

$\mathbb{P}_{max}^{\aleph_1}$ and other variations

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1 Introduction of $\mathbb{P}_{max}$ variations

$\mathbb{P}_{max}$ has been introduced by W. Hugh Woodin who says that in [11], $\mathbb{P}_{max}$ forces the *canonical model* of the negation of the Continuum Hypothesis CH over $L(\mathbb{R})$ with some large cardinal assumptions, e.g. $AD^{L(\mathbb{R})}$, or there are infinitely many Woodin cardinals with the measurable cardinal above. Under suitable large cardinal assumptions (in this paper, I abbreviate this to LC), $\mathbb{P}_{max}$ generically adds, over $L(\mathbb{R})$, a directed system of countable transitive models of ZFC (or its fragments) whose limit restricted to $H(\omega_2)$ (in this extension) is the whole $H(\omega_2)$, and $\mathbb{P}_{max}$ forces that the nonstationary ideal NS_{ω_1} on ω_1 is saturated. One of the important facts on $\mathbb{P}_{max}$ is absoluteness of Π_2 -sentences for the structure

$$\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$$

for some set R of reals in $L(\mathbb{R})$ as follows:

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If a Π_2 -sentence for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ is Ω_{ZFC} -consistent (e.g. forceable by set-forcing over ZFC), then it is true in $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ in the extension with $\mathbb{P}_{\text{max}}$ over $L(\mathbb{R})$ with LC.

(Under LC (e.g. there exist proper class many Woodin cardinals), every set of reals in $L(\mathbb{R})$ is universally Baire, and weakly homogeneously Suslin (see e.g. [5]). R is considered as an interpretation of its universally Baire set of reals in each universe. For more historical and technical remarks on $\mathbb{P}_{\text{max}}$, see [11, 7, 1].)

In [11], Woodin studied not only $\mathbb{P}_{\text{max}}$ but also conditional variations of $\mathbb{P}_{\text{max}}$ for e.g. Suslin trees and the Borel Conjecture. $\mathbb{P}_{\text{max}}$ variations have been studied by several set theorists: Feng–Woodin, Larson, Larson–Todorćević, Shelah–Zapletal and Yorioka [3, 4, 6, 8, 10, 12]. In [10], many variations of $\mathbb{P}_{\text{max}}$ for Σ_2 -statements in the structure $H(\omega_2)$ on cardinal invariants of the reals have been investigated. We should notice that all of them are derived from $\diamond$. For example, the $\mathbb{P}_{\text{max}}$ variation, say $\mathbb{P}_{\text{max}}^{\mathfrak{d}=\aleph_1}$, for the statement that the dominating number $\mathfrak{d}$ in ω^ω is $\aleph_1$ has been studied. It has been proved in [10, §2] that the extension with $\mathbb{P}_{\text{max}}^{\mathfrak{d}=\aleph_1}$ over $L(\mathbb{R})$ under LC satisfies ZFC, the continuum $\mathfrak{c}$ is $\aleph_2$, NS_{ω_1} is saturated, $\mathfrak{d} = \aleph_1$ holds, and maximality with respect to Π_2 -statements in $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ for some set R of reals in $L(\mathbb{R})$, that is, under LC, the extension with $\mathbb{P}_{\text{max}}^{\mathfrak{d}=\aleph_1}$ over $L(\mathbb{R})$ satisfies the following property, called Π_2 -compactness in [10]:

If ψ is a Π_2 -sentence for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ and the statement $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \models “\mathfrak{d} = \aleph_1 \wedge \psi”$ is Ω_{ZFC} -consistent, then it is true in $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$.

So this model can be considered as the canonical model of $\mathfrak{d} = \aleph_1$. In [10], there are many examples and counterexamples of Π_2 -compact statements. One non- Π_2 -compact statement, which does not appear in [10], is that the additivity $\text{add}(\mathcal{M})$ of the meager ideal is $\aleph_1$: By Miller–Truss’s characterization of $\text{add}(\mathcal{M})$, $\text{add}(\mathcal{M})$ is the minimum of the bounding number $\mathfrak{b}$ and the covering number $\text{cov}(\mathcal{M})$ of the meager ideal. However both “ $\aleph_1 = \text{add}(\mathcal{M}) < \mathfrak{b}$ ” and “ $\aleph_1 = \text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$ ” are consistent with ZFC, and both “ $\text{cov}(\mathcal{M}) > \aleph_1$ ” and “ $\mathfrak{b} > \aleph_1$ ” are Π_2 -statements in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$. (The statement that the additivity of the null ideal is $\aleph_1$ is not Π_2 -compact either. It is known that $\text{add}(\mathcal{N}) = \min\{\text{add}^*(\mathcal{N}), \mathfrak{b}\}$. See [2, Theorem 2.7.13.] or [9].)

