

Schrödinger equations on scattering manifolds and microlocal singularities

By

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§ 1. Scattering theory on scattering manifold

We take a two-space approach to the scattering theory on a noncompact manifold called *scattering manifold*. Let M be a noncompact manifold with cylindrical ends with base manifold N ;

$$M = M_0 \cup M_\infty : \quad M_0 \Subset M, \quad M_\infty \cong (0, \infty) \times N, \quad ((0, 1) \times N \hookrightarrow M_0).$$

In what follows we denote $N = \partial M$, since N gives the topological boundary of M at infinity. We assume ∂M is a closed manifold. We put an asymptotically Euclidean metric on M . Let (r, θ) be local coordinates on $M_\infty \cong (0, \infty) \times \partial M$.

Definition 1.1. The Riemannian manifold (M, g^{sc}) with M as above is called scattering manifold if there exists a Riemannian metric g^∂ on ∂M such that

$$m = g^{\text{sc}} - (dr^2 + r^2 g_{jk}^\partial d\theta^j d\theta^k)$$

satisfies for some $\mu > 0$

$$\begin{aligned} m &= m^0(r, \theta) dr^2 + r m_j^1(r, \theta) (dr d\theta^j + d\theta^j dr) + r^2 m_{jk}^2(r, \theta) d\theta^j d\theta^k, \\ |\partial_r^j \partial_\theta^\alpha m^l(r, \theta)| &\lesssim r^{-\mu-j}, \quad (r, \theta) \in (1, \infty) \times \partial M, \quad l = 0, 1, 2. \end{aligned}$$

Set

$$M_{\text{fr}} = \mathbb{R} \times \partial M,$$

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and we develop the two-space scattering theory for

$$H_{\text{fr}} = -\partial_r^2$$

on $\mathcal{H}_{\text{fr}} = L^2(M_{\text{fr}}, \sqrt{G_{\partial}} dr d\theta)$, $G_{\partial} = \det(g_{jk}^{\partial})$, and

$$H = -\Delta_{\text{sc}} + V$$

on $\mathcal{H} = L^2(M, \sqrt{G_{\text{sc}}} dx)$, $G_{\text{sc}} = \det(g_{jk}^{\text{sc}})$, where Δ_{sc} is the Laplace-Beltrami operator on M . Note that H_{fr} is not derived from any Riemannian structure on M_{fr} . As in the scattering theory on the Euclidean space, we have to put a short-range type assumption on H . For the potential V it would be natural to assume $V \in C^\infty(M; \mathbb{R})$ and

$$|\partial_r^j \partial_\theta^\alpha V(r, \theta)| \lesssim r^{-1-\mu-j} \quad (\text{subcoulomb}).$$

This is exactly what the condition $|\partial_x^\alpha V(x)| \lesssim \langle x \rangle^{-1-\mu-|\alpha|}$, $x \in \mathbb{R}^n$ implies for the polar coordinates on the Euclidean space. For the metric g^{sc} it automatically follows from the definition that the dual metric g_{sc} on T^*M is of the form

$$\begin{aligned} g_{\text{sc}}^{jk} &= \partial_r \otimes \partial_r + \frac{1}{r^2} g_{\partial}^{jk} \partial_{\theta^j} \otimes \partial_{\theta^k} \\ &\quad + a_0 \partial_r \otimes \partial_r + \frac{1}{r} a_1^j (\partial_r \otimes \partial_{\theta^j} + \partial_{\theta^j} \otimes \partial_r) + \frac{1}{r^2} a_2^{jk} \partial_{\theta^j} \otimes \partial_{\theta^k} \end{aligned}$$

on T^*M_∞ with

$$(1.1) \quad |\partial_r^j \partial_\theta^\alpha a_l(r, \theta)| \lesssim r^{-\mu-j}, \quad l = 0, 1, 2.$$

We assume that g^{sc} is *radially short-range* in the sense that it satisfies, in addition,

$$(1.2) \quad |\partial_r^j \partial_\theta^\alpha a_0(r, \theta)| \lesssim r^{-1-\mu-j}.$$

In contrast to the potential condition, this radially short-range condition is a little weaker than the usual short-range condition on the Euclidean space that suggests $|\partial_r^j \partial_\theta^\alpha a_l(r, \theta)| \lesssim r^{-1-\mu-j}$ for $l = 0, 1, 2$. The reason why we can weaken the assumption is that the conservation of angular momentum is available.

