

LOCALLY COERCIVE NONLINEAR EQUATIONS

TOSIO KATO

In this lecture we consider abstract nonlinear equations of the form

$$(1) \quad Au = f,$$

$$(2a) \quad (A+\xi)u = f, \quad (2b) \quad (1+\kappa A)u = f,$$

$$(3) \quad \partial u / \partial t + Au = f, \quad t \geq 0, \quad u(0) = \phi,$$

where A is a certain nonlinear operator. The two equations in (2) are slightly different, in that the same expression f appears on the right-hand side, while the left hand sides differ by a factor of $\kappa = 1/\xi$.

In what follows we shall give some general theorems that are convenient to solve the equations (1) to (3). For this purpose we introduce the usual triplet of real Banach spaces

$$(4) \quad V \subset H \subset V^*$$

in which H is a Hilbert space with the inner product $(\mid)_H$ and the norm $\|\mid_H$, and V , reflexive and separable, is densely and continuously embedded in H . $\langle, \rangle$ denotes the pairing between V and V^* . (4) implies that $\langle v, h \rangle = (v \mid h)_H$ whenever $v \in V$ and $h \in H$. (In problem (1), we may replace H by another pair of Banach spaces $Y \subset Y^*$, but we shall not go into such a generalization here.)

STANDING ASSUMPTION. A is a sequentially weakly continuous map of H into V^* . (In other words, $h_n \xrightarrow{w} h$ in H implies Ah_n

Ah in V^* , where $\xrightarrow{w}$ denotes weak convergence.)

THEOREM I. Assume that

$$5) \quad \langle v, Av \rangle \geq \beta \geq 0 \quad \text{for } v \in V \text{ with } \|v\|_H = r > 0.$$

Given any $f \in H$ with $\|f\|_H \leq \beta/r$, there is a solution $u \in H$ of (1) with $\|f\|_H \leq r$.

THEOREM II. Assume that

$$6) \quad \langle v, Av \rangle \geq - (\|v\|_H^2) \quad \text{for } v \in V,$$

where κ is a continuous function on $\mathbb{R}_+$ to $\mathbb{R}$. Given any $f \in H$, there is a solution $u \in H$ of (2b) if $|\kappa|$ is sufficiently small (depending on f).

THEOREM III. Assume (6). Given any $\phi \in H$, there is $T > 0$ and a solution $u \in C_w([0, T]; H)$ of (3). (Here C_w indicates weak continuity.) T may be any number such that the ordinary differential equation

$$(7) \quad d\rho/dt = 2\mathfrak{F}(\rho), \quad \mathfrak{F}(0) = \|\phi\|_H^2,$$

has a solution ρ on $t \in [0, T]$; then the solution u satisfies the inequality

$$(8) \quad \|u(t)\| \leq \rho(t), \quad t \in [0, T].$$

(If the solution to (7) is not unique, let ρ be its maximal solution.)

REMARKS. (a) In these theorems, H is the basic space; V and V^* are auxiliary and may be chosen more or less arbitrarily. By choosing V small (which makes V^* large), the STANDING ASSUMPTION and the conditions (5), (6) are more likely to be satisfied. But V must be dense in H .

(b) It is not assumed that A maps V into H . If this

happens, however, the left member of (5) and (6) can be replaced by $(v|Av)_H$ and we can forget V^* .

(c) There is in general no uniqueness in these theorems. But uniqueness may be proved under some additional assumptions. For Theorem I, such a condition is

$$(9) \quad (v-w|Av-Aw)_{H'} \geq \delta \|v-w\|^2 \quad \text{for } v, w \in K,$$

where H' is a Hilbert space such that $V^* \subset H'$ and δ is a positive constant. For Theorems II and III, a corresponding condition is

$$(10) \quad (v-w|Av-Aw)_{H'} \geq -\psi(\|v\|_H + \|w\|_H) \|v-w\|_H^2, \quad \text{for } v, w \in V,$$

where ψ is a function on $\mathbb{R}_+$ to $\mathbb{R}$.

(d) For the proof of these theorems, see [1].

