

A note on the stable $\mathbb{Z}_2$ - cohomotopy groups

by Goro Nishida

§ 0. Introduction

Let X and Y be based $\mathbb{Z}_2$ - complexes and let $\tilde{\pi}^{n,m}(X; Y)$ denote the group of stable $\mathbb{Z}_2$ - maps of degree (n, m) (see §1 for the definition). If $Y = \Sigma^{0,0}$ is the 0 - sphere, then $\tilde{\pi}^{n,m}(X : \Sigma^{0,0}) = \tilde{\pi}^{n,m}(X)$ is called the stable $\mathbb{Z}_2$ - cohomotopy group of X , and has been studied by various authors ([3], [4], [7] and [8]). The purpose of this note is to describe it in terms of non equivariant stable homotopy for certain X and Y .

To state our result recall that the stunted projective space P_a^b is defined for all integers a and b as a stable complex by the well known periodicity. We shall define in §2 a stable map $u : \Sigma^{-1} \rightarrow P_a^b$ for $b \geq -1$ and a well defined stable homotopy type P_a^b / Σ^{-1} . Let S^q denote the q - sphere with the antipodal involution. Then our results are,

Theorem 1. Let q be a positive integer and let X be a based $\mathbb{Z}_2$ - complex with the trivial $\mathbb{Z}_2$ - action. Then for any $n, m \in \mathbb{Z}$,

there is a natural stable isomorphism

$$\bar{\alpha} : \tilde{\pi}^{n,m}(X : S_+^q) \cong \tilde{\pi}^m(X : P_n^{n+q}).$$

Theorem 2. Let n, m and q be integers such that $n + q > 0$.

Then for any based $\mathbb{Z}_2$ -complex X with the trivial $\mathbb{Z}_2$ -action of $\dim. < n + m + q$, there is a natural stable isomorphism

$$\alpha : \tilde{\pi}^{n,m}(X) \cong \tilde{\pi}^m(X : P_n^{n+q} / \Sigma^{-1}).$$

Here $\tilde{\pi}^m(:)$ denotes the usual group of (non-equivariant) stable maps of degree m . If we fix $\check{\lambda}$, then the above groups are all generalized cohomology theories and by stable we mean that those isomorphisms commute with the suspension isomorphism. We should mention about the dimensional restriction in Theorem 2. If we use a homotopy type P_n^∞ / Σ^{-1} (defined in a non-canonical way), we can state that there is an isomorphism (not natural !)

$$\tilde{\pi}^{n,m}(X) \cong \tilde{\pi}^m(X : P_n^\infty / \Sigma^{-1})$$

for any finite trivial $\mathbb{Z}_2$ -complex X .

Let $n = m = 0$, then $P_0^\infty / \Sigma^{-1} \simeq \mathbb{R}P_+^\infty \vee \Sigma^0$. In this case our result is just the theorem of Segal [7]. When $n > 0$ and X is a sphere

similar results are obtained by [7].

Finally we state a conjecture which is seen to be equivalent to the conjecture of Mahowald [1] by using Theorem 1 and 2.

Let $\pi_{n,m} = \tilde{\pi}^{-n,-m}(\Sigma^{0,0})$ be the stable (n, m) -stem. Using the inclusion $i: \Sigma^{p,q} \rightarrow \Sigma^{p+1,q}$, one can define an inverse system $\{\pi_{n,m}\}_n$.

Conjecture. $\lim_{\leftarrow n} \pi_{n,m} = 0$ for all m .

§ 1. Some notations

First we recall some notations. If X is a $\mathbb{Z}_2$ -space, $X^{\mathbb{Z}_2}$ denotes the fixed point subspace. $R^{n,m}$ denotes the representation of $\mathbb{Z}_2$ on R^{n+m} given by

$$\tau(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) = (-x_1, \dots, -x_n, x_{n+1}, \dots, x_{n+m}).$$

$\Sigma^{n,m}$ denotes the one point compactification of $R^{n,m}$. The unit sphere in $R^{r+1,0}$ is a free $\mathbb{Z}_2$ -complex and denoted by S^r . Let X be a based $\mathbb{Z}_2$ -space, then $\Omega^{n,m}X$ denotes the function space $\text{Map}(\Sigma^{n,m}, * : X, *)$ with the compact open topology and the usual $\mathbb{Z}_2$ -action.

