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On the product of the terms of a finite arithmetic progression

by

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Let a, d and k be positive integers, $k \geq 3$. We consider the arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$ and in particular the product $\Delta = a(a+d)\ldots(a+(k-1)d)$.

There are two circles of problems we shall consider:

I: What can be said about the greatest prime factor $P(\Delta)$ of Δ and the number of distinct prime divisors $\omega(\Delta)$ of Δ?

II: Can Δ be an (almost) perfect power? Can each of $a, a+d, \ldots, a+(k-1)d$ be an ℓ-th power for some $\ell \geq 2$?

This lexture reports on joint work with T.N. Shorey. It can be considered as an updating of my first lecture given in Banff in 1988, [20]. Almost all results have effective proofs, but for this aspect I refer to the original papers. I am grateful to Shorey for his remarks on an earlier draft of the present paper.

I. Without loss of generality we may assume $\gcd(a, d) = 1$. A first general result on

I was obtained by Sylvester [19] in 1892. He proved

(1) $$\text{if} \quad a \geq d+k \quad \text{then } P(\Delta) > k.$$

Suppose $d = 1$. Then we consider the product of a block of k integers. If $a \geq 1+k$, then there is apparently at least one number in the block $a, a+1, \ldots, a+k-1$ which is not composed of primes $\leq k$. If $a = 1+k$, this is Bertrand's Postulate. If $a < 1+k$, the last term of the block is less than $2k$. Then the question becomes whether $k+1, \ldots, a+k-1$ contains a prime. This is the classical problem on gaps between consecutive primes. The theorem of Hoheisel, Ingham as improved by many others says that if $a > k^{23/42}$ and k is sufficiently large then there is a prime. (The exponent $23/42$ has been slightly improved, see [7]). Probably $a > (1+\epsilon)(\log k)^2$ for large k is sufficient according to a hypothesis of Cramér, but it will be extremely hard to prove this.

Suppose $d > 1$. Then (1) was slightly improved by Langevin in 1977 as follows.

(2) $$\text{If } a > k, \quad \text{then } P(\Delta) > k.$$

Shorey and I showed that in fact $2, 9, 16$ is the only exception:

Theorem 1. ([14]). *Let* $d > 1, k > 2, \gcd(a,d) = 1, (a,d,k) \neq (2,7,3)$. *Then* $P(\Delta) > k$.

The proof rests on a sharp upper bound for $\pi(x)$ due to Rosser and Schoenfeld and is further computational.

If a becomes large, then much better lower bounds are possible. Shorey and I improved upon some estimate of Langevin [5].

Theorem 2. ((a) and (c) in [15], (b) unpublished). *Let* $\chi = a + (k-1)d$, $\chi_0 = \max(\chi/k, 3)$,

$\epsilon > 0$.

(a) $P(\Delta) >> k \log\log \chi_0 \qquad (\geq k \log\log d)$.

(b) *if* $\chi > k(\log k)^{\epsilon}$ *then* $P(\Delta) >>_{\epsilon} k \log\log a$.

(c) *if* $\chi > k^{1+\epsilon}$ *then* $P(\Delta) >>_{\epsilon} k \log\log \chi$.

The proof is based on Baker's method, in particular a result on the Thue equation by Györy [4]. Note that some conditions in (b) and (c) are necessary. In (b) we can take $a = [k/2], d = 1$ and it follows that $P(\Delta) \leq a + (k-1)d < 3k/2 = o(k \log\log a)$. In (c) we can take $a = 1, d = [(\log\log \chi)^{1/2}]$ and it follows that $P(\Delta) \leq a + (k-1)d < k(\log\log \chi)^{1/2} = o(k \log\log \chi)$.

Very recently we studied $\omega(\Delta)$. If $a = d = 1$, then $\omega(\Delta) = \omega(k!) = \pi(k)$. There are more examples with $\omega(\Delta) = \pi(k)$, for example $1, 625, 1249$, and $1, 3, 5, 7, 9$.

Theorem 3. [16]

$$\omega(\Delta) \geq \pi(k).$$

The proof is similar to that of Theorem 1. Here only limited improvement is possible

if a becomes large. We cannot exclude that there are $k-1$ primes $p_1, \ldots, p_{k-1}$ in such a way that $1, p_1, \ldots, p_{k-1}$ are in arithmetic progression, so that we cannot prove anything better than $\omega(\Delta) \geq k-1$.

