f(Q_+ + Q_- + \varepsilon) - f(Q_+ + Q_-) - f'(Q_+ + Q_-)\varepsilon \\ &\quad + f(Q_+ + Q_-) - f(Q_+) - f(Q_-) \\ &\quad + (f'(Q_+ + Q_-) - f'(Q_+) - f'(Q_-))\varepsilon. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |f(Q_+ + Q_- + \varepsilon) - f(Q_+ + Q_-) - f'(Q_+ + Q_-)| &\lesssim |\varepsilon|^2, \\ |f(Q_+ + Q_-) - f(Q_+) - f(Q_-)| &\lesssim Q_+^{p-1}Q_- + Q_+Q_-^{p-1}, \\ |(f'(Q_+ + Q_-) - f'(Q_+) - f'(Q_-))\varepsilon| &\lesssim (Q_+^{p-2}Q_- + Q_+Q_-^{p-2})|\varepsilon|. \end{aligned}$$

By Lemma 5.2, we have

$$\|Q_+^{p-1}Q_- + Q_+Q_-^{p-1}\|_{L^2} \lesssim e^{-2z}.$$

Therefore, we have

$$\|(Q_+^{p-2}Q_- + Q_+Q_-^{p-2})\varepsilon\|_{L^2} \lesssim e^{-2(\min(p-2, \frac{1}{3}))z} \|\varepsilon\|_{\mathcal{H}} \lesssim e^{-2z} + \|\varepsilon\|_{\mathcal{H}}^{\theta_3},$$

and obtain (5.31). $\square$

Next, we estimate the distance between the origin and the center of the solitary wave.

Proposition 5.7. *Let $\vec{u}$ be a global solution of (DNKG) satisfying (5.19) and (5.25) for some $z(t)$. Then, we have*

$$|z'| \lesssim e^{-2z} + \|\varepsilon\|_{\mathcal{H}}^{\theta_2}. \quad (5.33)$$

Proof. Differentiating the second equation of (5.19), we obtain

$$\begin{aligned} 0 &= \int (\partial_t \eta + 2\alpha \partial_t \varepsilon) \partial_x Q_+ - z' \int (\eta + 2\alpha \varepsilon) \partial_x^2 Q_+ \\ &= z' \left(2\alpha \|Q'\|_{L^2}^2 - 2\alpha \sigma \int \partial_x Q_- \partial_x Q_+ - \int (\eta + 2\alpha \varepsilon) \partial_x^2 Q_+ \right) \\ &\quad + \gamma \int \delta_0(u \cdot \partial_x Q_+) + \sigma p \int Q_-^{p-1} \partial_x Q_+ \varepsilon + \int R \partial_x Q_+. \end{aligned}$$

By Lemma 5.2 and (5.29), we have

$$\begin{aligned} 0 &= z' \left(2\alpha \|Q'\|_{L^2}^2 + O(e^{-\frac{3}{2}z} + \|\varepsilon\|_{\mathcal{H}}) \right) \\ &\quad + O(\|\varepsilon\|_{\mathcal{H}}^{\theta_2} + e^{-2z}). \end{aligned}$$

Therefore, we obtain (5.33). $\square$

Remark 5.3. To prove Proposition 5.5, (5.33) is a sufficient estimate. However, when we compute z' more precisely by using (1.4) and (5.4), and

$$\begin{aligned} \int R \partial_x Q_+ &= \sigma \int \frac{d}{dx} (Q_+^p) Q_- + \int R_1 \partial_x Q_+ \\ &= 2\sigma c_Q^2 e^{-2z} + \int R_1 \partial_x Q_+ + O(e^{-(p+1)z}), \end{aligned}$$

we have

$$|2\alpha \|Q'\|_{L^2}^2 z' + \gamma c_Q e^{-z} u(t, 0) + 2\sigma c_Q^2 e^{-2z} + R_2| \lesssim e^{-\theta z} + \|\varepsilon\|_{\mathcal{H}}^3,$$

where $R_2 = \int R_1 \partial_x Q_+$ and $\theta = \min(p+1, 3)$. Furthermore, by direct computation, we have

$$|2\alpha \|Q'\|_{L^2}^2 z' + \{\gamma(1+\sigma) + 2\sigma\} c_Q^2 e^{-2z} + \gamma c_Q e^{-z} \varepsilon(t, 0) + R_2| \lesssim e^{-\theta z} + \|\varepsilon\|_{\mathcal{H}}^3.$$

In this paper, we can not prove whether ε decays better because of the delta potential. However, we conjecture that if $\sigma = 1, \gamma < -1$ or $\sigma = 0, \gamma < 0$, z satisfies $z \sim \frac{1}{2} \log t$.

Next we define certain notations to extract eigenmodes with real eigenvalues around the solitary wave. We define a_+, a_-, a_0 as

$$\begin{aligned} a_{\pm}(t) &= \int (\eta - \nu^{\mp} \varepsilon) \phi_{\pm}, \\ a_0(t) &= \int \eta \partial_x Q_+. \end{aligned}$$

Proposition 5.8. *Let $\vec{u}$ be a solution of (DNKG) satisfying (5.19) and (5.25) for some $z(t)$. Then, we have*

$$\left| \frac{d}{dt} a_{\pm} - \nu^{\pm} a_{\pm} \right| \lesssim e^{-2z} + \|\varepsilon\|_{\mathcal{H}}^{\theta_2}, \quad (5.34)$$

$$\left| \frac{d}{dt} a_0 + 2\alpha a_0 \right| \lesssim e^{-2z} + \|\varepsilon\|_{\mathcal{H}}^{\theta_2}. \quad (5.35)$$

Proof. As the proof of Proposition 5.7, we compute $a_{\pm}$ and a_0 . First, we have

$$\frac{d}{dt} a_{\pm} = \int (\partial_t \eta - \nu^{\mp} \partial_t \varepsilon) \phi_{\pm} - z' \int (\eta - \nu^{\mp} \varepsilon) \partial_x \phi_{\pm}. \quad (5.36)$$

We note that

$$\begin{aligned} \partial_t \eta + \nu^{\mp} \partial_t \varepsilon &= -(-\partial_x^2 \varepsilon + \varepsilon - pQ_+^{p-1} \varepsilon) - (2\alpha + \nu^{\mp}) \eta \\ &\quad + \sigma p Q_-^{p-1} \varepsilon + \gamma \delta_0 u + R(t) - \nu^{\mp} z' (\partial_x Q_+ - \sigma \partial_x Q_-). \end{aligned} \quad (5.37)$$