The equivariant infinite loop space of X is defined by $Q_{\mathbb{Z}_2}(X) = \varinjlim \Omega^{n,m}(\Sigma^{n,m}X)$ similarly to the non-equivariant case. It is known [2] that if X has a $\mathbb{Z}_2$ -homotopy type of a $\mathbb{Z}_2$ -complex, then so does $\Omega^{n,m}X$ (and hence $Q_{\mathbb{Z}_2}(X)$).

A $\mathbb{Z}_2$ -spectrum $\mathbb{X} = \{X_n, \mathcal{E}_n\}$ is defined [3] by $\mathbb{Z}_2$ -spaces X_n and structure maps $\mathcal{E}_n : \Sigma^{1,1}X_n \rightarrow X_{n+1}$. Given a $\mathbb{Z}_2$ -complex X the suspension spectrum (with a shifted dimension) $\Sigma^k X$ is defined by $(\Sigma^k X)_n = \Sigma^{n+k, n+k} X$ where $k \in \mathbb{Z}$, and is referred to a stable complex and sometimes written as $\Sigma^k X$. A $\mathbb{Z}_2$ -spectrum map (or stable

$\mathbb{Z}_2$ - map) is defined similarly as the non - equivariant case.

Let X and Y be stable complexes. Then the group $\tilde{\mathcal{H}}^{n,m}(X : Y)$ of stable homotopy classes of stable maps of degree (n, m) is defined by

$$\tilde{\mathcal{H}}^{n,m}(X : Y) = \varinjlim_p [\Sigma^{p,p} X, \Sigma^{n+p,m+p} Y]_{\mathbb{Z}_2}.$$

If $(n, m) = (0, 0)$ it is sometimes denoted by $\{X, Y\}_{\mathbb{Z}_2}$.

$\tilde{\mathcal{H}}^{n,m}(X : \Sigma^{0,0})$ is simply denoted by $\tilde{\mathcal{H}}^{n,m}(X)$ and called the (n, m) - dim. Stable $\mathbb{Z}_2$ - cohomotopy group.

Given a $\mathbb{Z}_2$ - spectrum X , we can define the associated

$\mathbb{Z}_2$ - Ω - spectrum QX by $(QX)_n = \varinjlim_q \Omega^{q,q} X_{n+q}$. Note that

$((QX)_n)^{\mathbb{Z}_2} = \varinjlim_q \Omega^q(\Omega^{q,0} X_{n+q})^{\mathbb{Z}_2}$ is an infinite loop space.

Hence the fixed point spectrum $(QX)^{\mathbb{Z}_2}$ is an Ω - spectrum which we

denote by $E(X)$. Evidently we see that $E(\Sigma^{0,n} X) \cong \Sigma^n E(X)$.

The infinite loop space $E(X)_0$ is denoted by $E(X)$.

Clearly E and E are functor. We remark that $E(X)$ is not equi-

valent to the fixed point subspectrum $X^{\mathbb{Z}_2}$. Given a $\mathbb{Z}_2$ - complex X

and $n, m \in \mathbb{Z}$, consider the stable complex $\Sigma^{n,m} X$ (i.e., X with dimen-

sion shifted). If n and m are positive, then $E(\Sigma^{n,m}X)$
 $= Q_{\mathbb{Z}_2}(\Sigma^{n,m}X)^{\mathbb{Z}_2}$. Therefore for negative n or m , we often write as
 $Q_{\mathbb{Z}_2}(\Sigma^{n,m}X)^{\mathbb{Z}_2}$ instead of $E(\Sigma^{n,m}X)$.

Under the above notations the following lemmas are obvious.

Lemma 3. Let Y be a $\mathbb{Z}_2$ - complex and X a $\mathbb{Z}_2$ - complex with the trivial $\mathbb{Z}_2$ - action. Then there is a natural isomorphism.

$$\tilde{\pi}^{n,m}(X : Y) \cong [X, E(\Sigma^{n,m}Y)].$$

Lemma 4. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a $\mathbb{Z}_2$ - cofibration (see [3] for definition) of stable $\mathbb{Z}_2$ - complexes. Then

$$E(X) \xrightarrow{E(f)} E(Y) \xrightarrow{E(g)} E(Z)$$

is a cofibration of spectra. Therefore the sequence

$$E(X) \xrightarrow{E(f)} E(Y) \xrightarrow{E(g)} E(Z)$$

is homotopy equivalent to a fibration.