Theorem 4. (a) *For any positive integer $t > 1$ we have*

$$\text{if} \quad \chi \geq k^{\frac{k}{t-1}+1} \qquad \text{then} \quad \omega(\Delta) > k - t.$$

(b) *there are infinitely many instances with $\chi \geq k^{2.7}$ and $\omega(\Delta) < ck$ with $c < 1$.*

The proof of (a) is elementary. For (b) we use estimates for the Dickman function $\psi(x, y)$. If we take $t = .6k$ in (a), then we find that $\omega(\Delta) > .4k$ if $\chi \geq k^{2.7}$ and (b) shows that ck for some c with $0 < c < 1$ is the actual order of magnitude.

II. How many ℓth powers can be in arithmetic progression? If $\ell = 2$, then there are infinitely many triples of squares in arithmetic progression, but Fermat proved that there are no four squares in arithmetic progression. Dénes [1] proved in 1952 that there are no three ℓ–th powers in arithmetic progression for $3 \leq \ell \leq 30$ and for 60 other prime values of ℓ and he conjectured that this is true for all ℓ. He used Kummer theory and his method is not applicable for irregular primes. The celebrated result of Faltings [3] implies that for any $\ell \geq 5$ there are only finitely many triples of coprime ℓ–th powers in arithmetic progression, but his result does not provide any bound for k independent of ℓ. In the sequel we assume $\gcd(a, d) = 1, k \geq 3$ and $\ell \geq 2$. Let d_1 be the maximal divisor of d composed of prime factors $\equiv 1 \pmod{\ell}$.

Theorem 5. ((a), (c) and (d) from [12], (b) from [13]).

Suppose $a, a+d, \ldots, a+(k-1)d$ *are all* ℓ*–th powers. Then*

(a) *if* d *is odd, then* $k = 3$,

(b) $k < (1+\epsilon)2^{\omega(d_1)}$ *for* $k \geq k_0(\epsilon)$,

(c) $k << \omega(d)\log\omega(d)$,

(d) $k << \sqrt{\log d}$.

A much weaker condition is that not each of the numbers is an ℓ–th power, but that the product of the numbers is an ℓ–th power. If $d = 1$ we find that

$$a(a+1)(a+2)\ldots(a+k-1) = y^{\ell} \qquad (\ell > 1). \tag{3}$$

It was proved by Erdös and Selfridge [2] in 1975, 36 years after Erdös started this research, that (3) has no solution in positive integers $a, k > 2, y,\ \ell > 1$. Later Erdös conjectured that if

$$a(a+d)(a+2d)\ldots(a+(k-1)d) = y^{\ell} \tag{4}$$

then k is bounded by an absolute constant. Still later he conjectured $k \leq 3$. Some special cases are in the literature. Euler proved that the product of four numbers in arithmetic progression cannot be a square. Of course, this implies Fermat's result that there are no four squares in arithmetic progression. Obláth [8] proved the result for the product of five numbers in arithmetic progression. He [9] also proved that

the product of three numbers in arithmetic progression cannot be a third, fourth of fifth power. Marszalek [6] was the first to deal with the general problem. He proved that k is bounded by a number depending only on d. He gave rather refined estimates, but a rough simplification of his result gives:

$$\begin{array}{rll} k \leq \exp(2d^{3/2}) & \text{if} & \ell = 2, \\ k \leq \exp(2d^{7/3}) & \text{if} & \ell = 3, \\ k \leq Cd^{5/2} & \text{if} & \ell = 4, \\ k \leq Cd & \text{if} & \ell \geq 5, \quad \text{where} \quad C = 3 \cdot 10^4. \end{array}$$

Shorey [10] proved that k is bounded by a number depending on $P(d)$, the greatest prime factor of d, provided that $\ell > 2$. Shorey [10] further proved that $d_1 > 1$ if $m > k$ and k large.

Shorey and I have obtained many results on equation (4). Actually we proved these results under the following weaker assumption.

$$(5) \qquad \begin{cases} \text{Let } a, d, k, b, y, \ell \text{ be positive integers such that } \gcd(a, d) = 1, \\ k > 2, \ell > 1, P(b) \leq k, P(y) > k \quad \text{and} \\ a(a+d)\dots(a+(k-1)d) = by^{\ell}. \end{cases}$$

In the sequel we assume that (5) holds.