In this paper, we work in ZFC except for the definition of $\mathbb{P}_{max}^\phi$ and the proof of Theorem Schemes because when we force by $\mathbb{P}_{max}^\phi$, we always consider $L(\mathbb{R})$ as the ground model which never satisfies the Axiom of Choice (by our assumption). $\mathbb{P}_{max}$ can be defined by various ways. One of them is defined by use of *iterable pairs*. Suppose a suitable large cardinal property, M is a countable transitive model of ZFC and I is a member of M which is a uniform normal ideal on ω_1^M in M . We can take a direct system $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \omega_1 \rangle$, called an *iteration* of (M, I) (of length ω_1), such that

- $M_0 = M$,
- G_β is an M_β -generic filter of the forcing notion $(\mathcal{P}(\omega_1^{M_\beta}) / j_{0,\beta}(I))^{M_\beta}$ (or $(\mathcal{P}(\omega_1^{M_\beta}) \setminus j_{0,\beta}(I))^{M_\beta}$) for every $\beta \in \omega_1$,
- $j_{\gamma,\gamma}$ is the identity on M_γ for every $\gamma \in \omega_1 + 1$,
- $M_{\beta+1}$ is (the transitive collapse of) the generic ultrapower of M_β by G_β (if it is wellfounded, otherwise we stop the construction), and $j_{\gamma,\gamma+1}$ is the ultrapower embedding induced by G_γ for every $\gamma \in \omega_1$, and
- if $\alpha \in \omega_1 + 1$ is a limit ordinal, then M_α is (the transitive collapse of) the direct limit of the system $\langle M_\gamma, j_{\gamma,\delta}; \gamma \leq \delta < \alpha \rangle$ and $j_{\gamma,\alpha}$ is the induced embedding for every $\gamma \in \alpha$.

(See [11, Definition 3.5. or Definition 4.1.] or [7, 1.2 Definition].) A pair (M, I) as above is called *iterable* if all M_γ , $\gamma \in \omega_1$, are wellfounded regardless of the choice of generic filters G_β . Woodin proved that if I is precipitous, then (M, I) is iterable (see [11, Lemma 3.10. and Lemma 4.5.]).

In many cases, we define the $\mathbb{P}_{max}$ variation $\mathbb{P}_{max}^\phi$ for a Σ_2 -sentence ϕ in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$ which is derived from $\diamond$. For example, $\mathfrak{d} = \aleph_1$ holds, and there exists a coherent Suslin tree, etc. In [10], variations of $\mathbb{P}_{max}$ are defined by use of stationary tower forcing ([5]). In this paper, we adopt a definition in [7, §10.2], however all of proofs in this paper can be applied to any type of $\mathbb{P}_{max}^\phi$ variations.

Definition of $\mathbb{P}_{max}^\phi$ ([7, §10.2]) *Let ϕ be a Σ_2 -statement for the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$, and say that ϕ forms $\exists u \forall v \phi_0(u, v)$. Conditions of the forcing notion $\mathbb{P}_{max}^\phi$ are defined by recursion on their ranks as follows. A triple $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{max}$ if*

1. (M, I) is an iterable pair,
2. $a \in H(\omega_2)^M$ and $\langle H(\omega_2), \in, I \rangle^M \models \text{"}\forall v \phi_0(a, v)\text{"}$, and
3. $\mathcal{X}$ is a member of M and a set (possibly empty) of pairs $\langle \langle (N, J), b, \mathcal{Y} \rangle, j \rangle$ such that
 - $\langle (N, J), b, \mathcal{Y} \rangle \in \mathbb{P}_{max}^\phi \cap H(\omega_1)^M$,
 - j is in M and an iteration of (N, J) of length ω_1^M such that $j(J) = I \cap j(\mathcal{P}(\omega_1^N)^N)$, $j(b) = a$ and $j(\mathcal{Y}) \subseteq \mathcal{X}$, and
 - $\mathcal{X}$ forms a function, i.e. for members $\langle p, j \rangle$ and $\langle p', j' \rangle$ in $\mathcal{X}$, if $p = p'$, then $j = j'$.