By (5.30), (5.33), (5.36), and (5.37), we obtain

$$\frac{d}{dt} a_{\pm} = \int \left\{ -(-\partial_x^2 \varepsilon + \varepsilon - pQ_+^{p-1} \varepsilon) - (2\alpha + \nu^{\mp}) \eta \right\} \phi_{\pm} + O(\|\varepsilon\|_{\mathcal{H}}^{\theta_2} + e^{-2z}),$$

and

$$\begin{aligned} &\int \left\{ -(-\partial_x^2 \varepsilon + \varepsilon - pQ_+^{p-1} \varepsilon) - (2\alpha + \nu^{\mp}) \eta \right\} \phi_{\pm} \\ &= \int \varepsilon \cdot \left(-\partial_x^2 \phi_{\pm} + \phi_{\pm} - pQ_+^{p-1} \phi_{\pm} \right) - (2\alpha + \nu^{\mp}) \phi_{\pm} \\ &= \int (\nu^2 \varepsilon + \nu^{\pm} \eta) \phi_{\pm} \\ &= \nu^{\pm} a_{\pm}. \end{aligned}$$

Therefore we obtain (5.34). Second we have

$$\frac{d}{dt} a_0 = -2\alpha \int \eta \partial_x Q_+ + \int (p\sigma Q_-^{p-1} \varepsilon + \gamma \delta_0 u + R) \partial_x Q_+.$$

By using the same argument as in the proof of Proposition 5.7, we have

$$\left| \int (p\sigma Q_-^{p-1} \varepsilon + \gamma \delta_0 u + R) \partial_x Q_+ \right| \lesssim e^{-2z} + \|\varepsilon\|_{\mathcal{H}}^{\theta_2}.$$

Thus, we obtain (5.35). □

5.5. Energy estimates. In this subsection, we assume that $\vec{u}$ is a solution of (DNKG) satisfying (5.19) and (5.25) for some $z(t)$.

For $\mu > 0$ small to be chosen, we denote $\rho = 2\alpha - \mu$. Furthermore, we define $\mathcal{E}$ as

$$\mathcal{E}(t) = \frac{1}{2} \int \{(\partial_x \varepsilon)^2 + (1 - \rho\mu)\varepsilon^2 + (\eta + \mu\varepsilon)^2 - (pQ_+^{p-1} + p\sigma Q_-^{p-1})\varepsilon^2\} - \frac{\gamma}{2} |u(t, 0)|^2. \quad (5.38)$$

Lemma 5.9. *Let $\vec{u}$ be a solution of (DNKG) satisfying (5.19) and (5.25) for some $z(t)$. Then, for $t > 0$,*

$$\mathcal{E}' + 2\mu\mathcal{E} + \mu\gamma c_Q(1 + \sigma)e^{-z}\varepsilon(t, 0) \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}. \quad (5.39)$$

Proof. By a formal calculation, we obtain

$$\begin{aligned} \mathcal{E}'(t) &= \int \partial_x \varepsilon \partial_x \partial_t \varepsilon + (1 - \rho\mu)\varepsilon \partial_t \varepsilon + (\eta + \mu\varepsilon)(\partial_t \eta + \mu \partial_t \varepsilon) - (pQ_+^{p-1} + p\sigma Q_-^{p-1})\varepsilon \partial_t \varepsilon \\ &\quad + \frac{p-1}{2} z' \left\{ \int (Q_+^{p-2} \partial_x Q_+ - \sigma Q_-^{p-2} \partial_x Q_-) \varepsilon^2 \right\} - \gamma u(t, 0) \partial_t u(t, 0). \end{aligned}$$

Notably, if the right term does not have a singular term, $\mathcal{E}$ becomes a differentiable function. By (5.18) and integration by parts, we have

$$\begin{aligned} &\int \partial_x \varepsilon \partial_x \partial_t \varepsilon + (1 - \rho\mu)\varepsilon \partial_t \varepsilon + (\eta + \mu\varepsilon)(\partial_t \eta + \mu \partial_t \varepsilon) - (pQ_+^{p-1} + p\sigma Q_-^{p-1})\varepsilon \partial_t \varepsilon \\ &= \int (-\partial_x^2 \varepsilon + (1 - \rho\mu)\varepsilon + \mu(\eta + \mu\varepsilon) - (pQ_+^{p-1} + p\sigma Q_-^{p-1})\varepsilon) \partial_t \varepsilon + (\eta + \mu\varepsilon) \partial_t \eta, \end{aligned}$$

and

$$\begin{aligned} &\int (-\partial_x^2 \varepsilon + (1 - \rho\mu)\varepsilon + \mu(\eta + \mu\varepsilon) - (pQ_+^{p-1} + p\sigma Q_-^{p-1})\varepsilon) \partial_t \varepsilon \\ &= \int (\mathcal{L}_t \varepsilon - \mu(\rho - \mu)\varepsilon + \mu\eta) \partial_t \varepsilon, \end{aligned}$$

and

$$\int (\eta + \mu\varepsilon) \partial_t \eta = \int (\eta + \mu\varepsilon) (-\mathcal{L}_t \varepsilon - 2\alpha\eta + \gamma\delta_0 u + R).$$

By (5.24), (5.18), (5.31) and (5.33), we obtain

$$\begin{aligned} &\int (\mathcal{L}_t \varepsilon - \mu(\rho - \mu)\varepsilon + \mu\eta) \partial_t \varepsilon = \int (\mathcal{L}_t \varepsilon - \mu(\rho - \mu)\varepsilon + \mu\eta) \eta + O(e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}), \\ &\int (\eta + \mu\varepsilon) (\gamma\delta_0 u + R) = \gamma \partial_t u(t, 0) u(t, 0) + \mu\gamma \varepsilon(t, 0) u(t, 0) + O(e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}). \end{aligned}$$

By combining these estimates, we obtain

$$\begin{aligned} \mathcal{E}'(t) &= \int [\{(\mathcal{L}_t \varepsilon - \mu(\rho - \mu)\varepsilon + \mu\eta)\eta\} + (\eta + \mu\varepsilon)(-\mathcal{L}_t \varepsilon - 2\alpha\eta)] \\ &\quad + \mu\gamma \varepsilon(t, 0) u(t, 0) + O(e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}) \\ &= -2\mu\mathcal{E}(t) - (\rho - \mu)\|\eta + \mu\varepsilon\|_{L^2}^2 + \mu\gamma(\varepsilon(t, 0) - u(t, 0))u(t, 0) + O(e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}) \\ &= -2\mu\mathcal{E}(t) - (\rho - \mu)\|\eta + \mu\varepsilon\|_{L^2}^2 - \mu\gamma c_Q(1 + \sigma)e^{-z}\varepsilon(t, 0) + O(e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}), \end{aligned}$$

and complete the proof using this estimate. $\square$

Next we define

$$b(t) = \left(\int \varepsilon \partial_x Q_- \right)^2 + \left(\int \varepsilon \phi_- \right)^2.$$

We then establish the coercivity and some uniform boundness.