§ 2. Stunted projective spaces.

Let ξ be the canonical line bundle over the projective space RP^a .

There is a canonical isomorphism

$$KO(RP^a) \cong KO_{\mathbb{Z}_2}(S^a)$$

induced from the projection $S^a \rightarrow RP^a$ (see [4]).

By this isomorphism ξ corresponds to the $\mathbb{Z}_2$ - vector bundle $S^a \times R^{1,0}$.

Therefore the space $S_+^a \wedge_{\mathbb{Z}_2} \Sigma^{n,0} = (S_+^a \wedge \Sigma^{n,0}) / \mathbb{Z}_2$ is identified with

the thom complex $T(n\xi)$. It is well known [4] that $T(n\xi)$ is

homeomorphic to the stunted projective space $P_n^{n+a} = RP^{n+a} / RP^{n-1}$.

It is well known that the order of $\xi - 1 \in \tilde{KO}(RP^a)$ is finite.

Then the following lemma is obvious.

Lemma 5. Let d be a multiple of the order of $\xi - 1 \in KO(RP^a)$.

Then there is a $\mathbb{Z}_2$ - vector bundle isomorphism

$$\eta : S^a \times R^{d,0} \rightarrow S^a \times R^{0,d}.$$

We denote by the same η the induced $\mathbb{Z}_2$ - homeomorphism

$S_+^a \wedge \Sigma^{d,0} \rightarrow S_+^a \wedge \Sigma^{0,d}$, and also the homeomorphism $P_a^{a+d} \rightarrow \Sigma_{\mathbb{Z}_2}^d P_0^a$. Now

for each $a \geq 0$ and $n \in \mathbb{Z}$ choose d as above satisfying $d+n \geq 0$,

and define a stable homotopy type P_n^{n+a} by $\sum^{-d} P_{n+d}^{n+d+a}$

Then by the periodicity η , P_n^{n+a} is well defined. For the stable

complex P_n^{n+a} , define an infinite loop space $Q(P_n^{n+a})$ by $\Omega^d Q(P_{n+d}^{n+d+a})$.

Next we define a stable homotopy type P_n^{n+a} / Σ^{-1} for $n+a \geq -1$.

First let $n \geq 0$, then we define P_n^{n+a} / Σ^{-1} to be $P_n^{n+a} \vee \Sigma^0$, namely

the cofibre of the unique homotopy class $\Sigma^{-1} \rightarrow P_n^{n+a}$, where Σ^m denotes

the m -sphere spectrum. Next let $n < 0$ and put $r = -n > 0$. Let

d be a multiple of the order of $\xi^{-1} \in \widetilde{KO}(RP^a)$, and choose a periodicity

$$\eta : S^a \times R^{d,0} \cong S^a \times R^{0,d}.$$

Clearly it restricts to a periodicity $\eta : S^{r-1} \times R^{d,0} \cong S^{r-1} \times R^{0,d}$.

Note that $a > r$. Let ν be the normal bundle of an embedding

$RP^{r-1} \subset R^N$, N large enough. It is well known that such embeddings

are isotopic to each other. Since we have canonical isomorphisms

$\tau(RP^{r-1}) \oplus \varepsilon^1 \cong r\xi$ and $\tau(RP^{r-1}) \oplus \nu \cong RP^{r-1} \times R^N$, we have a canonical

isomorphism

$$d\xi \oplus \nu \cong (d-r)\xi \oplus (N+1)\varepsilon^1.$$

Then by using $\eta : d\xi \cong d\varepsilon^1$, we have a bundle isomorphism $\eta : \nu \oplus d\varepsilon^1$

$\cong (d-r)\xi \oplus (N+1)\varepsilon^1$. Let $h : \Sigma^N \rightarrow T(\nu)$ be the Pontrjagin -