For conditions $\langle (M, I), a, \mathcal{X} \rangle$ and $\langle (N, J), b, \mathcal{Y} \rangle$ in $\mathbb{P}_{max}^\phi$, we define

$$\langle (M, I), a, \mathcal{X} \rangle <_{\mathbb{P}_{max}^\phi} \langle (N, J), b, \mathcal{Y} \rangle$$

if there exists j such that $\langle \langle (N, J), b, \mathcal{Y} \rangle, j \rangle \in \mathcal{X}$.

We have to note that the statement that a pair (M, I) is iterable is Π_2^1 about a real coding (M, I) , so is absolute (see e.g. [7, 1.3 Remark and 1.10 Remark]). Therefore the statement that a triple $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{max}^\phi$ is also Π_2^1 , and so is absolute. Since $L(\mathbb{R})$ has every real, it also has every countable transitive model. And since a condition of $\mathbb{P}_{max}^\phi$ can be coded by a real, $(\mathbb{P}_{max}^\phi)^{L(\mathbb{R})} = \mathbb{P}_{max}^\phi$. If ϕ is trivial (e.g. “ $0 = 0$ ”, or the statement that there exists the empty set), then $\mathbb{P}_{max}^\phi$ can be considered the standard $\mathbb{P}_{max}$. (However $\mathbb{P}_{max}^\phi$ and $\mathbb{P}_{max}$ are slightly different, see [11, §5.4, in particular Theorem 5.40].)

To analyze the extension by $\mathbb{P}_{max}^\phi$, we need some game theoretic lemmata. (On definitions of games $\mathcal{G}_1^\phi$, $\mathcal{G}_\omega^\phi$ and $\mathcal{G}_{\omega_1}^\phi$, I refer [7, §3 and §10.2].)

We define the game $\mathcal{G}_1^\phi$ as follows. Suppose that $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{max}^\phi$, J is a normal uniform ideal on ω_1 . Players **I** and **II** collaborate to build an iteration $\langle M_\gamma, G_\beta, j_{\gamma, \delta}; \beta < \gamma \leq \delta \leq \omega_1 \rangle$ of (M, I) with the following rule: In each round α , **II** chooses a set A in the set $\mathcal{P}(\omega_1^{M_\alpha})^{M_\alpha} \setminus j_{0, \alpha}(I)$, and then **I** chooses an $(M_\alpha, (\mathcal{P}(\omega_1^{M_\alpha}) \setminus j_{0, \alpha}(I))^{M_\alpha})$ -generic filter G_α with $A \in G_\alpha$. (To just simplify notation, we force by $\mathcal{P}(\omega_1) \setminus I$ instead of $\mathcal{P}(\omega_1)/I$ in this paper.) After all ω_1 many rounds have been played, **I** wins if

- $\langle H(\omega_2), \in, J \rangle \models \text{"}\forall v \phi_0(j_{0,\omega_1}(a), v) \text{"}$.

(We should note that player **II** has a strategy such that after all ω_1 rounds have been played whenever player **II** plays according to this strategy,

- $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ holds.

See [11, Lemma 4.36.], [7, 2.8 Lemma], [1, Lemma 1.8].)

To show σ -closedness of $\mathbb{P}_{max}^\phi$ and define the strategic iteration lemma for ϕ , we need to define an iterable limit sequence and two games $\mathcal{G}_\omega^\phi$ and $\mathcal{G}_{\omega_1}^\phi$. (On this paragraph, see [11, Chapter 4.1 and Lemma 4.43.], [7, §3] and [1, §2].) Let $\langle p_i; i \in \omega \rangle$ is a decreasing sequence of $\mathbb{P}_{max}^\phi$ and write $p_i := \langle (M_i, I_i), a_i, \mathcal{X}_i \rangle$. Let $j_{i,i+1} : (M_i, I_i) \rightarrow (M_i^*, I_i^*)$ be an iteration witnessing that $p_{i+1} <_{\mathbb{P}_{max}^\phi} p_i$ (and if $p_{i+1} = p_i$, then let $j_{i,i+1}$ be the identity map) and let $\{j_{i,i'}; i \leq i' \leq \omega\}$ be the commuting family of embeddings generated by $\{j_{i,i+1}; i \in \omega\}$. We write $j_{i,\omega} : (M_i, I_i) \rightarrow (N_i, J_i)$ for each $i \in \omega$. Let $a := \bigcup_{i \in \omega} a_i$ and $\mathcal{X} := \bigcup_{i \in \omega} \mathcal{X}_i$. In most cases, a forms a witness of ϕ in every N_i . (At least, every application in any present published paper, including this paper, on $\mathbb{P}_{max}^\phi$ and its variations is in this case.) Then we can show that