Lemma 5.10. *Let $\vec{u}$ be a solution of (DNKG) satisfying (5.19) and (5.25) for some $z(t)$. Then there exists $C > 0$ such that for all $t \geq 0$,*

$$C \{a_+(t)^2 + a_-(t)^2 + a_0(t)^2 + \sigma b(t)\} + \mathcal{E}(t) + \frac{\gamma}{2} |u(t, 0)|^2 \geq \frac{1}{C} \|\vec{\varepsilon}(t)\|_{\mathcal{H}}^2. \quad (5.40)$$

Furthermore if $\gamma < 0$, we have

$$\begin{aligned} & \mathcal{E} + C(a_+^2 + a_-^2 + a_0^2 + \sigma b) + e^{-2z} \\ & \lesssim (E_\gamma(\vec{u}) - (1 + \sigma)J_0(Q)) + (a_+^2 + a_-^2 + a_0^2 + \sigma b^2) + e^{-2z}. \end{aligned} \quad (5.41)$$

Remark 5.4. C is independent of $\mu > 0$.

For further details, you can refer to [9, Lemma 2.6].

Proof. First we note that

$$\langle \varepsilon, \phi_+ \rangle = \frac{a_+ - a_-}{\nu^+ - \nu^-}, \quad \langle \varepsilon, \partial_x Q_+ \rangle = -\frac{a_0}{2\alpha}$$

hold true. Therefore, we have

$$\langle \varepsilon, \phi_+ \rangle^2 + \langle \varepsilon, \partial_x Q_+ \rangle^2 \lesssim a_+^2 + a_-^2 + a_0^2. \quad (5.42)$$

First, we consider $\sigma = 0$. Then, by Lemma 5.1 there exists $C > 0$ such that (5.40) is satisfied since $\mu > 0$ is sufficiently small.

Next, we consider $\sigma = 1$. Let χ be a smooth function satisfying the following properties:

$$\chi = 1 \text{ on } [0, 1], \quad \chi = 0 \text{ on } [2, \infty), \quad \chi' \leq 0 \text{ on } \mathbb{R}.$$

For $\lambda = \frac{|z|}{2} \gg 1$, let

$$\chi_+(x) = \chi\left(\frac{|x - z|}{\lambda}\right), \quad \chi_-(x) = (1 - \chi_+^2(x))^{\frac{1}{2}}.$$

Furthermore, we define $\varepsilon_\pm = \varepsilon \cdot \chi_\pm$. Then, we have

$$\varepsilon^2 = \varepsilon_+^2 + \varepsilon_-^2. \quad (5.43)$$

In addition, we have

$$\|\varepsilon_+\|_{L^2}^2 + \|\varepsilon_-\|_{L^2}^2 = \|\varepsilon\|_{L^2}^2, \quad (5.44)$$

$$\|\partial_x \varepsilon\|_{L^2}^2 = \|\partial_x \varepsilon_+\|_{L^2}^2 + \|\partial_x \varepsilon_-\|_{L^2}^2 + O\left(\frac{\|\varepsilon\|_{H^1}^2}{|z|}\right), \quad (5.45)$$

$$|\langle \varepsilon - \varepsilon_-, \partial_x Q_- \rangle| + |\langle \varepsilon - \varepsilon_-, \phi_- \rangle| \lesssim e^{-z} \|\varepsilon\|_{L^2}, \quad (5.46)$$

$$|\langle \varepsilon - \varepsilon_+, \partial_x Q_+ \rangle| + |\langle \varepsilon - \varepsilon_+, \phi_+ \rangle| \lesssim e^{-\frac{z}{2}} \|\varepsilon\|_{L^2}. \quad (5.47)$$

By Lemma 5.1, we have

$$\langle \mathcal{L}_t \varepsilon_+, \varepsilon_+ \rangle + \langle pQ_-^{p-1}, \varepsilon_+^2 \rangle \geq c \|\varepsilon_+\|_{H^1}^2 - \frac{1}{c} (\langle \varepsilon_+, \phi_+ \rangle^2 + \langle \varepsilon_+, \partial_x Q_+ \rangle^2), \quad (5.48)$$

$$\langle \mathcal{L}_t \varepsilon_-, \varepsilon_- \rangle + \langle pQ_+^{p-1}, \varepsilon_-^2 \rangle \geq c \|\varepsilon_-\|_{H^1}^2 - \frac{1}{c} (\langle \varepsilon_-, \phi_- \rangle^2 + \langle \varepsilon_-, \partial_x Q_- \rangle^2). \quad (5.49)$$

Furthermore, we have

$$|\langle pQ_-^{p-1}, \varepsilon_+^2 \rangle| + |\langle pQ_+^{p-1}, \varepsilon_-^2 \rangle| \lesssim e^{-\frac{z}{2}} \|\varepsilon\|_{L^2}^2. \quad (5.50)$$

Notably, by (5.43) we have

$$\langle \mathcal{L}_t \varepsilon, \varepsilon \rangle = \langle \mathcal{L}_t \varepsilon_+, \varepsilon_+ \rangle + \langle \mathcal{L}_t \varepsilon_-, \varepsilon_- \rangle + O\left(\frac{\|\varepsilon\|_{H^1}^2}{|z|}\right). \quad (5.51)$$

Based on (5.44)-(5.51), we obtain

$$\langle \mathcal{L}_t \varepsilon, \varepsilon \rangle \geq \frac{c}{2} \|\varepsilon\|_{H^1}^2 - \frac{1}{c} \sum_{*=\pm} (\langle \varepsilon, \partial_x Q_* \rangle^2 + \langle \varepsilon, \phi_* \rangle^2).$$

Furthermore by (5.42) and $\mu > 0$ small enough, there exists $C > 0$ such that (5.40) is satisfied.