Define the smooth cutoff $J: \mathcal{H}_{\text{fr}} \rightarrow \mathcal{H}$ by

$$(Ju)(x) = \begin{cases} j(r(x)) [G_{\partial}(\theta(x))/G_{\text{sc}}(x)]^{1/4} u(r(x), \theta(x)), & \text{if } x \in M_\infty, \\ 0, & \text{if } x \notin M_\infty, \end{cases}$$

where $j \in C^\infty((0, \infty))$ is chosen to satisfy $j(r) = 1$ for $r \geq 2$ and $j(r) = 0$ for $r \leq 1$. Note that the factor $[G_{\partial}(\theta(x))/G_{\text{sc}}(x)]^{1/4}$ makes J unitary on $(2, \infty) \times \partial M$.

Theorem 1.2 ([3]). *Let (M, g^{sc}) be a scattering manifold of radially short-range type, and V a smooth subcoulomb potential. Then the wave operators*

$$W_{\pm} = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} J e^{-itH_{\text{fr}}} : \mathcal{H}_{\text{fr}} \rightarrow \mathcal{H},$$

exists, and they are partial isometries with initial spaces

$$\mathcal{H}_{\text{fr}, \pm} = \{u \in \mathcal{H}_{\text{fr}}; \text{supp } \mathcal{F}_{\text{fr}} u \subset \mathbb{R}_{\pm} \times \partial M\}, \quad \mathbb{R}_{\pm} = \{\rho \in \mathbb{R}; \pm\rho \geq 0\},$$

respectively, where $\mathcal{F}_{\text{fr}}$ is the Fourier transform in the radial direction:

$$(\mathcal{F}_{\text{fr}} u)(\rho, \theta) = \int e^{-ir\rho} u(r, \theta) dr.$$

Moreover, $W_{\pm}$ are complete, i.e., $\text{Ran } W_{\pm} = \mathcal{H}_{\text{ac}}(H)$. Hence the scattering operator $S = W_{+}^{} W_{-}$ is unitary as $\mathcal{H}_{\text{fr}, -} \rightarrow \mathcal{H}_{\text{fr}, +}$.*

The proof of Theorem 1.2 is similar to the standard one on the Euclidean space except for small modifications; The existence of the wave operator follows from the Cook-Kuroda method; Apply the Mourre theory, using the conjugation operator

$$A = \frac{1}{2i} \left(jr \frac{\partial}{\partial r} + \frac{\partial}{\partial r} jr + jr \frac{1}{2} \frac{\partial \log G_{\text{sc}}}{\partial r} \right),$$

and we obtain the limiting absorption principle; Then the abstract stationary theory ensures the completeness of the wave operators. In applying the abstract stationary theory, we encounter a difficulty that comes from the H_{fr} -unboundedness of the operator H , but it can be eluded by taking a smaller subspace than weighted L^2 space. We omit the detail here.

The restrictions

$$\mathcal{F}_{\text{fr}, \pm} = \mathcal{F}_{\text{fr}}|_{\mathcal{H}_{\text{fr}, \pm}} : \mathcal{H}_{\text{fr}, \pm} \rightarrow L^2(\mathbb{R}_{\pm}, \mathcal{H}_{\partial}, d\rho), \quad \mathcal{H}_{\partial} = L^2(\partial M, \sqrt{G_{\partial}} d\theta), \quad d\rho = \frac{d\rho}{2\pi}$$

are unitary, and they give the *spectral representations* for $H_{\text{fr}}|_{\mathcal{H}_{\text{fr}, \pm}}$:

$$(\mathcal{F}_{\text{fr}, \pm} H_{\text{fr}} u)(\rho) = \rho^2 (\mathcal{F}_{\text{fr}, \pm} u)(\rho), \quad u \in D(H_{\text{fr}}) \cap \mathcal{H}_{\text{fr}, \pm}.$$

Thus, from the general theory, we have the *S-matrix*:

Theorem 1.3. *For a.e. $\rho \in \mathbb{R}_{+}$ there exists a unitary operator, so-called *S-matrix*,*

$$\hat{S}(\rho) : \mathcal{H}_{\partial} \rightarrow \mathcal{H}_{\partial}$$

satisfying

$$(\mathcal{F}_{\text{fr}, +} S \mathcal{F}_{\text{fr}, -}^{*} f)(\rho) = \hat{S}(\rho) f(-\rho), \quad f \in L^2(\mathbb{R}_{-}, \mathcal{H}_{\partial}, d\rho).$$