Thom map of the embedding $\nu \subset \mathbb{R}^N$. By the uniqueness of normal bundles

up to isotopy the homotopy class of h is uniquely determined. Then

define a stable map $u : \Sigma^{-1} \rightarrow P_{-r}^{-r+a} = P_n^{n+a}$ by the composite

$$\Sigma^{N+d} \xrightarrow{h} \Sigma^d \wedge T(\nu) \xrightarrow{\mathcal{P}} \Sigma^{N+1} T((d-r)\mathbb{Z}) = \Sigma^{N+1} P_{d-r}^{d-1} \xrightarrow{i} \Sigma^{N+1} P_{d-r}^{d-r+a}.$$

If we change a periodicity by $\eta' : S^{r+a} \times \mathbb{R}^{d,0} \rightarrow S^{r+a} \times \mathbb{R}^{0,d}$,

then the resulting map u' differs from u by a self homotopy equivalence

of P_{d-r}^{d-r+a} . In fact note that ν is a restriction of a vector bundle

over $\mathbb{R}P^a$.

Then bundle isomorphisms $\mu \oplus \text{id}, \mu' \oplus \text{id} : d\mathbb{Z} \oplus \nu \rightarrow d\mathbb{Z}^1 \oplus \nu$

extend to bundle isomorphisms over $\mathbb{R}P^a$, and hence $(\mu \oplus \text{id}) \circ (\mu' \oplus \text{id})^{-1}$

is a bundle automorphism of $(d-r)\mathbb{Z} \oplus (N+1)\mathbb{Z}^1$ over $\mathbb{R}P^a$. This

shows the required property.

Now let P_n^{n+a}/Σ^{-1} be the cofibre of u . The above argument shows

that the stable homotopy type of P_n^{n+a}/Σ^{-1} does not depend on choices

of d and η . It is obvious that the $n+b$ skeleton $(P_n^{n+a}/\Sigma^{-1})^{(n+b)}$

is homotopy equivalent to P_n^{n+b}/Σ^{-1} . Therefore we can define (not

canonically) a stable homotopy type P_n^∞/Σ^{-1} .

It will be useful to give another description of u .

Let $S^{r-1} \subset S^{r-1} \times \Sigma^{d-r,0}$ be the obvious embedding. The normal bundle is then canonically isomorphic to $S^{r-1} \times R^{d-r,0}$.

Hence the normal bundle $\nu(RP^{r-1}, S^{r-1} \times \Sigma^{d-r,0})$ of the induced embedding $RP^{r-1} \subset S^{r-1} \times \Sigma^{d-r,0}$ is identified with $(d-r)\xi$.

We remark that $\mathcal{L}(S^{r-1} \times \Sigma^{d-r,0}) \oplus \mathcal{E} = S^{r-1} \times \Sigma^{d-r,0} \times R^{d,0}$, and hence by the periodicity μ we have an isomorphism

$$\mathcal{L}(S^{r-1} \times \Sigma^{d-r,0}) \oplus \mathcal{E} \cong d\xi.$$

Thus $S^{r-1} \times \Sigma^{d-r,0}$ is a framed manifold and μ gives a framing.

Then given an embedding $S^{r-1} \times \Sigma^{d-r,0} \subset R^N$, we have the Pontrjagin -

Thom map

$$\Sigma^N \rightarrow \Sigma^{N-d+1}(S^{r-1} \times \Sigma^{d-r,0}_+).$$

It is easy to see that the map u defined above is also given by the composite

$$\begin{aligned} \Sigma^N &\rightarrow \Sigma^{N-d+1}(S^{r-1} \times \Sigma^{d-r,0}_+) \rightarrow \Sigma^{N-d+1}(T((d-r)\xi)) \\ &= \Sigma^{N-d+1} P_{d-r}^{d-1} \hookrightarrow \Sigma^{N-d+1} P_{d-r}^{d-r+a} = \Sigma^{N+1} P_n^{n+a} \end{aligned}$$

where the second map is the Pontrjagin - Thom map of the embedding

$$RP^{r-1} \rightarrow S^{r-1} \times \Sigma^{d-r,0}.$$

§ 3. The space $Q_{\mathbb{Z}_2}(X)^{\mathbb{Z}_2}$

Let X be a finite $\mathbb{Z}_2$ -complex and X_+ denotes X with the disjoint base point. Given a continuous map $f : \Sigma^m \rightarrow X_+^{\mathbb{Z}_2} \wedge \Sigma^m$,

let $e(f)$ be the composite

$$\Sigma^{n,m} \xrightarrow{\Sigma^{n, \text{of}}} X_+^{\mathbb{Z}_2} \wedge \Sigma^{n,m} \subset X_+ \wedge \Sigma^{n,m}$$

This defines a continuous map $e : Q(X_+^{\mathbb{Z}_2}) \rightarrow Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2}$.