Next, we estimate (5.41). We denote $E = J_0(Q)$. By direct computation, we obtain

$$\begin{aligned} E_\gamma(\vec{u}) &= (1 + \sigma)E + \sigma \int Q_+^p Q_- + \|\vec{\varepsilon}\|_{\mathcal{H}} - \frac{\gamma}{2} |u(t, 0)|^2 \\ &\quad - \frac{1}{p+1} \int (|Q_+ + \sigma Q_- + \varepsilon|^{p+1} - |Q_+ + \sigma Q_-|^{p+1} - (p+1)(Q_+ + \sigma Q_-)^p \varepsilon) \\ &\quad - \int \{(Q_+ + \sigma Q_-)^p - Q_+^p - \sigma Q_-^p\} \varepsilon \\ &\quad - \frac{1}{p+1} \int (|Q_+ + \sigma Q_-|^{p+1} - |Q_+|^{p+1} - |\sigma Q_-|^{p+1}). \end{aligned}$$

We note that

$$\begin{aligned} &|Q_+ + \sigma Q_- + \varepsilon|^{p+1} - |Q_+ + \sigma Q_-|^{p+1} - (p+1)(Q_+ + \sigma Q_-)^p \varepsilon \\ &= \frac{(p+1)p}{2} (Q_+ + \sigma Q_-)^{p-1} \varepsilon^2 + O(\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1}). \end{aligned}$$

We consider $\sigma = 0$. Then, we have

$$E_\gamma(\vec{u}) = E + \frac{1}{2} \langle \mathcal{L}_t \varepsilon, \varepsilon \rangle + \frac{1}{2} \|\eta\|_{L^2}^2 - \frac{\gamma}{2} |u(t, 0)|^2 + O(\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1}).$$

Next, we consider $\sigma = 1$ and $p > 2$. Then, we have

$$\begin{aligned} &||Q_+ + Q_-|^{p+1} - |Q_+|^{p+1} - |Q_-|^{p+1}| \lesssim Q_+^p Q_- + Q_+ Q_-^p, \\ &|\{(Q_+ + Q_-)^p - Q_+^p - Q_-^p\} \varepsilon| \lesssim (Q_+^{p-1} Q_- + Q_+ Q_-^{p-1}) |\varepsilon| \lesssim e^{-2z} |\varepsilon|. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} E_\gamma(\vec{u}) &= 2E + \frac{1}{2} \langle \mathcal{L}_t \varepsilon, \varepsilon \rangle + \frac{1}{2} \|\eta\|_{L^2}^2 - \frac{\gamma}{2} |u(t, 0)|^2 \\ &\quad - p \int \{(Q_+ + Q_-)^{p-1} - Q_+^{p-1} - Q_-^{p-1}\} |\varepsilon|^2 + O(e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1}). \end{aligned}$$

Furthermore by Lemma 5.2, we have

$$\int \{(Q_+ + Q_-)^{p-1} - Q_+^{p-1} - Q_-^{p-1}\} |\varepsilon|^2 \lesssim e^{-2 \min(p-2, \frac{1}{3})z} \|\vec{\varepsilon}\|_{\mathcal{H}}^2 \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{2\theta_3}.$$

Thus, we obtain

$$E_\gamma(\vec{u}) = 2E + \frac{1}{2} \langle \mathcal{L}_t \varepsilon, \varepsilon \rangle + \frac{1}{2} \|\eta\|_{L^2}^2 - \frac{\gamma}{2} |u(t, 0)|^2 + O(\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_4} + e^{-2z}),$$

where $\theta_4 = \min(\theta_2 + 1, 2\theta_3) > 2$. Based on these arguments, we have

$$E_\gamma(\vec{u}) = (1 + \sigma)E + \frac{1}{2}\langle \mathcal{L}_t \varepsilon, \varepsilon \rangle + \frac{1}{2}\|\eta\|_{L^2}^2 - \frac{\gamma}{2}|u(t, 0)|^2 + O(\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_4} + e^{-2z}). \quad (5.52)$$

By the definition of (5.38), we have

$$E_\gamma(\vec{u}) - (1 + \sigma)E = \mathcal{E} + \frac{\mu}{2}\{(\rho - \mu)\|\varepsilon\|_{L^2}^2 - 2\langle \varepsilon, \eta \rangle\} + O(\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_4} + e^{-2z}).$$

We recall that $\mu > 0$ is sufficiently small and that γ is negative. By (5.40), we have

$$E_\gamma(\vec{u}) - (1 + \sigma)E \geq \frac{1}{2}\mathcal{E} - \frac{C}{2}(a_+^2 + a_-^2 + a_0^2 + \sigma b^2) - C'e^{-2z} \quad (5.53)$$

for some $C' > 0$. By (5.53), we obtain (5.41). $\square$

To ensure that $\vec{u}$ is even when $\sigma = 1$, we refine the condition (5.25) as follows:

$$\sigma = 0 \quad \text{or} \quad \sigma = 1 \text{ and } p > 2 \text{ and } \vec{u} \text{ is even.} \quad (5.54)$$

Notably, an even solution $\vec{u}$ satisfies (5.19) for some $z(t)$ and $\sigma = 1$, $\vec{\varepsilon}$ becomes to be even, and we have

$$\langle \varepsilon, \partial_x Q_- \rangle = -\langle \varepsilon, \partial_x Q_+ \rangle, \quad \langle \varepsilon, \phi_+ \rangle = \langle \varepsilon, \phi_- \rangle.$$

Thus we refine Lemma 5.10 as follows:

Corollary 5.11. *We assume that $\vec{u}$ satisfies (5.19) and (5.54) for some $z(t)$. Then there exists $C > 0$ such that*

$$C\{a_+(t)^2 + a_-(t)^2 + a_0(t)^2\} + \mathcal{E}(t) + \frac{\gamma}{2}|u(t, 0)|^2 \gtrsim \|\vec{\varepsilon}\|_{\mathcal{H}}^2. \quad (5.55)$$

Furthermore if $\gamma < 0$, we have

$$\begin{aligned} & \mathcal{E} + C(a_+^2 + a_-^2 + a_0^2) + e^{-2z} \\ & \lesssim \{(E_\gamma(\vec{u}) - (1 + \sigma)J_0(Q)) + (a_+^2 + a_-^2 + a_0^2) + e^{-2z}\}. \end{aligned}$$

Let L large to be chosen. We define $\mathcal{G}$ as

$$\mathcal{G}(t) = \mathcal{E}(t) + L(a_+^2(t) + a_0^2(t)). \quad (5.56)$$

Notably, $\mathcal{G}$ is the size of $\|\vec{\varepsilon}\|_{\mathcal{H}}$ except the unstable mode a_+ . Here, we estimate $\mathcal{G}$.