Theorem 6. $\log k << \dfrac{\log d_1}{\log\log d_1}$.

Proof. For $\ell \geq 7$, see [18]. For $\ell \leq 5$ see [17], formula (2.14).

Observe that this is a considerable improvement of Marszalek's result. We see that $k/d \to 0$ as $d \to \infty$ and even $\log k / \log d \to 0$ as $d \to \infty$.

Theorem 7. [17] $\qquad k << d_1^{1/(\ell-2)}$.

This implies Shorey's estimate $d_1 > 1$ for $\ell > 2$ and k large.

Theorem 8. [17]

$$\frac{k}{\log k} << \ell^{\omega(d)} \qquad (even << \ell^{\omega(d_1)} for \ \ell \geq 7).$$

Thus k is bounded by a number depending only on ℓ and $\omega(d)$. Actually we tried to prove that k is bounded by a number depending only on $\omega(d)$, but we did not succeed.

Suppose $\ell > 2$ and $P(d)$ is bounded. Then k is bounded or every prime factor of d_1 is bounded by Theorem 7. However, by definition every prime factor of d_1 is larger than ℓ. So we obtain that $\omega(d)$ and ℓ are bounded, hence, by Theorem 8, k is bounded. Thus Theorems 7 and 8 generalize the results of Shorey mentioned above.

Theorems 7 and 8 imply a slightly weaker inequality than Theorem 6 gives. It is well known that $\omega(n) << \log n / \log\log n$ for all $n > e$. Suppose $\ell \geq 7$. If $\ell \leq \log\log k / \log\log\log k$ then Theorem 8 implies

$$\log k << \frac{\log d}{\log\log d_1} . \log\log\log k$$

and if $\ell > \log\log k / \log\log\log k$ then Theorem 7 implies

$$\log k << \frac{\log d_1}{\ell} < \log d_1 \frac{\log\log\log k}{\log\log k}.$$

Theorems 7 and 8 have proofs based on multiple application of the box principle. For $\ell = 2$ the proof is elementary, but complicated. For $\ell \geq 3$ the proof is completely

different. For $\ell \geq 7$ we obtain the best results via elementary arguments, but for $\ell = 3$ and 5 we reach the best estimates when we use Brun's sieve and some result of Evertse on the number of solutions of the equation $ax^\ell - by^\ell = c$, proved by using hypergeometric functions. I want to stress that many lemmas and arguments are due to Erdös.

We have proved that $P(d)$, and even $P(d_1)$, tends to ∞ when $k \to \infty$. In fact we can prove

Theorem 9. [18]

$$P(d_1) >> \ell \log k \log\log k \qquad \text{for } \ell \geq 7,$$

$$P(d) >> \ell \log k \log\log k \qquad \text{for } \ell \in \{2,3,5\}.$$

In [18] we give also lower bounds for the smallest prime factor and the greatest square free divisor of d_1.

Up to now I have restricted myself to dependence on d, d_1, k and ℓ. Of course, a can also been taken into account.

Theorem 10 [17].

(a) *There is an absolute constant ℓ_0 such that for $\ell \geq \ell_0$ we have*

$$k <<_a 1.$$

(b)

$$k <<_{a,\omega(d)} 1.$$

For (a) see Shorey [11]. Further (b) follows from the combination of (a) and Theorem 8.

The last theorem concerns upper bounds for the largest term in the arithmetic progression.

Theorem 11. [17]. *There is an absolute constant k_0 such that $k \geq k_0$ implies*

$$a + (k-1)d \leq 17d^2 k(\log k)^4 \qquad \text{if} \quad \ell = 2$$

and

$$a + (k-1)d << k(\frac{d}{\ell})^{\ell/(\ell-2)} \qquad \text{if} \quad \ell > 2.$$

Finally I want to state a conjecture for the general situation in the line of the conjectures of Dénes and Erdös stated above.

Conjecture. If (5) holds, then $k + \ell \leq 6$.

If $k + \ell \leq 6$, then $(k, \ell) = (3,3)$ or $(4,2)$. It is shown in [20] that in these cases there are infinitely many solutions. As a more moderate target I challenge the reader to prove that k is bounded by a function of only $\omega(d)$ if (5) is satisfied.

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