§ 2. Classical trajectories on scattering manifold

§ 2.1. Classical wave operators without potential term

Define the (*inverse of the*) *classical wave operators of finite time* by

$$w_{\text{sc},t}^* = \exp(-tH_{K_{\text{fr}}}) \circ J_{\text{cl}}^* \circ \exp tH_{K_{\text{sc}}},$$

where K_{fr} and K_{sc} are the classical Hamiltonians:

$$K_{\text{fr}}(r, \rho, \theta, \omega) = \rho^2, \quad K_{\text{sc}}(x, \xi) = \sum_{j,k=1}^n g_{\text{sc}}^{jk}(x) \xi_j \xi_k,$$

for $(r, \rho, \theta, \omega) \in T^*M_{\text{fr}}$ and $(x, \xi) \in T^*M$, respectively, and

$$J_{\text{cl}}: T^*M_{\text{fr}} \supset T^*((0, \infty) \times \partial M) \xrightarrow{\cong} T^*M_{\infty} \subset T^*M, \quad J_{\text{cl}}^* = (J_{\text{cl}})^{-1}.$$

Then the limits

$$w_{\text{sc},\pm}^* = \lim_{t \rightarrow \pm\infty} w_{\text{sc},t}^*: T^*M \setminus \mathcal{T}_{\text{sc},\pm} \rightarrow T^*M_{\text{fr}} \setminus \mathcal{T}_{\text{fr},\pm}$$

exist and are diffeomorphisms, where

$$\begin{aligned} \mathcal{T}_{\text{sc},\pm} &= \{(x, \xi) \in T^*M; \{\exp tH_{K_{\text{sc}}}(x, \xi); \pm t \geq 0\} \Subset T^*M\}, \\ \mathcal{T}_{\text{fr},\pm} &= \{(r, \rho, \theta, \omega) \in T^*M_{\text{fr}}; \pm \rho \leq 0\}. \end{aligned}$$

We note that $\mathcal{T}_{\text{sc},\pm}$ and $\mathcal{T}_{\text{fr},\pm}$ are closed sets.

Since the Hamiltonians are homogeneous of degree 2 in the fiber variable, we have, λ^{-1} denoting the multiplication in fibers,

$$w_{\text{sc},\lambda t}^*(x, \xi) = \lambda^{-1} w_{\text{sc},t}^*(x, \lambda \xi), \quad \lambda > 0,$$

as long as they are well-defined. Thus we note that the classical wave operators coincide with the high energy limit of the classical wave operators of finite time:

$$w_{\text{sc},\pm}^*(x, \xi) = \lim_{\lambda \rightarrow \infty} \lambda^{-1} w_{\text{sc},t}^*(x, \lambda \xi), \quad \pm t > 0.$$

In particular the $w_{\text{sc},\pm}^*$ is homogeneous in ξ , and the canonical relations

$$\mathcal{C}_{\text{sc},\pm} = \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus \mathcal{T}_{\text{sc},\pm}) \times (T^*M_{\text{fr}} \setminus \mathcal{T}_{\text{fr},\pm}); (r, \rho, \theta, \omega) = w_{\text{sc},\pm}^*(x, \xi)\},$$

are conic.

§ 2.2. High energy limit of classical wave operators with sublinear potentials

We consider

$$w_t^* = \exp(-tH_{K_{\text{fr}}}) \circ J_{\text{cl}}^* \circ \exp tH_K, \quad K(x, \xi) = \sum_{j,k=1}^n g_{\text{sc}}^{jk}(x) \xi_j \xi_k + V,$$

here allowing the potential V to grow sublinearly:

$$(2.1) \quad |\partial_r^j \partial_\theta^\alpha V(r, \theta)| \lesssim r^{1-\mu-j}.$$

Since V may grow at infinity in every direction, $\lim_{t \rightarrow \pm\infty} w_t^*$ might not exist on any subset of T^*M except for the zero section 0. But the high energy limit exists:

Theorem 2.1. *Let (M, g^{sc}) be a scattering manifold of radially short-range type, and V a smooth sublinear potential. For any $(x_0, \xi^0) \in T^*M \setminus \mathcal{T}_{\text{sc}, \pm}$ and $\pm t > 0$ the following limits in the right-hand side converges in C^∞ -topology, and the equalities hold:*

$$(2.2) \quad w_{\text{sc}, \pm}^*(x_0, \xi^0) = \lim_{\lambda \rightarrow \pm\infty} \lambda^{-1} w_t^*(x_0, \lambda \xi^0).$$