Lemma 5.12. *Let $\gamma < 0$. We assume that $\vec{u}$ satisfies (5.19) and (5.54) for some $z(t)$. Then the followings hold:*

(1) *There exists $C > 0$ such that*

$$\frac{1}{C}\|\vec{\varepsilon}\|_{\mathcal{H}}^2 \leq \mathcal{G} + La_+^2 \leq C\{(E_\gamma(\vec{u}) - (1 + \sigma)J_0(Q)) + a_+^2 + a_-^2 + a_0^2 + e^{-2z}\}. \quad (5.57)$$

In particular, we have

$$|\mathcal{G}| \lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}. \quad (5.58)$$

(2) *There exists $\delta > 0$ such that*

$$\mathcal{G}' \leq -\delta\mathcal{G} + \frac{1}{\delta}a_+^2 + \frac{1}{\delta}\left(e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}\right). \quad (5.59)$$

Proof. We prove (1). If $\gamma < 0$ and $\vec{u}$ satisfies (5.19), (5.54) for some $z(t)$, then by Corollary 5.11, we obtain (5.57) for some $C > 0$. Furthermore by (5.56) and (5.57), we obtain (5.58).

Next, we prove (2). By Proposition 5.8 and $|a_0| + |a_+| + |a_-| \lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}$, we have

$$\left| \frac{d}{dt}(a_{\pm})^2 - 2\nu^{\pm}(a_{\pm})^2 \right| \lesssim |a_{\pm}|(e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2}) \lesssim e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1}, \quad (5.60)$$

$$\left| \frac{d}{dt}(a_0)^2 + 4\alpha(a_0)^2 \right| \lesssim e^{-3z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1}. \quad (5.61)$$

By (5.56), we have

$$\mathcal{G}' = \mathcal{E}' + L((a_0^2)' + (a_-^2)').$$

Furthermore by (5.24), (5.39), (5.60), and (5.61), we obtain

$$\mathcal{G}' + 2\mu\mathcal{E} + \mu\gamma c_Q(1 + \sigma)e^{-z}\varepsilon(t, 0) - 2L\nu^-a_-^2 + 4\alpha La_0^2 \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}. \quad (5.62)$$

Here, we choose $\delta_1 > 0$ small so that

$$2(\mu - \delta_1)\mathcal{E}(t) + 2\mu L(a_-^2(t) + a_+^2(t) + a_0^2(t)) + \frac{\gamma}{2}|u(t, 0)|^2 \geq \delta_1|\varepsilon(t, 0)|^2, \\ \nu^- + \mu < -\delta_1, \quad -2\alpha + \mu < -\delta_1.$$

Notably, the above estimate holds true by (5.55). Then, we have

$$2\mu\mathcal{E} + \mu\gamma c_Q(1 + \sigma)e^{-z}\varepsilon(t, 0) - 2L\nu^-a_-^2 + 4\alpha La_0^2 \\ \geq 2\delta_1\mathcal{G} - 2\mu La_+^2 + \delta_1|\varepsilon(t, 0)|^2 + \mu\gamma c_Q(1 + \sigma)e^{-z}\varepsilon(t, 0) \\ \geq 2\delta_1\mathcal{G} - 2\mu La_+^2 - \frac{\mu^2\gamma^2 c_Q^2(1 + \sigma)^2}{4\delta_1}e^{-2z}.$$

Thus, by (5.62), there exists $C_1 > 0$ such that

$$\mathcal{G}' \leq -2\delta_1\mathcal{G} + 2\mu La_+^2 + \frac{\mu^2\gamma^2 c_Q^2(1 + \sigma)^2}{4\delta_1}e^{-2z} + C_1(e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}).$$

Therefore by choosing

$$\delta = \min\left(2\delta_1, \frac{1}{2\mu L}, \frac{1}{C_1 + \frac{\mu^2\gamma^2 c_Q^2(1 + \sigma)^2}{4\delta_1}}\right),$$

we obtain (5.59). $\square$

5.6. Estimate of the instability term. In this subsection, we estimate a_+ .

Proposition 5.13. *Let $\gamma < 0$. We assume that $\vec{u}$ satisfies (5.19) and (5.54) for some $z(t)$. Then for any $M > 0$, there exists $T_M > 0$ such that for all $t \geq T_M$*

$$|a_+| \leq \frac{1}{M}(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z})^{\frac{1}{2}}. \quad (5.63)$$

Proof. First, according to Lemma 5.12, there exists $C > 0$ such that

$$\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z} \leq C\{(E_{\gamma}(\vec{u}) - (1 + \sigma)J_0(Q)) + a_+^2 + a_-^2 + a_0^2 + e^{-2z}\}. \quad (5.64)$$

We fix $M > 0$. Let $\delta_M > 0$ be sufficiently small. By (5.19), there exists $T_M > 0$ such that for $t \geq T_M$

$$e^{-z(t)} + \|\vec{\varepsilon}(t)\|_{\mathcal{H}} + |a_+(t)| < \delta_M. \quad (5.65)$$

We assume that $T_1 > T_M$ satisfies

$$\frac{1}{M} \left(\|\vec{\varepsilon}(T_1)\|_{\mathcal{H}}^2 + e^{-2z(T_1)} \right)^{\frac{1}{2}} < |a_+(T_1)|.$$

We then introduce the following bootstrap estimate

$$\frac{1}{M_1} \left(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z} \right)^{\frac{1}{2}} \leq |a_+| \leq \delta_M, \quad (5.66)$$

where M_1 satisfies

$$(M_1^2 - C)\nu^+ - 2C > 0, \quad M_1 > M.$$

We define $T_2 \geq T_1$ as

$$T_2 = \sup \{t \in [T_1, \infty) \text{ such that holds (5.66) on } [T_1, t]\}.$$

Since δ_M is sufficiently small, by (5.60) a_+ satisfies for $T_1 \leq t \leq T_2$

$$\frac{d}{dt} |a_+|^2 \geq \nu^+ |a_+|^2. \quad (5.67)$$

Furthermore by (5.33), (5.60), and (5.61) we have

$$\begin{aligned} \frac{d}{dt} a_0^2 &\lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1} + e^{-3z}, \\ \frac{d}{dt} a_-^2 &\lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2+1} + e^{-3z}, \\ |z'| &\lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_2} + e^{-2z}. \end{aligned}$$

Since δ_M is sufficiently small, we have

$$\frac{d}{dt} \{ (a_0)^2 + (a_-^2) \} \leq a_+^2, \quad (5.68)$$

$$\left| \frac{d}{dt} e^{-2z} \right| \leq a_+^2. \quad (5.69)$$

Here, we define

$$\mathcal{I} = M_1^2 a_+^2 - C (E_\gamma(\vec{u}) - (1 + \sigma)J_0(Q) + a_+^2 + a_-^2 + a_0^2 + e^{-2z}).$$