- for each $i \in \omega$, (N_i, J_i) is an iterable pair,
- for each $i \in \omega$, $N_i \in N_{i+1}$ and $\omega_1^{N_i} = \omega_1^{N_0}$,
- for each $i \in \omega$, $J_{i+1} \cap N_i = J_i$,
- $a \in H(\omega_2)^{N_0}$ and for each $i \in \omega$, $\langle H(\omega_2), \in, J_i \rangle^{N_i} \models \text{"}\forall b \phi_0(a, b) \text{"}$.

We call $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ a limit sequence if it is constructed as above. For a limit sequence $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$, when an ultrafilter G on the set

$$\bigcup_{i \in \omega} \mathcal{P}(\omega_1^{N_i})^{N_i} \setminus J_i$$

satisfies that for every regressive function f on $\omega_1^{N_i}$ in $\bigcup_{i \in \omega} N_i$, f is constant on some condition in G , we call it a $\bigcup \{N_i; i \in \omega\}$ -normal ultrafilter for $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$. Then we form the ultrapower of $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ formed from G and all functions $f : \omega_1^{N_0} \rightarrow N_i$ in $\bigcup_{i \in \omega} N_i$. (More precisely, see [11, Definition 4.15.].) Using this ultrapower, we define the iteration of the sequence $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$, and the iterability of the sequence $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ as in the iterable pair. We note that for a limit sequence $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ constructed as above,

- $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ is iterable.

We define the game $\mathcal{G}_\omega^\phi$ as follows. Suppose that $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ is a limit sequence, J is a normal uniform ideal on ω_1 . Players **I** and **II** collaborate to build an iteration of $\langle \langle (N_i, J_i); i \in \omega \rangle, a, \mathcal{X} \rangle$ consisting of limit sequences $\langle \langle (N_i^\alpha, J_i^\alpha); i \in \omega \rangle, a^\alpha, \mathcal{X}^\alpha \rangle$, $\bigcup \{N_i^\alpha; i \in \omega\}$ -normal ultrafilters G_α for $\langle \langle (N_i^\alpha, J_i^\alpha); i \in \omega \rangle, a^\alpha, \mathcal{X}^\alpha \rangle$ and a commuting family of embeddings $j_{\alpha, \beta}$ for $\alpha \leq \beta \leq \omega_1$ with the following rule: In each round α , **II** chooses a set A in the set $\bigcup \left\{ \mathcal{P}(\omega_1^{N_i^\alpha})^{N_i^\alpha} \setminus J_i^\alpha; i \in \omega \right\}$, and then **I** chooses a $\bigcup \{N_i^\alpha; i \in \omega\}$ -normal ultrafilter G_α for $\langle \langle (N_i^\alpha, J_i^\alpha); i \in \omega \rangle, a^\alpha, \mathcal{X}^\alpha \rangle$ with $A \in G_\alpha$. After all ω_1 many rounds have been played, **I** wins if

- $\langle H(\omega_2), \in, J \rangle \models \text{"}\forall v \phi_0(j_{0, \omega_1}(a), v) \text{"}$.

(We should note that **II** has a strategy such that after all ω_1 rounds have been played whenever player **II** plays according to this strategy,

- $J_i^{\omega_1} = J \cap N_i^{\omega_1}$ holds for every $i \in \omega$.

We can prove σ -closedness of $\mathbb{P}_{max}^\phi$ using strategies for both players **I** and **II**. See [11, Lemma 4.43.], [7, 3.4 Lemma and 3.5 Lemma], [1, Lemma 2.5].)

