

Foundation of Quantum Entropy

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§ 1 Mathematical Description of CDS and QDS

Let fix the notations used throughout this paper. Let μ be a probability measure on a measurable space $(\Omega, \mathfrak{F})$, $P(\Omega)$ be the set of all probability measures on Ω and $M(\Omega)$ be the set of all measurable functions on Ω . We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$, and the set of all density operators on $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$. Moreover, let $\mathfrak{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$ (C*-algebra or von Neumann algebra). Therefore the descriptions of classical dynamical systems, quantum dynamical systems and general quantum dynamical systems are given in the following Table:

	CDS	QDS	GQDS
obs.	real r.v.f in $M(\Omega)$	Hermitian op. A on $\mathcal{H}$ (s.a. op. in $B(\mathcal{H})$)	self-adjoint A in C*-algebra $\mathcal{A}$
state	prob. meas. $\mu \in P(\Omega)$	density op. ρ on $\mathcal{H}$	p.l.fnal $\varphi \in \mathfrak{S}$ with $\varphi(I) = 1$
expec -tation	$\int_{\Omega} f d\omega$	$\text{tr} \rho A$	$\varphi(A)$

Table. 1.1 Description of CDS, QDS and GQDS

§ 2 Classical Entropy

2.1 Discrete Case (Shannon's Theory)

A state in a discrete classical system is given by a probability distribution such that

$$\Delta_n = \left\{ p = \{p_i\}_{i=1}^n; \sum_i p_i = 1, p_i \geq 0 \right\}$$

The entropy of a state $p = \{p_i\} \in \Delta_n$ is

$$S(p) = -\sum_i p_i \log p_i$$

The relative uncertainty (relative entropy) is defined by Kullback-Leibler as

$$S(p, q) = \begin{cases} \sum_i p_i \log \frac{p_i}{q_i} & (p \ll q) \\ \infty & (p \not\ll q) \end{cases}$$

for any $p, q \in \Delta_n$. Once a state p is changed through a channel Λ^* , the information transmitted from a initial state p to a final state $q \equiv \Lambda^* p$ is described by the mutual entropy defined by

$$I(p; \Lambda^*) = S(p \otimes q) = \sum_{ij} r_{ij} \log \frac{r_{ij}}{p_i q_j}$$

where $\Lambda^* : \Delta_n \rightarrow \Delta_m$; $q = \Lambda^* p$ is a channel (e.g., $\Lambda = (p(j|i))$ transition matrix), $r_{ij} = p(j|i)p_i$ and $p \otimes q = \{p_i q_j\}$. The fundamental inequality of Shannon is

$$0 \leq I(p; \Lambda^*) \leq \min\{S(p), S(q)\}$$

According to this inequality, the ratio

$$r(p; \Lambda^*) = \frac{I(p; \Lambda^*)}{S(p)}$$

represents the efficiency of the channel transmission

2.2 Continuous Case

In classical continuous systems, a state is described by a probability measure μ . Let $(\Omega, \mathfrak{F}, P(\Omega))$ be an input probability space and $(\overline{\Omega}, \overline{\mathfrak{F}}, P(\overline{\Omega}))$ be an output probability space. A channel is a map Λ^* from $P(\Omega)$ to $P(\overline{\Omega})$, in particular, Λ^* is a Markov type if it is given by

$$\Lambda^* \varphi(Q) = \int_{\Omega} \lambda(x, Q) \varphi(dx), \varphi \in P(\Omega), Q \in \overline{\mathfrak{F}}$$

where $\lambda : \Omega \times \overline{\mathfrak{F}} \rightarrow R^+$ with (i) $\lambda(x, \bullet) \in P(\overline{\Omega})$, (ii) $\lambda(\bullet, Q) \in M(\Omega)$. In continuous case, the

entropies are defined as follows: Let $F(\Omega)$ be the set of all finite partitions $\{A_k\}$ of Ω . For any $\varphi \in P(\Omega)$, the entropy is defined by