Moreover, if $\pm t > 0$ is fixed, then

$$w_{\text{sc}, \pm}^* = (r_{\text{sc}, \pm}, \rho^{\text{sc}, \pm}, \theta_{\text{sc}, \pm}, \omega^{\text{sc}, \pm}), \quad w_t^* = (r_t, \rho^t, \theta_t, \omega^t)$$

satisfy locally in $(x_0, \xi^0/|\xi^0|) \in T^*M \setminus \mathcal{T}_{\text{sc}, \pm}$ and for large $|\xi^0|$

$$(2.3) \quad \begin{aligned} |\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (r_{\text{sc}, \pm} - r_t)| &\leq C \langle \xi^0 \rangle^{-\mu-|\alpha|}, \\ |\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (\rho^{\text{sc}, \pm} - \rho^t)| &\leq C \langle \xi^0 \rangle^{1-\mu-|\alpha|}, \\ |\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (\theta_{\text{sc}, \pm} - \theta_t)| &\leq C \langle \xi^0 \rangle^{-\mu-|\alpha|}, \\ |\partial_{x_0}^\alpha \partial_{\xi^0}^\beta (\omega^{\text{sc}, \pm} - \omega^t)| &\leq C \langle \xi^0 \rangle^{1-\mu-|\alpha|}. \end{aligned}$$

From the estimates 2.3 it follows that the canonical relation

$$\mathcal{C}_t = \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus \mathcal{T}_{\text{sc}, \pm}) \times (T^*M_{\text{fr}} \setminus \mathcal{T}_{\text{fr}, \pm}); (r, \rho, \theta, \omega) = w_t^*(x, \xi)\}$$

is not necessarily conic but *asymptotically conic* with asymptotes $\mathcal{C}_{\text{sc}, \pm}$ for $\pm t > 0$, respectively.

§ 2.3. Classical wave operators at infinity

We study the asymptotics of the classical wave operator $w_{\text{sc}, \pm}$ (without potential term) at spatial infinity, that is,

$$(2.4) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-1} w_{\text{sc}, \pm}^*(\lambda r, \rho, \theta, \lambda \omega),$$

where the multiplication λ^{-1} here acts on the (r, ω) -variables. Since the scattering metric approaches the underlying conic metric $g^{\text{cn}} = dr^2 + r^2 g_{jk}^\partial d\theta^j d\theta^k$, the limits (2.4) would approach

$$w_{\text{cn}, \pm}^* = \lim_{t \rightarrow \pm\infty} \exp(-tH_{K_{\text{fr}}}) \circ J_{\text{cl}}^* \circ \exp tH_{K_{\text{cn}}}, \quad K_{\text{cn}}(x, \xi) = \sum g_{\text{cn}}^{jk}(x) \xi_j \xi_k,$$

respectively. Precisely,

Theorem 2.2. *Let (M, g^{sc}) be a scattering manifold of radially short-range type. For any*

$$(r_0, \rho^0, \theta_0, \omega^0) \in \mathcal{U}_0 = \{(r, \rho, \theta, \omega) \in T^*M_\infty; r > 0, \omega \neq 0\},$$

the following limits in the right-hand side converge in C^∞ -topology, and the equalities hold:

$$w_{\text{cn}, \pm}^*(r_0, \rho^0, \theta_0, \omega^0) = \lim_{\lambda \rightarrow \infty} \lambda^{-1} w_{\text{sc}, \pm}^*(\lambda r_0, \rho^0, \theta_0, \lambda \omega^0).$$

Moreover, if one denotes

$$w_{\text{cn}, \pm}^* = (r_{\text{cn}, \pm}, \rho^{\text{cn}, \pm}, \theta_{\text{cn}, \pm}, \omega^{\text{cn}, \pm}), \quad w_{\text{sc}, \pm}^* = (r_{\text{sc}, \pm}, \rho^{\text{sc}, \pm}, \theta_{\text{sc}, \pm}, \omega^{\text{sc}, \pm}),$$

then, locally in $(r_0/|(r_0, \omega^0)|, \rho^0, \theta_0, \omega^0/|(r_0, \omega^0)|) \in \mathcal{U}_0$,