APPLICATIONS. 1. Periodic solutions. Consider a nonlinear equation

$$(11) \quad a(x, u, \partial_1 u, \dots, \partial_m u) = f(x), \quad x \in T^m,$$

where T^m is the m -dimensional torus, the unknown u is a real-valued function on T^m , and $\partial_j = \partial/\partial x_j$. a is a sufficiently smooth, real-valued function of its arguments, which we denote by $x, u, p_1, \dots, p_m$. To describe the assumptions on a , we introduce the notations

$$(12) \quad a_{x_j} = \partial a / \partial x_j, \quad a_u = \partial a / \partial u, \quad a_{x_j p_k} = \partial^2 a / \partial x_j \partial p_k, \text{ etc.},$$

$$(13) \quad a^0(x) = a(x, 0, \dots, 0), \quad a_u^0(x) = a_u(x, 0, \dots, 0), \text{ etc.}$$

We now assume

$$(14) \quad a^0(x) = 0 \quad \text{identically,}$$

$$(15) \quad a^{00}(x) \equiv a_u^0(x) - \frac{1}{2} \sum_{j=1}^m a_{x_j p_j}^0(x) \geq 2\gamma ,$$

$$(16) \quad a^{00}(x) |\xi|^2 + s \sum_{j,k=1}^m a_{x_j p_k}^0(x) \xi_j \xi_k \geq 2\gamma |\xi|^2$$

for $\xi = (\xi_j) \in \mathbb{R}^m$,

where γ is a positive constant and s is an integer such that

$$(17) \quad s \geq [m/2] + 3 .$$

If we choose $H = H^s = H^s(T^m)$, $V = H^{s+1}$, $V^* = H^{s-1}$, with

$$(18) \quad (u|v)_H = (\Lambda^s u | \Lambda^s v)_0 + \lambda^2 (u|v)_0 ,$$

where $(\cdot | \cdot)_0$ is the L^2 -inner product. If the constant λ is chosen sufficiently large, it can be shown that condition (5) is satisfied with certain constants β and $r > 0$. Thus it follows from Theorem I that (11) has a solution $u \in H^s$ if $f \in H^s$ with sufficiently small $\|f\|_s$.

Moreover, the solution u is unique, since (9) is seen to hold with $H' = H^{s-1}$ or H^0 . Thus we have obtained a partial refinement of a result of Moser [2]. (It is expected that these results can be extended to symmetric systems of the form (11).)

2. Another type of nonlinear equations (see Rabinowitz [3])

$$(19) \quad (1-\Delta)u + \kappa b(x, u, \partial u, \partial^2 u, \partial^3 u) = f , \quad x \in T^m ,$$

can be handled in the same way, in which κ is a small parameter, b is a smooth function of its arguments, and $\partial^r u$ denotes the aggregate of all the derivatives of u of order r , etc. (19) can be written

$$(20) \quad u + \kappa Au = g = (1-\Delta)^{-1} f , \quad Au = (1-\Delta)^{-1} b(x, u, \dots) .$$

Theorem II is applicable to (20) with $H = H^s$, $V = H^{s+1}$, $V^* = H^{s-1}$, provided

$$(21) \quad s \geq [m/2] + 5.$$

Indeed, it is not difficult to show that (6) is satisfied. It follows that (19) is solvable for any $f \in H^s$, with $u \in H^s$, if $|\kappa|$ is sufficiently small. There is no restriction on b except smoothness. The solution u is unique; this can be proved by verifying (10) with $H' = H^{s-1}$ (or $H' = H^0$).

3. An equation of evolution

$$(22) \quad \partial_t u + a(x, u, \partial u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^m,$$

where a is a smooth real-valued function of its arguments. If we set $Au = a(x, u, \partial u)$, Theorem II is applicable with $H = H^s = H^s(\mathbb{R}^m)$, $V = H^{s+1}$, $V^* = H^{s-1}$ with s satisfying (17). Thus (22) has a short-time solution with $u(t) \in H^s$ for any initial value $u(0) = \phi \in H^s$.

4. The Euler equation in $\Omega \subset \mathbb{R}^m$. Theorem II can be applied to the Euler equation. For details see [4].

REFERENCES. 1. T. Kato, Locally coercive nonlinear equations, with applications to some periodic solutions, Preprint, 1984.

2. J. Moser, A rapidly convergent iteration method and non-linear partial differential equations - I., Ann. Scuola Normale Sup. Pisa, III. 20 (1966), 265-314.

3. P. Rabinowitz, A rapid convergence method for a singular perturbation problem, Ann. Inst. Henri Poincaré 1 (1984), 1-17.

4. T. Kato and C. Y. Lai, Nonlinear evolution equations and the Euler flow, J. Functional Anal. 56 (1984), 15-28.