Conversely by assigning to a $\mathbb{Z}_2$ -map $f : \Sigma^{n,m} \rightarrow X_+ \wedge \Sigma^{n,m}$

the map $f^{\mathbb{Z}_2}$ of fixed point sets, we obtain a continuous map

$$\psi : Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2} \rightarrow Q(X_+^{\mathbb{Z}_2}).$$

It is obvious that $\psi \circ e = \text{id}$. More precisely we have

Proposition 6. There is a natural homotopy equivalence

$$\lambda : Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2} \rightarrow Q(X_+^{\mathbb{Z}_2}) \times Q((S^\infty \times \mathbb{Z}_2 X)_+).$$

Moreover via λ , the maps e and ψ are homotopic to the canonical inclusion and projection, respectively.

Proof. The existence of λ is shown in [8]. In [8], λ

is defined by a geometric method. That is, we may suppose that X is

a G -manifold and let Y be a manifold. Then any element of

$[Y_+, Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2}]$ is represented by a pair

$$X \xleftarrow{f} E \xrightarrow{h} Y$$

!!

where E is a G -manifold, f is a G -map and h is a framed map. (see [8] for definition).

Then it is known that E is decomposed into a disjoint sum of submanifolds $E_0 \amalg E_1$, where E_0 is trivial and E_1 is free as $\mathbb{Z}_2$ -space.

Then the map λ is induced from this decomposition.

Then checking for a geometric representative, we see that the homomorphism

$$e_* : [Y_+, Q(X_+^{\mathbb{Z}_2})] \longrightarrow [Y_+, Q_{\mathbb{Z}_2}(X_+)^{\mathbb{Z}_2}]$$

coincides with the canonical inclusion. For the map φ the proof is similar.

Now we shall stabilize the above result. Let n and m be positive integers. We have a $\mathbb{Z}_2$ -cofibration

$$X_+ \xrightarrow{i} (X \times \Sigma^{n,m})_+ \xrightarrow{\pi} X_+ \wedge \Sigma^{n,m}$$

and the projection $p : (X \times \Sigma^{n,m})_+ \longrightarrow X_+$ such that $poi = \text{id}_X$.

Applying the functor $Q_{\mathbb{Z}_2}(\)^{\mathbb{Z}_2} = E(\)$, we obtain a fibration (up to equivalence)

$$E(X_+) \xrightarrow{E(i)} E((X \times \Sigma^{n,m})_+) \xrightarrow{E(\pi)} E(X_+ \wedge \Sigma^{n,m}).$$

It is obvious that the fibration is trivial, and by using the map $E(p)$ we can define a canonical splitting

$$s : E(X_+ \wedge \Sigma^{n,m}) \longrightarrow E((X \times \Sigma^{n,m})_+).$$

Since the homotopy equivalence of Proposition 6 is natural, we easily see the following generalization of Proposition 6.

Lemma 7. Let n and m be positive integers, then there exists a natural homotopy equivalence

$$\lambda : E(X_+ \wedge \Sigma^{n,m}) \longrightarrow Q(X_+ \wedge \Sigma^m) \times Q((X \times S^\infty)_+ \wedge \Sigma^{n,m}).$$

Lemma 8. The map λ is an infinite loop map.