$$\begin{aligned} |\partial_{r_0}^\alpha \partial_{\rho^0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega^0}^\delta (r_{\text{cn}, \pm} - r_{\text{sc}, \pm})| &\leq C |(r_0, \omega^0)|^{1-\mu-|\alpha|-|\delta|}, \\ |\partial_{r_0}^\alpha \partial_{\rho^0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega^0}^\delta (\rho^{\text{cn}, \pm} - \rho^{\text{sc}, \pm})| &\leq C |(r_0, \omega^0)|^{-\mu-|\alpha|-|\delta|}, \\ |\partial_{r_0}^\alpha \partial_{\rho^0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega^0}^\delta (\theta_{\text{cn}, \pm} - \theta_{\text{sc}, \pm})| &\leq C |(r_0, \omega^0)|^{-\mu-|\alpha|-|\delta|}, \\ |\partial_{r_0}^\alpha \partial_{\rho^0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega^0}^\delta (\omega^{\text{cn}, \pm} - \omega^{\text{sc}, \pm})| &\leq C |(r_0, \omega^0)|^{1-\mu-|\alpha|-|\delta|} \end{aligned}$$

hold for large $|(r_0, \omega^0)|$.

Note that $w_{\text{cn}, \pm}^*$ is explicitly computed and is a diffeomorphism as

$$\begin{aligned} \mathcal{U}_0 &= \{(r, \rho, \theta, \omega) \in T^*M_\infty; r > 0, \omega \neq 0\} \\ &\rightarrow \mathcal{U}_{\text{fr}, \pm} = \{(r, \rho, \theta, \omega) \in T^*M_{\text{fr}}; \pm\rho > 0, \omega \neq 0\}. \end{aligned}$$

We define the *classical scattering operators* analogously to the scattering operator $S = W_+^* W_-$ by

$$\begin{aligned} s_{\text{sc}} &= w_{\text{sc}, +}^* \circ w_{\text{sc}, -} : T^*M_{\text{fr}} \setminus \mathcal{T}_{\text{fr}, -} \rightarrow T^*M_{\text{fr}} \setminus \mathcal{T}_{\text{fr}, +}, & w_{\text{sc}, -} &= (w_{\text{sc}, -}^*)^{-1}, \\ s_{\text{cn}} &= w_{\text{cn}, +}^* \circ w_{\text{cn}, -} : \mathcal{U}_{\text{fr}, -} \rightarrow \mathcal{U}_{\text{fr}, +}, & w_{\text{cn}, -} &= (w_{\text{cn}, -}^*)^{-1}. \end{aligned}$$

Since we have the explicit formula:

$$s_{\text{cn}}(r, \rho, \theta, \omega) = (-r, -\rho, \exp \pi H_{\sqrt{K_{\partial}}}(\theta, \omega)), \quad K_{\partial}(\theta, \omega) = \sum_{j,k=1}^{n-1} g_{\partial}^{jk}(\theta) \omega_j \omega_k,$$

the canonical relation

$$\begin{aligned} \hat{\mathcal{D}}_{\text{cn}} &= \{(\rho, r, \theta, \omega; \rho', r', \theta', \omega') \in (T^* \hat{M}_{\text{fr},+} \setminus 0) \times (T^* \hat{M}_{\text{fr},-} \setminus 0); \\ &\quad (r, \rho, \theta, \omega) = s_{\text{cn}}(r', \rho', \theta', \omega'), \omega \neq 0, \omega' \neq 0\}, \end{aligned}$$

where $\hat{M}_{\text{fr},\pm} = \{\pm\rho > 0\} \times \partial M$, is (r, ω) -conic. Theorem 2.2 implies that the canonical relation

$$\begin{aligned} \hat{\mathcal{D}}_{\text{sc}} &= \{(\rho, r, \theta, \omega; \rho', r', \theta', \omega') \in (T^* \hat{M}_{\text{fr},+} \setminus 0) \times (T^* \hat{M}_{\text{fr},-} \setminus 0); \\ &\quad (r, \rho, \theta, \omega) = s_{\text{sc}}(r', \rho', \theta', \omega'), \omega \neq 0, \omega' \neq 0\}, \end{aligned}$$

is asymptotically (r, ω) -conic with asymptote $\hat{\mathcal{D}}_{\text{cn}}$.

§ 3. Microlocal structure of the wave operators

In this and the following section we state the theorems concerning the microlocal structure of the wave operators and the S-matrix. The wave front set of $u \in \mathcal{S}'(\mathbb{R}^n)$ is characterized as follows: Let $(x_0, \xi^0) \in T^*\mathbb{R}^n \setminus 0 \cong \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Then $(x_0, \xi^0) \notin \text{WF}(u)$ is equivalent to that for some $a \in C_0^\infty(T^*\mathbb{R}^n)$ we have

$$a(x_0, \xi^0) \neq 0, \quad \|a^w(x, hD_x)u\|_{L^2} = O(h^\infty) (= O(h^N) \text{ for any } N > 0) \text{ as } h \downarrow 0,$$

where

$$a^w(x, hD_x)u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

Hence the wave front set is the set of directions in the phase space $T^*\mathbb{R}^n$ in which the function u is decaying rapidly. Note that u is a function of *only* x , but the fact that the wave front set is well-defined means that we may consider u as a function of x and ξ modulo small errors for large $|\xi|$. The wave front set for a function on a manifold is characterized similarly by using the local coordinates.