$$S(\varphi) = \sup \left\{ -\sum_k \varphi(A_k) \log \varphi(A_k); \{A_k\} \in F(\Omega) \right\},$$

which is often infinite. For any $\varphi, \psi \in P(\Omega)$, the relative entropy is given by

$$\begin{aligned} S(\varphi, \psi) &= \sup \left\{ \sum_k \varphi(A_k) \log \frac{\varphi(A_k)}{\psi(A_k)}; \{A_k\} \in F(\Omega) \right\} \\ &= \begin{cases} \int_{\Omega} \log \left(\frac{d\varphi}{d\psi} \right) d\psi & (\varphi \ll \psi) \\ +\infty & (\varphi \not\ll \psi) \end{cases} \end{aligned}$$

Let Φ, Φ_0 be two compound states (measures) defined as follows :

$$\begin{aligned} \Phi(Q_1, Q_2) &= \int \lambda(x, Q_2) \varphi(dx), Q_1 \in \mathfrak{F}, Q_2 \in \overline{\mathfrak{F}} \\ \Phi_0(Q_1, Q_2) &= \int_{\Omega} (\varphi \otimes \Lambda^* \varphi)(Q_1, Q_2) = \varphi(Q_1) \Lambda^* \varphi(Q_2) \end{aligned}$$

For $\varphi \in P(\Omega)$ and a channel Λ^* , the mutual entropy is given by

$$I(\varphi; \Lambda^*) = S(\Phi, \Phi_0).$$

§ 3 Quantum Entropy

3.1 Entropies for density operators

A state in quantum systems is described by a density operator on a Hilbert space $\mathcal{H}$. The entropies are defined as follows: For a state $\rho \in \mathfrak{S}(\mathcal{H})$, the entropy [N.1] is given by

$$S(\rho) = -\text{tr} \rho \log \rho.$$

If $\rho = \sum_k p_k E_k$ (Schatten decomposition, $\dim E_k = 1$), then

$$S(\rho) = -\sum_k p_k \log p_k.$$

Let us summarize the properties of the entropy $S(\rho)$.

Theorem 3.3 For any density operator $\rho \in \mathfrak{S}(\mathcal{H})$, the followings hold:

- (1) Positivity : $S(\rho) \geq 0$.
- (2) Symmetry : Let $\rho' = U^{-1}\rho U$ for an invertible operator U . Then

$$S(\rho') = S(\rho)$$
- (3) Concavity : $S(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda S(\rho_1) + (1-\lambda)S(\rho_2)$ for any $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H})$.
- (4) Additivity : $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$ for any $\rho_k \in \mathfrak{S}(\mathcal{H})$.
- (5) Subadditivity : For the reduced states ρ_1, ρ_2 of $\rho \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$,

$$S(\rho) \leq S(\rho_1) + S(\rho_2).$$
- (6) Lower Semicontinuity : If $\|\rho_n - \rho\|_1 (\equiv \text{tr}|\rho_n - \rho|) \rightarrow 0$, then

$$S(\rho) \leq \liminf S(\rho_n).$$
- (7) Continuity : Let ρ_n, ρ be elements in $\mathfrak{S}(\mathcal{H})$ which satisfy the following conditions :
 - (i) $\rho_n \rightarrow \rho$ weak as $n \rightarrow \infty$, (ii) $\rho_n \leq A$ ($\forall n$) for some compact operator A , and
 - (iii) $-\sum a_k \log a_k < +\infty$ for the eigenvalues $\{a_k\}$ of A , Then $S(\rho_n) \rightarrow S(\rho)$.
- (8) Strong Subadditivity : Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and denote the reduced states $\text{tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \rho$ by ρ_{ij} and ρ_k , respectively. Then

$$S(\rho) + S(\rho_2) \leq S(\rho_{12}) + S(\rho_{23}) \text{ and } S(\rho_1) + S(\rho_2) \leq S(\rho_{13}) + S(\rho_{23}).$$

For two states $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$, the relative entropy [U.2, L.1] is given by

$$S(\rho, \sigma) = \begin{cases} \text{tr} \rho (\log \rho - \log \sigma) & (\rho \ll \sigma) \\ +\infty & (\rho \not\ll \sigma) \end{cases}$$

where $\rho \ll \sigma \Leftrightarrow$ for any $A \geq 0$, $\text{tr} \sigma A = 0 \Rightarrow \text{tr} \rho A = 0$.