Proof. We are enough to show that the following diagram is commutative.

$$\begin{array}{ccc} [Y_+, E(X_+)] & \xrightarrow{\lambda_*} & [Y_+, Q(X_+ \wedge \Sigma^m)] \oplus [Y_+, Q((X \times S^\infty)_+ \wedge \Sigma^{n,m})] \\ \sigma \downarrow \cong & & \cong \downarrow \sigma \\ [Y_+ \wedge \Sigma^{o,n}, E(X_+ \wedge \Sigma^{o,n})] & \xrightarrow{\lambda_*} & [Y_+ \wedge \Sigma^n, Q(X_+ \wedge \Sigma^m)] \oplus [Y_+ \wedge \Sigma^n, Q((X \times S^\infty)_+ \wedge \Sigma^{o,n})], \end{array}$$

where σ is the suspension isomorphism. We may suppose that X and

Y are manifolds as before, and we can take a pair

$$x = (X \xleftarrow{f} E \xrightarrow{h} Y) \text{ for a representative of an element of } [Y_+, E(X_+)].$$

Note that $h \times \text{id} : E \times \Sigma^{o,n}$ is canonically framed. Hence define a

homomorphism

$$\bar{\sigma} : [Y_+, E(X_+)] \longrightarrow [(Y \times \Sigma^{0,n})_+, E((X \times \Sigma^{0,n})_+)]$$

by $\bar{\sigma}(x) = (X \times \Sigma^{0,n} \xleftarrow{f \times \text{id}} E \times \Sigma^{0,n} \xrightarrow{h \times \text{id}} Y \times \Sigma^{0,n})$. Then we easily see that the diagram

$$\begin{array}{ccc} & & [Y_+ \wedge \Sigma^{0,n}, E(X_+ \wedge \Sigma^{0,n})] \\ & \nearrow \sigma & \downarrow j \\ [Y_+, E(X_+)] & & \\ & \searrow \bar{\sigma} & \downarrow \\ & & [(Y \times \Sigma^{0,n})_+, E((X \times \Sigma^{0,n})_+)] \end{array}$$

is commutative, where j is the split monomorphism induced from the map s . Then the commutativity of the rectangle diagram is shown easily for a geometric representative $(X \xleftarrow{f} E \xrightarrow{h} Y)$. q. e. d.

Given a CW - complex X , let $\underline{\Sigma} X$ denotes the suspension spectrum.

Then from the above lemmas, we obtain a homotopy equivalence of spectra

$$\bar{\lambda} : E(X_+ \wedge \Sigma^{n,m}) \simeq \underline{\Sigma}(X_+ \wedge \Sigma^m) \vee \underline{\Sigma}((X \times S^\infty)_+ \wedge \Sigma^{n,m}).$$

Let $n(>0)$ and m be integers. Then

Proposition 9. There exists an equivalence of spectra

$$\bar{\lambda} : E(X_+ \wedge \Sigma^{n,m}) \simeq \underline{\Sigma}(X_+ \wedge \Sigma^m) \vee \underline{\Sigma}((X \times S^\infty)_+ \wedge \Sigma^{n,m})$$

§ 4. Proof of the theorems

In this section we prove Theorem 1 and Theorem 2 simultaneously.

By Lemma 3 we have natural isomorphisms

$$\tilde{\pi}^{n,m}(X : S_+^q) \cong [X, E(\Sigma^{n,m} S_+^q)]$$

and

$$\tilde{\pi}^{n,m}(X) \cong [X, E(\Sigma^{n,m})].$$

Thus the problem is to determine those spectra $E(\Sigma^{n,m})$ and $E(\Sigma^{n,m} S_+^q)$.

for any $n, m \in \mathbb{Z}$.

First we suppose that $n \geq 0$. Note that $(S^q)^{\mathbb{Z}_2} = \phi$ and $S^q \times S^\infty$ is $\mathbb{Z}_2$ -homotopy equivalent to S^q . Then by Lemma 7 and Lemma 8, we easily see that

$$E(\Sigma^{n,m} S_+^q) \simeq \Sigma(\Sigma^m P_n^{n+q})$$

and

$$E(\Sigma^{n,m}) \simeq \Sigma(\Sigma^m) \vee \Sigma(\Sigma^m P_n^*)$$

as spectra for any $m \in \mathbb{Z}$. This immediately implies the theorems for $n \geq 0$.

Next suppose that $n < 0$, and put $r = -n > 0$.