We define the game $\mathcal{G}_{\omega_1}^\phi$ as follows. Let p_0 is a condition of $\mathbb{P}_{max}^\phi$. Players **I** and **II** collaborate to build a decreasing ω_1 -chain $\langle p_\alpha; \alpha \in \omega_1 \rangle$ of conditions with the following rule: In each round α , if α is a successor ordinal, **II** chooses a condition p_α below $p_{\alpha-1}$. If α is a limit ordinal, then **II** chooses a cofinal ω -sequence of α and, letting $\langle \langle (N_i^\alpha, J_i^\alpha); i \in \omega \rangle, a_\alpha^*, \mathcal{X}_\alpha^* \rangle$ be the induced limit sequence, **II** chooses a set A_α in the set $\bigcup \left\{ \mathcal{P}(\omega_1^{N_i^\alpha})^{N_i^\alpha} \setminus J_i^\alpha; i \in \omega \right\}$, and then **I** chooses a condition $p_\alpha = \langle \langle M_\alpha, I_\alpha \rangle, a_\alpha, \mathcal{X}_\alpha \rangle$ below every p_β such that for some iteration k of $\langle \langle (N_i^\alpha, J_i^\alpha); i \in \omega \rangle, a_\alpha^*, \mathcal{X}_\alpha^* \rangle$, $k[\mathcal{X}_\alpha^*] \subseteq \mathcal{X}_\alpha$ and $\omega_1^{N_0^\alpha} \in k(A_\alpha)$. After all ω_1 rounds have been played, **I** wins if, letting $j_{\alpha, \beta}$ ($\alpha < \beta \leq \omega_1$) be the induced commuting family of embeddings on the sequence $\langle p_\alpha; \alpha \in \omega_1 \rangle$,

- $\langle H(\omega_2), \in, j_{0, \omega_1}(I_0) \rangle \models \text{"}\forall v \phi_0(j_{0, \omega_1}(a), v) \text{"}$.

(In [10], the strategic iteration lemma for ϕ is the following lemma scheme:

(ZFC + $\diamond$) Player **I** has a winning strategy in $\mathcal{G}_{\omega_1}^\phi$.

This is related to [7, 5.2 Theorem].)

The following theorem is a basic theorem of $\mathbb{P}_{max}^\phi$.

Theorem Scheme 1 ([11, Chapter 4], [1, §§3-5], [7, §§5-7], [10, §1]) (ZFC + LC)
Let ϕ be a Σ_2 -sentence in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$. Assume that the following three statements

- (1) *player I has a winning strategy in $\mathcal{G}_1^\phi$,*
- (ω) *player I has a winning strategy in $\mathcal{G}_\omega^\phi$,*
- (ω_1) *player I has a winning strategy in $\mathcal{G}_{\omega_1}^\phi$,*

are all Ω_{ZFC} -consistent. Let G be a $(L(\mathbb{R}), \mathbb{P}_{\text{max}}^\phi)$ -generic filter. Then in $L(\mathbb{R})[G]$, ZFC holds, $\mathfrak{c} = \aleph_2$, NS_{ω_1} is saturated and $\langle H(\omega_2), \in, NS_{\omega_1} \rangle \models \phi$ holds.

In the above theorem scheme, the phrase that (1), (ω) and (ω_1) are all Ω_{ZFC} -consistent are usually considered as the slightly stronger following statement:

(ZFC + $\diamond$) Both (1), (ω) and (ω_1) hold.

One of important conclusions of $\mathbb{P}_{\text{max}}^\phi$ extensions is Π_2 -maximality. To show this, we need a more technical lemma. For a sentence Φ in the language of set theory, the iteration lemma for ϕ from Φ is defined as follows:

Lemma Scheme; The Iteration Lemma for ϕ from Φ (ZFC + Φ) *If*

- *(M, I) is an iterable pair,*
- *$a \in H(\omega_2)^M$ and $\langle H(\omega_2), \in, I \rangle^M \models \forall b \phi_0(a, b)$ "*
- *J is a normal uniform ideal on ω_1 , and*
- *$\langle H(\omega_2), \in, J \rangle \models \phi$,*

then there exists an iteration $j : (M, I) \rightarrow (M^, I^*)$ of length ω_1 such that*

- *$I^* = J \cap M^*$, and*
- *$\langle H(\omega_2), \in, J \rangle \models \forall v \phi_0(j(a), v)$.*

Of course, the case that Φ contradicts ϕ does not make sense. In [10], the simple iteration lemma for ϕ is the iteration lemma for ϕ from $\diamond$, and the optimal iteration lemma for ϕ is the iteration lemma for ϕ from any trivial statement. We note that if (under ZFC) player I has a winning strategy in $\mathcal{G}_1^\phi$, then the optimal iteration lemma for ϕ holds. We should notice that for some Σ_2 -sentence ϕ in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$, the simple iteration lemma for ϕ fails. For example, the simple iteration lemma for CH, and for the statement that the almost disjointness number is $\aleph_1$ fail (see [10, §1.3] and [11, Lemma 5.29.]).