Then by (1.3), (5.68), and (5.69) we have

$$\mathcal{I}' \geq \{ (M_1^2 - C)\nu^+ - 2C \} a_+^2 > 0.$$

Therefore as long as (5.66) holds, we have

$$C (E_\gamma(\vec{u}) - (1 + \sigma)J_0(Q) + a_+^2 + a_-^2 + a_0^2 + e^{-2z}) < M_1^2 a_+^2.$$

Thus by (5.64), we obtain

$$\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z} < M_1^2 a_+^2. \quad (5.70)$$

This implies that as long as (5.66) holds, we have (5.70). Therefore for $T_1 \leq t \leq T_2$, (5.70) holds, and a_+^2 increases exponentially. Thus $a_+(T_3) = \delta_M$ holds for some $T_3 > 0$, which contradicts (5.65). Therefore we have for all $t \geq T_M$,

$$|a_+| \leq \frac{1}{M} \left(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z} \right)^{\frac{1}{2}}.$$

□

5.7. Proof of Theorem 1.3. In this subsection we prove Theorem 1.3. First we organize the estimates of $\mathcal{G}$. Gathering (5.57)-(5.63), we obtain the following lemma:

Lemma 5.14. *Let $\gamma < 0$. We assume that $\vec{u}$ satisfies (5.19) and (5.54) for some $z(t)$. Then, there exist $T > 0$, $C > 0$, $c > 0$, $\delta > 0$ such that for $t \geq T$*

$$c(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}) \leq \mathcal{G} + e^{-2z} \leq C(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}), \quad (5.71)$$

$$\mathcal{G}' \leq -\delta\mathcal{G} + \frac{1}{\delta}e^{-2z}. \quad (5.72)$$

Proof. Notably, $\mathcal{G} + e^{-2z} \lesssim \|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}$ holds true obviously. Thus we show $\mathcal{G} + e^{-2z} \gtrsim \|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}$. By (1) of Lemma 5.12, there exists $C_1 > 0$ such that

$$\frac{1}{C_1}\|\vec{\varepsilon}\|_{\mathcal{H}}^2 \leq \mathcal{G} + La_+^2, \quad (5.73)$$

By Proposition 5.13, there exists $T_1 > 0$ such that for $t > T_1$

$$a_+^2 \leq \frac{1}{2C_1L}(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}). \quad (5.74)$$

By (5.73) and (5.74), we have for $t > T_1$

$$\frac{1}{C_1}\|\vec{\varepsilon}\|_{\mathcal{H}}^2 \leq \mathcal{G} + \frac{1}{2C_1}(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}),$$

and obtain

$$\mathcal{G} + \frac{1}{C_1}e^{-2z} \geq \frac{1}{2C_1}(\|\vec{\varepsilon}\|_{\mathcal{H}}^2 + e^{-2z}).$$

Thus, there exist $T > 0$, $C > 0$, $c > 0$ such that (5.71) is satisfied.

Next we prove (5.72). By (2) of Lemma 5.12, there exists $\delta_1 > 0$ such that

$$\begin{aligned} \mathcal{G}' &\leq -\delta_1\mathcal{G} + \frac{1}{\delta_1}a_+^2 + \frac{1}{\delta_1}\left(e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}\right) \\ &= -\frac{\delta_1}{2}\mathcal{G} + \frac{1}{\delta_1}e^{-2z} - \left(\frac{\delta_1}{2}\mathcal{G} - \frac{1}{\delta_1}a_+^2 - \frac{1}{\delta_1}\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1}\right). \end{aligned}$$

By $\|\vec{\varepsilon}\|_{\mathcal{H}} \ll 1$ and Proposition 5.13 and (5.71), there exist $T_2 > T_1$ and $C_2 > 0$ such that for $t \geq T_2$, we have

$$\frac{\delta_1}{2}\mathcal{G} - \frac{1}{\delta_1}a_+^2 - \frac{1}{\delta_1}\|\vec{\varepsilon}\|_{\mathcal{H}}^{\theta_3+1} > -C_2e^{-2z}.$$

Thus, we define $T = T_2$ and $\delta = \min(\frac{\delta_1}{2}, \frac{\delta_1}{C_2\delta_1+1})$. Then we obtain (5.72). Therefore, we complete the proof. $\square$

Here, we prove Proposition 5.5.

Proof of Proposition 5.5. First, by Lemma 5.14, there exist $\delta > 0$ and $T' > 0$ such that for $t > T'$ (5.72) is satisfied.

Since we assume $p \geq 2$, (5.33) is rewritten as

$$|z'| \lesssim e^{-2z} + \|\vec{\varepsilon}\|_{\mathcal{H}}^2. \quad (5.75)$$

By (5.71) and (5.75), there exists $\hat{C} > 0$ such that

$$z' \leq \hat{C}(\mathcal{G} + e^{-2z}). \quad (5.76)$$

We define $w(t) = e^{2z(t)}$. Then, by using (5.76) we have

$$w' \leq 2\hat{C}\mathcal{G}w + 2\hat{C}. \quad (5.77)$$

Since $\lim_{t \rightarrow \infty} w(t) = \infty$ and $\lim_{t \rightarrow \infty} \mathcal{G}(t) = 0$ hold true, there exist $0 < \hat{\delta} < \delta$ and $T > T'$ such that for $t \geq T$,

$$w' \leq \hat{\delta}w.$$

Then, we have for $t \geq s \geq T$,

$$w(t) \leq e^{\hat{\delta}(t-s)}w(s). \quad (5.78)$$

By (5.72) and (5.78), we have for $t > T$

$$\begin{aligned} \mathcal{G}(t) &\leq e^{-\delta(t-T)} + \frac{1}{\delta} \int_T^t e^{\delta(s-t)} \cdot \frac{1}{w(s)} ds \\ &\leq e^{-\delta(t-T)} + \frac{1}{\delta w(t)} \int_T^t e^{(\delta-\hat{\delta})(s-t)} ds \\ &\leq e^{-\delta(t-T)} + \frac{1}{\delta(\delta-\hat{\delta})} \cdot \frac{1}{w(t)}. \end{aligned}$$

Based on these arguments, we have

$$w' \lesssim 1. \quad (5.79)$$

Integrating (5.79) on $[0, t]$, there exists $C_\star > 0$ such that for all $t > 0$,

$$w(t) \leq C_\star(t+1),$$

which implies $z(t) - \frac{1}{2} \log t \lesssim 1$. $\square$

Remark 5.5. When we prove Proposition 5.5, we impose the restriction $p \geq 2$ in order to obtain the condition (5.75).