Given a positive integer q , let d be an integer ($d \geq r$) and

$\mu : \Sigma^{o,d,q}_{+} \longrightarrow \Sigma^{d,o,q}_{+}$ be a periodicity as in Lemma 5. Let N, M be

integers large enough. Given a $\mathbb{Z}_2$ - map $f : \Sigma^{N+r,M}_{+} \longrightarrow \Sigma^{N,M+m,q}_{+}$,

let $\mu^*(f)$ be the composite

$$\Sigma^{N+r,M}_{+} \xrightarrow{f} \Sigma^{N,M+m,q}_{+} \xrightarrow{\mu} \Sigma^{N+d,M-d+m,q}_{+}.$$

Then we obtain an isomorphism (periodicity) of spectra.

$$\mu^* : E(\Sigma^{n,m,q}_{+}) \longrightarrow E(\Sigma^{n+d,m-d,q}_{+}).$$

Since $n + d \geq 0$, we can reduce to the first case and we have

$$E(\Sigma^{n+d,m-d,q}_{+}) \simeq E(\Sigma^{m-d,n+d+q}_{P_{n+d}}) = E(\Sigma^{m,n+q}_{P_n}).$$

This shows Theorem 1 for $n < 0$.

Next let n, m and q be as above, and suppose that $n + q > 0$.

From the standard $\mathbb{Z}_2$ - cofibration $S^{q}_{+} \xrightarrow{p} \Sigma^{o,o} \xrightarrow{i} \Sigma^{q+1,o} \xrightarrow{\pi} \Sigma^{o,1,q}_{+}$,

where P is the unique non-trivial map, i is the standard inclusion

and $\pi : \Sigma^{q+1,0} \longrightarrow \Sigma^{q+1,0}/\Sigma^{0,0} \cong \Sigma^{0,1,q}_{+}$ is the projection, we obtain

a stable $\mathbb{Z}_2$ - cofibration

$$\Sigma^{n,m,q}_{+} \xrightarrow{p} \Sigma^{n,m} \xrightarrow{i} \Sigma^{n+q+1,m} \xrightarrow{c} \Sigma^{n,m+1,q}_{+}.$$

Then by Lemma 4 we obtain a stable cofibration

$$E(\Sigma^{n+q+1,m-1}) \xrightarrow{E(c)} E(\Sigma^{n,m,q}_{+}) \xrightarrow{E(p)} E(\Sigma^{n,m}).$$

Choose d such as $n + d \geq 0$ and a periodicity

$$\mu_* : E(\Sigma^{n,m} S_+^q) \cong E(\Sigma^{n+d,m-d} S_+^q).$$

$$\text{Recall that } E(\Sigma^{n+d,m-d} S_+^q) = \Sigma(\Sigma^{m-d} P_{n+d}^{n+d+q}) = \Sigma(\Sigma^m P_n^{n+q}).$$

By assumption $n + q + 1 > 0$ and by Lemma 8 we have an equivalence of spectra

$$\bar{\lambda} : E(\Sigma^{n+q+1,m-1}) \simeq \Sigma(\Sigma^{m-1}) \vee \Sigma(\Sigma^{m-1} P_{n+q+1}^\infty)$$

Using the equivalences μ_* and $\bar{\lambda}$, the map $E(c)$ is homotopic to a

$$\text{map } w : \Sigma(\Sigma^{m-1}) \vee \Sigma(\Sigma^{m-1} P_{n+q+1}^\infty) \longrightarrow \Sigma(\Sigma^m P_n^{n+q}). \quad \text{Let}$$

$$u' : \Sigma(\Sigma^{m-1}) \longrightarrow \Sigma(\Sigma^m P_n^{n+q})$$

be the restriction of w to $\Sigma(\Sigma^{m-1})$. Note that $\Sigma(\Sigma^{m-1} P_{n+q+1}^\infty)$ is

$n + m + q$ - connected. Hence the cofibre of u' is $n + m + q$ - equivalent to $E(\Sigma^{n,m})$.

Then Theorem 2 for $n < 0$ follows immediately from the following

Lemma 9 The cofibre of u' is stably homotopy equivalent to

$$\Sigma^m(P_n^{n+q}/\Sigma^{-1}) \text{ of } \S 2.$$

Proof. We may suppose that $m = 0$. Since we have defined u' using

a periodicity $\mu : S^q \times R^{d,0} \longrightarrow S^q \times R^{0,d}$, it is enough to show that

$u' : \Sigma(\Sigma^{-1}) \longrightarrow \Sigma(P_n^{n+q})$ is homotopic to u for the same choice of μ .