We let $I_\rho^m(M, N; \mathcal{C})$ be the set of Fourier integral operators from functions on N to those on M that have amplitudes in $S_\rho^m = S_{\rho, 1-\rho}^m$ and a canonical relation $\mathcal{C}$. Note that, in general, Fourier integral operators move the wave front set around according to the associated canonical relations: If $\mathcal{A}$ is a Fourier integral operator and $\mathcal{C}$ is the associated canonical relation, then we have

$$\text{WF}(\mathcal{A}u) \subset \mathcal{C} \circ \text{WF}(u) = \{(x, \xi); \exists (y, \eta) \text{ s.t. } (x, \xi, y, \eta) \in \mathcal{C}\}$$

Theorem 3.1 ([5], cf. [4]). *Suppose (M, g^{sc}) is a scattering manifold of radially short-range type, and V is a smooth sublinear potential, and let μ be as in (1.1), (1.2) and (2.1). Then for $u \in \mathcal{H}_{\text{fr}}$ and $\pm t > 0$*

$$(3.1) \quad \text{WF}(W_t u) \setminus \mathcal{T}_{\text{sc}, \pm} = (w_{\text{sc}, \pm}^*)^{-1}[\text{WF}(u) \setminus \mathcal{T}_{\text{fr}, \pm}],$$

respectively. In addition, if g^{sc} is nontrapping, that is, $\mathcal{T}_{\text{sc}, +} = \mathcal{T}_{\text{sc}, -} = 0$, then the wave operator W_t of finite time $\pm t > 0$ belongs to $I_1^0(M, M_{\text{fr}}; \mathcal{C}_t) \cap I_\mu^0(M, M_{\text{fr}}; \mathcal{C}_{\text{sc}, \pm})$ with

$$\begin{aligned} \mathcal{C}_t &= \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus 0) \times (T^*M_{\text{fr}} \setminus 0); (r, \rho, \theta, \omega) = w_t^*(x, \xi)\}, \\ \mathcal{C}_{\text{sc}, \pm} &= \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus 0) \times (T^*M_{\text{fr}} \setminus 0); (r, \rho, \theta, \omega) = w_{\text{sc}, \pm}^*(x, \xi)\}, \end{aligned}$$

Remarks. 1. Theorem 3.1 is an analogue of the result by Hassell-Wunsch [1], and would actually be a refinement.

2. Since W_t is classically described by $w_t = (w_t^*)^{-1}$, it should be natural to use the canonical relation $\mathcal{C}_t$ to conclude $W_t \in I_1^0(M, M_{\text{fr}}; \mathcal{C}_t)$. However, $\mathcal{C}_t$ is just *asymptotically* conic. If we are forced to use the *exactly* conic canonical relations $\mathcal{C}_{\text{sc}, \pm}$, which are the asymptotes of $\mathcal{C}_t$, then the amplitudes get worse and we have $W_t \in I_\mu^0(M, M_{\text{fr}}; \mathcal{C}_{\text{sc}, \pm})$ for $\pm t > 0$, respectively.

3. If the potential V is subconstant, i.e.,

$$|\partial_r^j \partial_\theta^\alpha V(r, \theta)| \leq C_{j\alpha} r^{-\mu-j},$$

then we have $W_t \in I_1^0(M, M_{\text{fr}}; \mathcal{C}_t) \cap I_\mu^0(M, M_{\text{fr}}; \mathcal{C}_{\text{sc}, \pm}) \cap I_1^0(M, M_{\text{fr}}; \mathcal{C}_{\text{sc}, t})$, where

$$\mathcal{C}_{\text{sc}, t} = \{(x, \xi; r, \rho, \theta, \omega) \in (T^*M \setminus 0) \times (T^*M_{\text{fr}} \setminus 0); (r, \rho, \theta, \omega) = w_{\text{sc}, t}^*(x, \xi)\}.$$

4. Even if g^{sc} is not nontrapping, W_t composed with a microlocal cut off of the trapping region would belong to $I_1^0(M, M_{\text{fr}}; \mathcal{C}_t) \cap I_\mu^0(M, M_{\text{fr}}; \mathcal{C}_{\text{sc}, \pm})$.