Theorem Scheme 2 ([11, Chapter 4], [1, §§3-5], [7, §§5-7], [10, §1]) (ZFC + LC)
Let ϕ be a Σ_2 -sentence in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$ and Φ a sentence in the language of set theory such that the iteration lemma for ϕ from Φ holds. Assume that both (ω) and (ω_1) are Ω_{ZFC} -consistent. Let G be a $(L(\mathbb{R}), \mathbb{P}_{\text{max}}^\phi)$ -generic filter. Then in $L(\mathbb{R})[G]$, ZFC holds, $\mathfrak{c} = \aleph_2$, NS_{ω_1} is saturated and $\langle H(\omega_2), \in, NS_{\omega_1} \rangle \models \phi$ holds, and for any Π_2 -sentence ψ in the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle$ for some set R of reals in $L(\mathbb{R})$, if the statement $\Phi + \langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \models \phi \wedge \psi$ is Ω_{ZFC} -consistent, then $\langle H(\omega_2), \in, NS_{\omega_1}, R \rangle \models \psi$ holds.

Therefore under the assumption in Theorem Scheme 1, if the optimal iteration lemma for ϕ holds, then ϕ is Π_2 -compact in the extension by $\mathbb{P}_{\text{max}}^\phi$ over $L(\mathbb{R})$. We have some examples of Σ_2 -statements for which the optimal iteration lemma fails, e.g. for the existence of a Suslin tree. (See [10, §1.3].) However we should notice that even if the optimal iteration lemma for ϕ fail, we *cannot* conclude that ϕ cannot be Π_2 -compact.

In this note, we prove the optimal iteration lemma for $\mathfrak{d} = \aleph_1$. This proof is prototypical for any other $\mathbb{P}_{\text{max}}$ variations of $\mathfrak{x} = \aleph_1$ where $\mathfrak{x}$ is a cardinal invariant which is the smallest size of the cofinality of some ordered structure, or some ideal on the reals. The point whether we can adopt the proof for $\mathfrak{d} = \aleph_1$ to the optimal iteration lemma for $\mathfrak{x} = \aleph_1$ is whether we have a Suslin ccc Amoeba forcing for this structure and we can show the subgenericity lemma (i.e. a variation of Proposition 2.3).

2 The optimal iteration lemma for $\mathfrak{d} = \aleph_1$

We don't prove the optimal iteration lemma for $\mathfrak{d} = \aleph_1$ usually. We find an equivalent statement of $\mathfrak{d} = \aleph_1$ and we show the optimal iteration lemma for it.

Definition 2.1 ([10, Lemma 2.6.]). *Let I be a normal uniform ideal on ω_1 . A sequence $\langle f_\xi; \xi \in \omega_1 \rangle$ of functions in ω^ω is an I -good scale if*

- *it is a scale, i.e. a well-ordered with respect to the eventually dominance, and*
- *for every $f \in \omega^\omega$, the set $\{\xi \in \omega_1; f_\xi \text{ dominates } f \text{ everywhere } (f \leq f_\xi)\}$ is I -positive.*

Proposition 2.2. *Assume that I is a normal uniform ideal on ω_1 . $\mathfrak{d} = \aleph_1$ holds iff there exists an I -good scale.*

Proof. Suppose that $\mathfrak{d} = \aleph_1$ holds, and let $\langle g_\xi; \xi \in \omega_1 \rangle$ be a scale, i.e.

- if $\xi < \eta$ in ω_1 , then $g_\xi \leq^* g_\eta$, and
- for any $h \in \omega^\omega$, there exists $\xi \in \omega_1$ such that $h \leq^* g_\xi$.

Let

$$\langle X_{s,\alpha}; s \in \omega^{<\omega} \text{ \& } \alpha \in \omega_1 \rangle$$

be a sequence of pairwise disjoint I -positive subsets of ω_1 .