Let $\Lambda^* : \mathfrak{S}(\mathcal{H}) \rightarrow \mathfrak{S}(\overline{\mathcal{H}})$ be a channel and set

$$\sigma = \Lambda^* \rho, \theta_E = \sum_k p_k E_k \otimes \Lambda^* E_k, \theta_0 = \rho \otimes \Lambda^* \rho.$$

The mutual entropy [O.1] is given by

$$\begin{aligned} I(\rho; \Lambda^*) &= \sup \{ S(\theta_E, \theta_0); E = \{E_k\} \} \\ &= \sup \left\{ \sum_k p_k S(\Lambda^* E_k, \Lambda^* \rho); E = \{E_k\} \right\} \end{aligned}$$

for any state $\rho \in \mathfrak{S}(\mathcal{H})$ and any channel Λ^* . When the decomposition of ρ is fixed such that

$\rho = \sum_k \lambda_k \rho_k$, then

$$I(\rho; \Lambda^*) = \sum_k \lambda_k S(\Lambda^* \rho_k, \Lambda^* \rho).$$

where $\theta_\lambda = \sum_k \lambda_k \rho_k \otimes \Lambda^* \rho_k$. The fundamental inequality of Shannon type is obtained:

$$0 \leq I(\rho; \Lambda^*) \leq \min\{S(\rho), S(\Lambda^* \rho)\}.$$

3.2 Channeling Transformations

A general quantum system containing continuous cases is described by a C^* -algebra or a von Neumann algebra. Let $\mathcal{A}$ be a C^* -algebra (complex normed algebra with involution $*$ such that $\|A\| = \|A^*\|$, $\|A^*A\| = \|A\|^2$ and complete w.r.t. $\|\cdot\|$) and $\mathfrak{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$ (positive continuous linear functionals φ on $\mathcal{A}$ s.t. $\varphi(I) = 1$ if $I \in \mathcal{A}$)

A channel $\Lambda^* : \mathfrak{S}(\mathcal{A}) \rightarrow \mathfrak{S}(\overline{\mathcal{A}})$ contains several physical transformations as special cases. First give the mathematical definitions of channels.

Definition

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha)$ be an input system and $(\overline{\mathcal{A}}, \mathfrak{S}(\overline{\mathcal{A}}), \overline{\alpha})$ be an output system. Take any $\varphi, \phi \in \mathfrak{S}(\mathcal{A})$.

- (1) Λ^* is linear if $\Lambda^*(\lambda\varphi + (1-\lambda)\phi) = \lambda\Lambda^*\varphi + (1-\lambda)\Lambda^*\phi$ for any $\lambda \in [0,1]$.
- (2) Λ^* is completely positive (C.P.) if Λ^* is linear and its dual $\Lambda : \overline{\mathcal{A}} \rightarrow \mathcal{A}$ satisfies

$$\sum_{i,j=1}^n A_i^* \Lambda(\overline{A_i^* A_j}) A_j \geq 0$$

for any $n \in \mathbb{N}$ and any $\overline{A_i} \in \overline{\mathcal{A}}$, $A_i \in \mathcal{A}$.