Combining Theorem 3.1 with the microlocal smoothing property of the Schrödinger propagator, we can restate the former part of Theorem 3.1 as follows:

Corollary 3.2 ([2]). *If g^{sc} is radially short-range and V is sublinear, then for any $(x_0, \xi^0) \in T^*M \setminus \mathcal{T}_{\text{sc}, \mp}$, $\pm t > 0$ and $u \in \mathcal{H}$*

$$(x_0, \xi^0) \in \text{WF}(e^{-itH} u) \iff w_{\text{sc}, \mp}^*(x_0, \xi^0) \in \text{WF}(e^{-itH_{\text{fr}}} J^* u),$$

respectively.

$\text{WF}(e^{-itH_{\text{fr}}} J^* u)$ can be computed explicitly from u by using

$$\|a^w(r, hD_r, \theta, hD_\theta) e^{-itH_{\text{fr}}} J^* u\|_{\mathcal{H}_{\text{fr}}} = \|a^w(r + 2tD_r, hD_r, \theta, hD_\theta) J^* u\|_{\mathcal{H}_{\text{fr}}},$$

and hence, Corollary 3.2 gives a characterization of $\text{WF}(e^{-itH} u)$ in terms of the initial data u .

Theorem 3.1 holds also for $t = \pm\infty$.

Theorem 3.3 ([5], cf. [4]). *Suppose (M, g^{sc}) is a scattering manifold of radially short-range type and V is a smooth subcoulomb potential, and let $u \in \mathcal{H}_{\text{fr}}$. Then*

$$(3.2) \quad \text{WF}(W_{\pm}u) \setminus \mathcal{T}_{\text{sc}, \pm} = (w_{\text{sc}, \pm}^*)^{-1}[\text{WF}(u) \setminus \mathcal{T}_{\text{fr}, \pm}],$$

respectively. In addition, if g^{sc} is nontrapping, then the wave operators $W_{\pm}$ belong to $I_1^0(M, M_{\text{fr}}; \mathcal{C}_{\text{sc}, \pm})$, respectively.

§ 4. Microlocal structure of the scattering matrix

The S-matrix $\hat{S}(\rho)$ is obtained by restricting the scattering operator to the fixed energy ρ^2 . (Such a restriction is possible since we have the conservation of energy.) Fixing ρ , we lose freedom in the radial direction, and we can say that the S-matrix is the scattering operator S at infinity. Thus, instead of studying the wave front set of the S-matrix directly, we study the *scattering wave front set* of the scattering operator:

Definition 4.1. Let $u \in \mathcal{S}'(M_{\text{fr}})$ be a tempered distribution. The *scattering wave front set* $\text{WF}_{\text{sc}, \text{fr}}(u) \subset T^*M_{\text{fr}}$ is the complement of the set of $(r_0, \rho^0, \theta_0, \omega^0) \in T^*M_{\text{fr}}$ satisfying for some $a \in C_0^\infty(T^*M_{\text{fr}})$

$$(4.1) \quad a(r_0, \rho^0, \theta_0, \omega^0) \neq 0, \quad \|a^{\text{w}}(hr, D_r, \theta, hD_\theta)u\|_{\mathcal{H}_{\text{fr}}} = O(h^\infty) \quad \text{as } h \downarrow 0.$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be equal 1 near the origin and set

$$(4.2) \quad \eta(r, \theta, \omega) = [1 - \chi(r^{-2}g_\partial^{jk}(\theta)\omega_j\omega_k)][1 - \chi(r^2 + g_\partial^{jk}(\theta)\omega_j\omega_k)].$$

The first factor in the right-hand side equals 0 or 1 for $g_\partial^{jk}(\theta)\omega_j\omega_k < cr^2$ or $g_\partial^{jk}(\theta)\omega_j\omega_k > Cr^2$, respectively, while the second kills the singularity of the first near $(r, \omega) = 0$. Hence (4.2) is a cutoff function of an (r, ω) -conic neighborhood of $T^*M_{\text{fr}} \setminus \mathcal{U}_{\text{fr}, \pm}$.