Remark 5.6. By the proof of Proposition 5.5, we have $\mathcal{G} \lesssim e^{-2z}$. Furthermore, by (5.34), (5.35), and (5.71), we have

$$\begin{aligned} \left| \frac{d}{dt} a_- - \nu^- a_- \right| &\lesssim e^{-2z}, \\ \left| \frac{d}{dt} a_0 + 2\alpha a_0 \right| &\lesssim e^{-2z}. \end{aligned}$$

Hence, by the same argument as in the proof of Proposition 5.5, we have $|a_-| \lesssim e^{-2z}$ and $|a_0| \lesssim e^{-2z}$. Furthermore, by (5.55), (5.63), and (5.71), we have

$$\mathcal{E} + \frac{\gamma}{2} |u(t, 0)|^2 \gtrsim \|\bar{\varepsilon}\|_{\mathcal{H}}^2.$$

Here, we estimate $\mathcal{E}$. When $|u(t, 0)|^2 \ll e^{-2z}$, we have $\|\bar{\varepsilon}\|_{\mathcal{H}}^2 \gtrsim e^{-2z}$ and obtain $\mathcal{E} \gtrsim e^{-2z}$. When $|u(t, 0)|^2 \gtrsim e^{-2z}$, $\mathcal{E}$ satisfies $\mathcal{E} \gtrsim \|\bar{\varepsilon}\|_{\mathcal{H}}^2 + |u(t, 0)|^2 \gtrsim e^{-2z}$ since $\gamma < 0$. Thus, we obtain

$$\mathcal{E} \sim e^{-2z}.$$

If we obtain a sharp estimate of $u(t, 0)$, we improve the estimate of $\|\bar{\varepsilon}\|_{\mathcal{H}}$, refine the condition p and γ more precisely and estimate $z(t)$ more strictly. However, obtaining a sharp estimate of $u(t, 0)$ is difficult because the delta potential is delicate.

Last, we prove Theorem 1.3.

Proof of Theorem 1.3. We prove (1) of Theorem 1.3. We assume that $\vec{u}$ is a solution of (DNKG) satisfying (1.8) for some $z(t)$ and $\sigma = \pm 1$. Then by (1.1), there exist a solution $\vec{u}_+$ of (DNKG) and z_+ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} (\|u_+(t) - Q(\cdot - z_+(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) &= 0, \\ \lim_{t \rightarrow \infty} z_+(t) &= \infty. \end{aligned} \quad (5.80)$$

Notably, by (1.1) we have for t large enough,

$$|z(t)| = z_+(t). \quad (5.81)$$

By (5.80), there exists $T > 0$ such that for $t > T$,

$$(\|u_+(t) - Q(\cdot - z_+(t))\|_{H^1} + \|\partial_t u(t)\|_{L^2}) \ll 1, \quad z_+(t) \gg 1.$$

Therefore, by Lemma 5.4, there exist $T' > 0$ and $\tilde{z}_+(t)$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} (\|u_+(t) - Q(\cdot - \tilde{z}_+(t))\|_{H^1} + \|\partial_t u_+(t)\|_{L^2}) &= 0, \\ \lim_{t \rightarrow \infty} \tilde{z}_+(t) &= \infty, \\ \int \{\partial_t u_+(t) + 2\alpha(u_+(t) - Q(\cdot - \tilde{z}_+(t)))\} Q'(\cdot - \tilde{z}_+(t)) &= 0 \text{ for } t > T', \\ \lim_{t \rightarrow \infty} |z_+(t) - \tilde{z}_+(t)| &= 0. \end{aligned}$$

Then we define $\vec{u}_*$ as the solution of (DNKG) with initial data $\vec{u}_*(0) = \vec{u}_+(T')$. Then by (2.13) $\vec{u}_*$ satisfies (5.15) for $z_*(t) = \tilde{z}_+(t + T')$. Thus, by Proposition 5.5, there exists $C_* > 0$ such that for $t > 0$,

$$z_*(t) - \frac{1}{2} \log t \leq C_*.$$

We note that by (2.13), we have for $t > T' + 1$

$$\begin{aligned} \tilde{z}_+(t) - \frac{1}{2} \log t &= z_*(t - T') - \frac{1}{2} \log t \\ &\leq \frac{1}{2} \log(t - T') - \frac{1}{2} \log t + C_* \\ &\leq C_*, \end{aligned}$$

and we have $\tilde{z}_+ - \frac{1}{2} \log t \lesssim 1$. Here, we estimate $z(t)$. By $\lim_{t \rightarrow \infty} |z_+(t) - \tilde{z}_+(t)| = 0$ and (5.81), we have $|z(t)| - \frac{1}{2} \log t \lesssim 1$. We prove (2) of Theorem 1.3 using the same argument. Therefore, we complete the proof. $\square$

ACKNOWLEDGEMENT

The author wishes to thank Kenji Nakanishi for many helpful comments, discussions and encouragement on the present paper. The author also would like to thank Nobu Kishimoto, Masaya Maeda and Takahisa Inui for their helpful discussions. This work was supported by JST, the establishment of university fellowships towards the creation of science technology innovation, Grant Number JPMJFS2123.