- (3) Λ^* is Schwarz type if $\Lambda(\overline{A^*}) = \Lambda(\overline{A})^*$ and $\Lambda(\overline{A})^* \Lambda(\overline{A}) \leq \Lambda(\overline{A^* A})$.
- (4) Λ^* is stationary if $\Lambda \circ \alpha_t = \overline{\alpha}_t \circ \Lambda$ for any $t \in \mathbb{R}$.
- (5) Λ^* is ergodic if Λ^* is stationary and $\Lambda^*(\text{exl}(\alpha)) \subset \text{exl}(\overline{\alpha})$.
- (6) Λ^* is orthogonal if any two orthogonal states $\varphi_1, \varphi_2 \in \mathfrak{S}(\mathcal{A})$ (denoted by $\varphi_1 \perp \varphi_2$) implies $\Lambda^* \varphi_1 \perp \Lambda^* \varphi_2$.
- (7) Λ^* is deterministic if Λ is orthogonal and bijection.
- (8) For a subset S of $\mathfrak{S}(\mathcal{A})$, Λ^* is chaotic for S if $\Lambda^* \varphi_1 = \Lambda^* \varphi_2$ for any $\varphi_1, \varphi_2 \in S$.
- (9) Λ^* is chaotic if Λ^* is chaotic for $\mathfrak{S}(\mathcal{A})$.

Most of channels appeared in physical processes are C.P. channels. Examples of channels are the followings [O.2, D.1]:

(1) Unitary evolution :

For any density operator $\rho \in \mathfrak{S}(\mathcal{H})$

$$\rho \rightarrow \Lambda_t^* \rho = \text{Ad} U_t(\rho) \equiv U_t^* \rho U_t, t \in \mathbb{R}, U_t = \exp(itH)$$

(2) Semigroup evolution :

$$\rho \rightarrow \Lambda_t^* \rho = V_t^* \rho V_t, t \in \mathbb{R}^+, \text{ where } (V_t; t \in \mathbb{R}^+) \text{ is a one parameter semigroup on } \mathcal{H}$$

(3) Measurement :

When we measure an observable $A = \sum_n a_n P_n$ (spectral decomposition) in a state ρ , the state ρ changes to a state $\Lambda^* \rho$ by this measurement such as

$$\rho \rightarrow \Lambda^* \rho = \sum_n P_n \rho P_n$$

(4) Reduction :

If a system Σ_1 interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ and σ , respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between two systems is given by

$$\theta_t = U_t^*(\rho \otimes \sigma)U_t,$$

where $U_t = \exp(itH)$ with the total Hamiltonian H of Σ_1 and Σ_2 . A channel is obtained by taking the partial trace w.r.t. $\mathcal{K}$ such as

$$\rho \rightarrow \Lambda_t^* \rho = \text{tr}_{\mathcal{K}} \theta_t.$$

3.3 Entropies in GQDS

The entropy (uncertainty) of a state $\varphi \in \mathcal{S}$ seen from the reference system $\mathcal{S}$, a weak *-compact convex subset of $\mathfrak{S}$, is given by [O.2,O.3].

Every state $\varphi \in \mathcal{S}$ has a maximal measure μ pseudosupported on $\text{ex}\mathcal{S}$ such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu$$

The measure μ giving the above decomposition is not unique unless $\mathcal{S}$ is a Choquet simplex, so that we denote the set of all such measures by $M_{\varphi}(\mathcal{S})$. Put

$$D_{\varphi}(\mathcal{S}) = \left\{ M_{\varphi}(\mathcal{S}); \exists \{\mu_k\} \subset \mathbb{R}^+ \text{ and } \{\varphi_k\} \subset \text{ex}\mathcal{S} \text{ s.t. } \sum_k \mu_k = 1, \mu = \sum_k \mu_k \delta(\varphi_k) \right\},$$

where $\delta(\varphi)$ is the Dirac measure concentrated on $\{\varphi\}$, and put

$$H(\mu) = -\sum_k \mu_k \log \mu_k$$

for a measure $\mu \in D_\varphi(\mathcal{S})$. Then the entropy of a state $\varphi \in \mathcal{S}$ w.r.t. $\mathcal{S}$ is defined by

$$S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu); \mu \in D_\varphi(\mathcal{S})\} \\ +\infty & \text{if } D_\varphi(\mathcal{S}) = \emptyset \end{cases}$$

The entropy (mixing entropy) of a general state φ satisfies the following properties [O.2,O.3].