Theorem 4.2 ([6], cf. [4]). *Suppose (M, g^{sc}) is a scattering manifold of radially short-range type and V is a smooth subcoulomb potential. Let $S = W_+^*W_-$ be the scattering operator, and $s_{\text{cn}} = w_{\text{cn}, +}^* \circ w_{\text{cn}, -} : \mathcal{U}_{\text{fr}, -} \rightarrow \mathcal{U}_{\text{fr}, +}$ be the classical scattering operator with respect to the underlying conic structure. Then, for any $u \in \mathcal{H}_{\text{fr}}$,*

$$(4.3) \quad \text{WF}_{\text{sc}, \text{fr}}(Su) \cap \mathcal{U}_{\text{fr}, +} = s_{\text{cn}}(\text{WF}_{\text{sc}, \text{fr}}(u) \cap \mathcal{U}_{\text{fr}, -}).$$

Moreover, set

$$\hat{S} = \mathcal{F}_{\text{fr}} S \mathcal{F}_{\text{fr}}^* : \hat{\mathcal{H}}_{\text{fr}, -} \rightarrow \hat{\mathcal{H}}_{\text{fr}, +}, \quad s_{\text{sc}} = w_{\text{sc}, +}^* \circ w_{\text{sc}, -}, \quad \hat{M}_{\text{fr}, \pm} = \{\pm\rho > 0\} \times \partial M.$$

Then, for any microlocal cutoff function $\eta \in C^\infty(T^*\hat{M}_{\text{fr}})$ given by (4.2), the operator (composed with restrictions)

$$\hat{S} \circ \eta^{\text{w}}(D_\rho, \theta, D_\theta) : C_0^\infty(\hat{M}_{\text{fr}, -}) \rightarrow C^\infty(\hat{M}_{\text{fr}, +})$$

belongs to $I_1^0(\hat{M}_{\text{fr},+}, \hat{M}_{\text{fr},-}; \hat{\mathcal{D}}_{\text{sc}}) \cap I_\mu^0(\hat{M}_{\text{fr},+}, \hat{M}_{\text{fr},-}; \hat{\mathcal{D}}_{\text{cn}})$ with

$$\begin{aligned}\hat{\mathcal{D}}_{\text{sc}} &= \{(\rho, r, \theta, \omega; \rho', r', \theta', \omega') \in (T^*\hat{M}_{\text{fr},+} \setminus 0) \times (T^*\hat{M}_{\text{fr},-} \setminus 0); \\ &\quad (r, \rho, \theta, \omega) = s_{\text{sc}}(r', \rho', \theta', \omega'), \omega \neq 0, \omega' \neq 0\}, \\ \hat{\mathcal{D}}_{\text{cn}} &= \{(\rho, r, \theta, \omega; \rho', r', \theta', \omega') \in (T^*\hat{M}_{\text{fr},+} \setminus 0) \times (T^*\hat{M}_{\text{fr},-} \setminus 0); \\ &\quad (r, \rho, \theta, \omega) = s_{\text{cn}}(r', \rho', \theta', \omega'), \omega \neq 0, \omega' \neq 0\}.\end{aligned}$$

Remark. As in Theorem 3.1, the canonical relation $\hat{\mathcal{D}}_{\text{sc}}$ is just asymptotically conic, and, if we replace it with the exactly conic asymptote $\hat{\mathcal{D}}_{\text{cn}}$, then the class of the amplitude gets worse. If g^{sc} is *short-range* in the sense that

$$(4.4) \quad |\partial_r^j \partial_\theta^\alpha a_l(r, \theta)| \leq C_{j\alpha} r^{-1-\mu-j} \quad l = 0, 1, 2,$$

then we obtain $\hat{S} \circ \eta^w(D_\rho, \theta, D_\theta) \in I_1^0(\hat{M}_{\text{fr},+}, \hat{M}_{\text{fr},-}; \hat{\mathcal{D}}_{\text{cn}})$.

Corollary 4.3. *The S -matrix $\hat{S}(\rho)$ belongs to $I_\mu^0(\hat{M}_{\text{fr},+}, \hat{M}_{\text{fr},-}; \mathcal{D}_\partial)$ with*

$$\mathcal{D}_\partial = \{(\theta, \omega; \theta', \omega') \in (T^*\partial M \setminus 0) \times (T^*\partial M \setminus 0); (\theta, \omega) = \exp \pi H_{\sqrt{K_\partial}}(\theta', \omega')\}$$

for a.e. $\rho \in \mathbb{R}_+$.

Remarks. 1. Using the Legendrian distributions, Melrose-Zworski [7] proved the above corollary.

2. If g^{sc} is short-range in the sense of (4.4), then $\hat{S}(\rho) \in I_1^0(\hat{M}_{\text{fr},+}, \hat{M}_{\text{fr},-}; \mathcal{D}_\partial)$.

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