Let

$$\beta = (u')_* : \tilde{\pi}^{-1}(Y_+) \longrightarrow \tilde{\pi}^0(Y_+, P_n^{n+q})$$

be the induced homomorphism. Let $r = -n > 0$. and let

$z \in \tilde{\pi}^{-r,1}(\Sigma^{0,0} ; S_+^q)$ be the class of the composite

$$\Sigma^{r,0} \xrightarrow{\pi} \Sigma^{0,1} S_+^{r-1} \subset \Sigma^{0,1} S_+^q.$$

By the smash product we have a pairing

$$\wedge : \tilde{\pi}^{a,b}(X : Y) \otimes \tilde{\pi}^{a',b'}(X' : Y') \longrightarrow \tilde{\pi}^{a+a',b+b'}(X \wedge X' : Y \wedge Y').$$

Let j be the composite

$$\begin{aligned} \tilde{\pi}^{-r,1}(\Sigma^{0,0} : S_+^q) \otimes \tilde{\pi}^0(Y_+) &\xrightarrow{\text{id} \otimes e} \tilde{\pi}^{-r,1}(\Sigma^{0,0} : S_+^q) \otimes \tilde{\pi}^0(Y_+) \\ &\xrightarrow{\wedge} \tilde{\pi}^{-r,1}(Y_+, S_+^q) \xrightarrow[\cong]{\mu} \tilde{\pi}^{d-r,1-d}(Y_+, S_+^q) \xrightarrow{\wedge} \tilde{\pi}(Y_+, P_n^{n+q}). \end{aligned}$$

Then we see that $\beta(x) = j(z \otimes x)$, and we are enough to show that

$j(Z \otimes 1) = \{u\}$. We remark that the element Z is given by the Pontrjagin

- Thom construction of the pair $(* \longleftarrow S^{r-1} \hookrightarrow S^q)$, where the unique map $S^r \longrightarrow *$ is, say, $(r, -1)$ - framed.

That is, $\mathcal{U}(S^{r-1}) \otimes \mathcal{E} = S^{r-1} \times R^{r,0}$. Then as in § 2, $S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,0}$

is a framed manifold by e of the periodicity μ . Then by the definition

of λ we easily see that the class $j(Z \otimes 1)$ is given by the composite

$$\begin{aligned} \Sigma^N &\xrightarrow{c} \Sigma^{N-d+1}((S^{r-1} \times_{\mathbb{Z}_2} \Sigma^{d-r,o})_+) \xrightarrow{i} \Sigma^{N-d+1}((S^q \times_{\mathbb{Z}_2} \Sigma^{d-r,o})_+) \\ &\xrightarrow{\pi} \Sigma^{N-d+1}(S^{q+1}_{+1} \times_{\mathbb{Z}_2} \Sigma^{d-r,o}) = \Sigma^{N+1}(P^{n+q}_n) \end{aligned}$$

where c is the Pontrjagin - Thom map, and i and π are obvious maps.

Now by the second description of the map u , we easily see that

$$\{u'\} = r(Z\emptyset 1) = \{u\}.$$

This completes the proof.

References

- [1] J. F. Adams: Operations of the n -th kind in K -theory and what we don't know about RP^∞ , New developments in topology, London Math. Soc. Lecture Note 11(1974).
- [2] S. Araki - M. Murayama: G -homotopy types of G -complexes and representation of G -cohomology theories, Publ. RIMS, Kyoto Univ. 14(1978), 203-222.
- [3] S. Araki - M. Murayama: $\mathbb{T}$ -cohomology theories, Japan. J. Math. 4(1978), 363-416.
- [4] M. F. Atiyah: Thom complexes, Proc. London Math. Soc. 11(1961), 291-310.
- [5] G. Bredon: Equivariant homotopy, Proc. Conference on Transformation Groups, 1967, Springer-Verlag 1968, 281-292.
- [6] C. Kosniowski: Equivariant cohomology and stable cohomotopy, Math. Ann. 210(1974), 83-104.
- [7] P. S. Landweber: On equivariant maps between spheres with involutions, Ann. of Math. 89(1969), 125-137.

- [8] G. Nishida: On the equivariant Barratt-Priddy-Quillen theorem,
to appear.
- [9] G. B. Segal: Equivariant stable homotopy theory, Actes. Congress
Intern. Math. 2(1970), 59-63.