Theorem When $\mathcal{A} = B(\mathcal{H})$ and $\alpha_i = \text{Ad}(U_i)$ with an unitary operator U_i , for any state φ given by $\varphi(\cdot) = \text{tr} \rho \cdot$ with a density operator ρ , the followings hold:

- (1) $S(\varphi) = -\text{tr} \rho \log \rho$.
- (2) If φ is an α -invariant state and every eigenvalue of ρ is non-degenerate, then $S'(\varphi) = S(\varphi)$.
- (3) If $\varphi \in K(\alpha)$, then $S^K(\varphi) = 0$.

Theorem For any $\varphi \in K(\alpha)$,

- (1) $S^K(\varphi) \leq S'(\varphi)$.
- (2) $S^K(\varphi) \leq S(\varphi)$.

This $\mathcal{S}$ (or mixing) entropy gives a measure of the uncertainty observed from the reference system $\mathcal{S}$. Similar properties as $S(\rho)$ hold (see [O.3]).

The relative entropy for two general states φ and ψ has been introduced by Araki [A.1,A.2] and Uhlmann [U.1] and their relation is considered in [H.1,H.2].

<Araki's definition>

Let $\mathfrak{N}$ be σ -finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and φ, ψ be normal states on $\mathfrak{N}$ given by $\varphi(\cdot) = \langle x, \cdot x \rangle$ and $\psi(\cdot) = \langle y, \cdot y \rangle$ with $x, y \in \mathcal{H}$. The operator S_{xy} is defined by

$$S_{xy}(Ay + z) = s^{\mathfrak{N}}(y)A^*x, \quad A \in \mathfrak{N}, \quad s^{\mathfrak{N}}(y)z = 0.$$

on the domain $\mathfrak{N}y + (I - s^{\mathfrak{N}}(y))\mathcal{H}$, where $s^{\mathfrak{N}}(y)$ is the projection from $\mathcal{H}$ to $\{\mathfrak{N}'y\}^\perp$, the

$\{\mathfrak{N}'y\}^-$ -support of y . Using this $S_{x,y}$, the relative modular operator $\Delta_{x,y}$ is defined as $\Delta_{x,y} = (S_{x,y})^* S_{x,y}^-$, with spectral decomposition denoted by $\int_0^\infty \lambda d e_{x,y}(\lambda)$. Then the relative entropy is given by

$$S(\psi | \varphi) = \begin{cases} \int_0^\infty \log \lambda d \langle y, e_{x,y}(\lambda) y \rangle & \text{if } \psi \ll \varphi \\ +\infty & \text{otherwise,} \end{cases}$$

where $\psi \ll \varphi$ means that $\varphi(A^*A) = 0$ implies $\psi(A^*A) = 0$ for $A \in \mathfrak{N}$.

< Uhlmann's definition >

Let $\mathcal{L}$ be a linear space and p, q be seminorms on $\mathcal{L}$, α a positive Hermitian form on $\mathcal{L}$. Put $\mathcal{G} \equiv \{\alpha; |\alpha(x, y)| \leq p(x)q(y), x, y \in \mathcal{L}\}$ and $QM(p, q) \equiv \sup\{\alpha(x, x)^{1/2}; \alpha \in \mathcal{G}\}$. There exists a quadratical interpolation $t \in [0, 1] \rightarrow p_t$ from p to q ($p_t \equiv QI_t(p, q)$) such that

- (1) p_t cont.
- (2) $t = \frac{1}{2}(t_1 + t_2) \Rightarrow p_t = QM(p_{t_1}, p_{t_2})$
- (3) $p_{1/2} = QM(p, q)$
- (4) $p_{t/2} = QM(p, p_t)$
- (5) $p_{\frac{t_1}{2}} = QM(p_{t_1}, q)$