REFERENCES

- [1] S. Aryan, *Existence of two-solitary waves with logarithmic distance for the nonlinear Klein-Gordon equation*, Commun. Contemp. Math. **24** (2022) no.1, 2050091
- [2] S. Aryan, *Soliton resolution for the energy-critical nonlinear heat equation in the radial case*, preprint, arXiv:2405.06005.
- [3] N. Burq, G. Raugel, W. Schlag, *Long time dynamics for damped Klein-Gordon equations*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no.6, 1447-1498.
- [4] T. Cazenave, A. Haraux, *An introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and its Applications **13**, Oxford University Press, 1988.
- [5] S. M. Chang, S. Gustafson, K. Nakanishi, T. P. Tsai, *Spectra of linearized operators for NLS solitary waves*, SIAM J. Math. Anal. **39** (2007/08), no.4, 1070-1111.
- [6] C. Collot, T. Duyckaerts, C. Kenig, F. Merle, *Soliton Resolution for the Radial Quadratic Wave Equation in Space Dimension 6*, Vietnam J. Math. **52** (2024), 735-773.
- [7] R. Côte, Y. Martel, *Multi-traveling waves for the nonlinear Klein-Gordon equations*, Trans. Amer. Math. Soc. **370** (2018), no.10, 7461-7487.
- [8] R. Côte, Y. Martel, X. Yuan, *Long time asymptotics of the one-dimensional damped nonlinear Klein-Gordon equation*, Arch. Ration. Mech. Anal. **239** (2021), no.3, 1837-1874.
- [9] R. Côte, Y. Martel, X. Yuan, L. Zhao, *Description and classification of 2-solitary waves for nonlinear damped Klein-Gordon equations*, Comm. Math. Phys. **388** (2021), 1557-1601.
- [10] R. Côte, C. Muñoz, *Multi-solitons for nonlinear Klein-Gordon equations*, Forum Math. Sigma **2** (2014), Paper No.e15, 38pp.
- [11] E. Cobo, F. Genoud, M. Ohta, J. Royer, *Stability of standing waves for a nonlinear Klein-Gordon equation with delta potentials*, J. Differential Equations **268** (2019), no. 1, 353-388.
- [12] R. Côte, X. Yuan, *Asymptotics of solutions with a compactness property for the nonlinear damped Klein-Gordon equation*, Nonlinear Anal. **218** (2022), Paper No. 112768, 34pp.
- [13] T. Duyckaerts, H. Jia, C. Kenig, F. Merle, *Soliton resolution along a sequence of times for the focusing energy critical wave equation*, Geom. Funct. Anal. **27** (2017), 798-862.
- [14] T. Duyckaerts, C. Kenig, Y. Martel, F. Merle, *Soliton resolution for critical co-rotational wave maps and radial cubic wave equation*, Comm. Math. Phys. **391** (2022), no.2, 779-871.
- [15] T. Duyckaerts, C. Kenig, F. Merle, *Classification of radial solutions of the focusing, energy-critical wave equation*, Camb. J. Math. **1**(1) (2013), 75-144.
- [16] T. Duyckaerts, C. Kenig, F. Merle, *Soliton resolution for the radial critical wave equation in all odd space dimensions*, Acta Math. **230** (2023), no. 1, 1-92.
- [17] E. Feireisl, *Finite energy travelling waves for nonlinear damped wave equations*, Quart. Appl. Math. **56** (1998), 5570.
- [18] R. Fukuzumi, L. Jeanjean, *Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential*, Discrete Contin. Dyn. Syst. **21** (2008), no.1, 121-136.
- [19] R. Fukuzumi, M. Ohta, T. Ozawa, *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008) 837-845.
- [20] S. Gustafson, T. Inui, *Two-solitons with logarithmic separation for 1D NLS with repulsive delta potential*, preprint, arXiv:2310.08865.
- [21] S. Gustafson, T. Inui, I. Shimizu, *Multi-solitons for the nonlinear Schrödinger equation with repulsive Dirac delta potential*, preprint, arXiv:2310.08862.
- [22] J. Gu, L. Zhao, *Soliton resolution for the energy critical damped wave equations in the radial case*, preprint, arXiv:2401.04114.
- [23] M. Ikeda, T. Inui, *Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential*, Anal. PDE **10** (2017), no.2, 481-512.
- [24] S. Ibrahim, N. Masmoudi, K. Nakanishi, *Scattering threshold for the focusing nonlinear Klein-Gordon equation*, Anal. PDE **4** (2011), no.3, 405-460.
- [25] K. Ishizuka, K. Nakanishi, *Global dynamics around 2-solitons for the nonlinear damped Klein-Gordon equations*, Ann. PDE **9** (2023), no.1, Paper No. 2, 79pp.
- [26] L. Jeanjean, K. Tanaka, *A positive solution for a nonlinear Schrödinger equation on $\mathbb{R}^N$* , Indiana Univ. Math. J. **54** (2005), 443-464.
- [27] J. Jendrej, *Construction of two-bubble solutions for the energy-critical NLS*, Anal. PDE **10**(8) (2017), 1923-1959.

- [28] J. Jendrej, *Construction of two-bubble solutions for energy-critical wave equations*, Am. J. Math. **141**(1) (2019), 55-118.
- [29] J. Jendrej, A. Lawrie, *Two-bubble dynamics for threshold solutions to the wave maps equation*, Invent. Math. **213**(3) (2018), 1249-1325.
- [30] J. Jendrej, A. Lawrie, *An asymptotic expansion of two-bubble wave maps in high equivariance classes*, Anal. PDE **15** (2022), no.2, 327-403.
- [31] J. Jendrej, A. Lawrie, *Soliton resolution for the energy-critical nonlinear wave equation in the radial case*, Ann. PDE **9** (2023) no.2, Paper No. 18, 117pp.
- [32] J. Jendrej, A. Lawrie, *Soliton resolution for energy-critical wave maps in the equivariant case*, preprint, arXiv:2106.10738.
- [33] H. Jia, B. Liu, G. Xu, *Long time dynamics of defocusing energy critical 3+1 dimensional wave equation with potential in the radial case*, Commun. Math. Phys. **339**(2) (2015), 353-384.
- [34] C. Keller, *Stable and unstable manifolds for the nonlinear wave equation with dissipation*, J. Differ. Equ. **50**(3), (1983), 330-347.
- [35] T. Kim, S. Kwon, *Soliton resolution for Calgero-Moser derivative nonlinear Schrödinger equation*, preprint, arXiv:2408.12893.
- [36] M. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$* , Arch. Ration. Mech. Anal. **105** (1989), 243-266.
- [37] P. L. Lions, *Solutions of Hartree-Fock equations for Coulomb systems*, Comm. Math. Phys. **109** (1987), 33-97.
- [38] P. L. Lions, *On positive solutions of semilinear elliptic equations in unbounded domains*, Math. Sci. Res. Inst. Publ., **13**, Springer New York, 1998.
- [39] X. Li, C. Miao, L. Zhao, *Soliton resolution for the energy critical wave equation with inverse-square potential in the radial case*, preprint, arXiv:2201.12957.
- [40] Y. Martel, T. V. Nguyen, *Construction of 2-solitons with logarithmic distance for the one-dimensional cubic Schrödinger system*, Discrete Contin. Dyn. Syst. **40** (2020), no.3, 1595-1620.
- [41] T. V. Nguyen, *Strongly interacting multi-solitons with logarithmic relative distance for the gKdV equation*, Nonlinearity **30** (2017), no.12, 4614-4648.
- [42] T. V. Nguyen, *Existence of multi-solitary waves with logarithmic relative distances for the NLS equation*, C. R. Math. Acad. Sci. Paris **357** (2019), no.1, 13-58.
- [43] K. Nakanishi, W. Schlag, *Global dynamics above the ground state for the nonlinear Klein-Gordon equation without a radial assumption*, Arch. Ration. Mech. Anal. **203** (2012), no. 3, 809-851.
- [44] X. Tang, G. Xu, *Minimal mass blow-up solutions for the L^2 -critical NLS with the delta potential for even data in one dimension*, SIAM J. Math. Anal. **56** (2024), no.2, 1727-1769.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: `ishizuka@kurims.kyoto-u.ac.jp`