Let $\mathcal{L} = \mathcal{A}$ and for any states $\varphi, \psi \in \mathfrak{S}(\mathcal{A})$

$$p(A) = \varphi(AA^*)^{1/2}$$

$$q(A) = \psi(A^*A)^{1/2}$$

Then the relative entropy for φ and ψ is given by

$$S(\varphi | \psi) = -\liminf_{t \rightarrow \infty} \frac{1}{t} \{QI_t(p, q)^2(I) - p^2(I)\}$$

For $\varphi \in \mathcal{S}(\mathcal{A}) \subset \mathfrak{S}(\mathcal{A})$, $\Lambda^* : \mathfrak{S}(\mathcal{A}) \rightarrow \mathfrak{S}(\overline{\mathcal{A}})$, let us define the compound states by

$$\Phi_\mu^{\mathcal{S}} = \int_{\mathcal{S}} \omega \otimes \Lambda^* \omega d\mu \quad \text{and}$$

$$\Phi_0 = \varphi \otimes \Lambda^* \varphi$$

The mutual entropy w.r.t. $\mathcal{S}$ and μ is

$$I_\mu^{\mathcal{S}}(\varphi; \Lambda^*) = S(\Phi_\mu^{\mathcal{S}} | \Phi_0)$$

and the mutual entropy w.r.t. $\mathcal{S}$ is defined as [O.3]

$$\begin{aligned} I^{\mathcal{S}}(\varphi; \Lambda^*) &= \liminf_{\varepsilon \rightarrow 0} \{I_{\mu}^{\mathcal{S}}(\varphi; \Lambda^*); \mu \in F_{\varphi}^{\varepsilon}(S)\} \\ &= \sup \left\{ \sum_k \mu_k S(\Lambda^* \omega_k | \Lambda^* \varphi); \varphi = \sum_k \mu_k \varphi_k \right\} \end{aligned}$$

where

$$\begin{aligned} F_{\varphi}^{\varepsilon}(S) &= \left\{ \begin{array}{l} \mu \in D_{\varphi}(S); S^{\mathcal{S}}(\varphi) \leq H(\mu) \leq S^{\mathcal{S}}(\varphi) + \varepsilon < +\infty \\ M_{\varphi}(S) \quad \text{if } S^{\mathcal{S}}(\varphi) = +\infty \end{array} \right\} \\ D_{\varphi}(S) &= \left\{ \mu \in M_{\varphi}(S); \exists \{\mu_k\} \subset R^+ \text{ s.t. } \mu = \sum_k \mu_k \delta(\varphi_k), \varphi_k \in exS, \sum_k \mu_k = 1 \right\} \end{aligned}$$

when a state φ is expressed as $\varphi = \sum_k \mu_k \omega_k$ (fixed), the mutual entropy is given by

$$I^{\mathcal{S}}(\varphi; \Lambda^*) = \sum_k \mu_k S(\Lambda^* \omega_k, \Lambda^* \varphi)$$

This entropy and $\mathcal{S}$ -entropy contains Connes-Thiring-Narnhofer entropy as a special case [M.1].

The inequality is satisfied for almost all physical cases.

$$0 \leq I^{\mathcal{S}}(\varphi; \Lambda^*) \leq S^{\mathcal{S}}(\varphi)$$

The fundamental properties of the relative entropy and the mutual entropy are the followings [A.1, A.2, U.1, H.1, O.3, O.4].

Theorem

- (1) Positivity : $S(\varphi | \psi) \geq 0$.
- (2) Joint Convexity : $S(\lambda \psi_1 + (1-\lambda)\psi_2 | \lambda \varphi_1 + (1-\lambda)\varphi_2) \leq \lambda S(\psi_1 | \varphi_1) + (1-\lambda)S(\psi_2 | \varphi_2)$.
- (3) Additivity : $S(\psi_1 \otimes \psi_2 | \varphi_1 \otimes \varphi_2) = S(\psi_1 | \varphi_1) + S(\psi_2 | \varphi_2)$.
- (4) Lower Semicontinuity : If $\lim_{n \rightarrow \infty} \|\psi_n - \psi\| = 0$ and $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\| = 0$, then $S(\psi | \varphi) \leq \liminf_{n \rightarrow \infty} S(\psi_n | \varphi_n)$. Moreover, if there exists a positive number λ satisfying $\psi_n \leq \lambda \varphi_n$, then $\lim_{n \rightarrow \infty} S(\psi_n | \varphi_n) = S(\psi | \varphi)$.
- (5) Monotonicity : For a channel Λ^* from $\mathfrak{S}$ to $\overline{\mathfrak{S}}$, $S(\Lambda^* \psi | \Lambda^* \varphi) \leq S(\psi | \varphi)$.
- (6) Lower Bound : $\|\psi - \varphi\|^2 \leq 2S(\psi | \varphi)$.

Theorem [O.3]

- (1) If Λ^* is deterministic, then $I(\varphi; \Lambda^*) = S(\varphi)$.
- (2) If Λ^* is chaotic, then $I(\varphi; \Lambda^*) = 0$.
- (3) For a state $\varphi = \text{tr} \rho \bullet$, if Λ^* is ergodic and the state is stationary for a time evolution $\alpha_t = \text{Ad} U_t$, and if every eigenvalue of ρ is nonzero and nondegenerate, then $I(\rho; \Lambda^*) = S(\Lambda^* \rho)$.

This mutual entropy is extensively used for analysing optical communication processes [B.1].

The CNT entropy $H_\varphi(\mathfrak{N})$ of C^* -subalgebra $\mathfrak{N} \subset \mathcal{A}$ is defined by [C.1].

$$H_\varphi(\mathfrak{N}) \equiv \sup_{\varphi = \sum_j \mu_j \varphi_j} \sum_j \mu_j S(\varphi_j | \mathfrak{N} | \varphi | \mathfrak{N})$$

where the supremum is taken over all finite decomposition $\varphi = \sum_j \mu_j \varphi_j$ of φ . This entropy is a mutual entropy when a channel is the restriction to subalgebra. There are some relations between the mixing entropy $S^{\mathcal{S}}(\varphi)$ and the CNT entropy.

Theorem [M.1]

- (1) For any state φ on a unital C^* -algebra $\mathcal{A}$,

$$S^{\mathcal{S}}(\varphi) = H_\varphi(\mathcal{A})$$

- (2) Let $(\mathfrak{M}, G, \alpha)$ be a G -finite W^* -dynamical system, φ be a G -invariant normal state of $\mathfrak{M}$, then

$$S^{\ell(\alpha)}(\varphi) = H_\varphi(\mathfrak{M}^\alpha)$$

- (3) Let $\mathcal{A}$ be the C^* -algebra $C(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$, and G be a group, α be a $*$ -automorphic action of G -invariant density operator. Then

$$S^{\ell(\alpha)}(\rho) = H_\rho(\mathcal{A}^\alpha)$$

The pseudo-mutual $\mathcal{S}$ -entropy $J^{\mathcal{S}}(\varphi; \Lambda^*)$ is given by

$$J^{\mathcal{S}}(\varphi; \Lambda^*) = \sup \left\{ \sum_j \mu_j S(\Lambda^* \varphi_j | \Lambda^* \varphi); \varphi = \sum_j \mu_j \varphi_j, \varphi_j \in \mathcal{S} \right\}.$$

Theorem [M.1]

$$(1) \quad 0 \leq I^{\mathcal{S}}(\varphi; \Lambda^*) \leq J^{\mathcal{S}}(\varphi; \Lambda^*) \leq \min\{H^{\mathcal{S}}(\varphi), H^{\Lambda^* \mathcal{S}}(\Lambda^* \varphi)\}.$$

(2) Let Λ^* be a G-stationary channel from $\mathcal{A}$ to $\overline{\mathcal{A}}$ and G be compact. Then

$$0 \leq I^{I(\alpha)}(\varphi; \Lambda^*) \leq \min\{S^{I(\alpha)}(\varphi), S^{I(\bar{\alpha})}(\Lambda^* \varphi)\}.$$

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