By recursion on $\xi \in \omega_1$, we construct $f_\xi \in \omega^\omega$ such that

- $f_\xi \leq^*$ -dominates g_η and f_η for all $\eta < \xi$, and
- if ξ is in some $X_{s,\alpha}$, then $f_\xi \sqsubseteq$ -dominates the function $s^\frown(g_\alpha \upharpoonright [|s|, \infty))$.

Then we note that $\langle f_\xi; \xi \in \omega_1 \rangle$ is a scale. So what we need to check is I -goodness.

Let $f \in \omega^\omega$. Then since $\langle g_\xi; \xi \in \omega_1 \rangle$ is a scale, we can find $\alpha \in \omega_1$ so that f is $\leq^*$ -dominated by g_α . Let $n \in \omega$ be such that $f(i) \leq g_\alpha(i)$ for every $i \geq n$ and let $s := f \upharpoonright n$. Then

$$\{\xi \in \omega_1; f \leq g_\xi\} \supseteq X_{s,\alpha},$$

that is, the set $\{\xi \in \omega_1; f \leq g_\xi\}$ is I -positive.

The other direction is trivial. □

We have a Suslin ccc Amoeba forcing for the structure $\langle \omega^\omega, \leq^* \rangle$, the Hechler forcing $\mathbb{D} := \omega^{<\omega} \times \omega^\omega$. For $p = \langle s^p, f^p \rangle$ and $q = \langle s^q, f^q \rangle$, $p \leq_{\mathbb{D}} q$ if $s^p \supseteq s^q$, $f^q \leq f^p$ and for every $i \in [|s^q|, |s^p|)$, $s^p(i) \geq f^q(i)$. For a condition $p \in \mathbb{D}$, we define

$$\text{body}(p) := s^p \frown f^p \restriction [|s^p|, \infty),$$

and let $\mathbb{D} \restriction f := \{p \in \mathbb{D}; \text{body}(p) \leq f\}$.

Proposition 2.3. *Suppose that M is a model of a large enough fragment of ZFC. (ZFC – Powerset + $\exists \mathcal{P}(2^\omega)$ is sufficient.) Suppose that f eventually dominates all functions in $\omega^\omega \cap M$, and $D \in M$ is such that D is dense in $\mathbb{D}$ in M . Then $D \cap (\mathbb{D} \restriction f)$ is dense in $(\mathbb{D} \restriction f) \cap M$.*

Proof. Let $p_0 = \langle s_0, f_0 \rangle \in (\mathbb{D} \restriction f) \cap M$. Working in M , we choose $p_i = \langle s_i, f_i \rangle \in \mathbb{D} \cap M$ by induction on $i \in \omega$ such that

- $p_{i+1} \in D$, and
- $p_{i+1} \leq_{\mathbb{D}} \langle \text{body}(p_0) \restriction |s_i|, f_i \rangle$. (We must note that $\langle \text{body}(p_0) \restriction |s_i|, f_i \rangle$ is a condition in $\mathbb{D} \cap M$ which extends p_0 .)

Then we define $g \in \omega^\omega$ such that

$$g(i) := \begin{cases} s_0(i) & \text{if } i < |s_0| \\ s_{k+1}(i) & \text{if } |s_k| \leq i < |s_{k+1}| \text{ for some } k \in \omega \end{cases}$$

Since $\langle p_i; i \in \omega \rangle$ is in M , g is also in M . Thus $g \leq^* f$ holds, hence for large enough $k \in \omega$, $g \restriction [|s_k|, \infty) \leq f \restriction [|s_k|, \infty)$. Then for a fixed such a k , p_{k+1} is in $D \cap (\mathbb{D} \restriction f)$. $\square$

Corollary 2.4. *Suppose that $\mathbb{P}$ is a forcing notion and $\dot{g}$ is a $\mathbb{P}$ -name such that*

1. $\Vdash_{\mathbb{P}} \dot{g} \in \omega^\omega$ & $\dot{g}$ eventually dominates all functions in $\omega^\omega \cap \mathbf{V}$, (where $\mathbf{V}$ is the ground model) and
2. for every condition $r \in \text{SLOC}$, $\|\text{body}(\check{r}) \leq \dot{g}\|_{ro(\mathbb{P})}$ is non-zero.

Then $\mathbb{D}$ is completely embeddable into $\mathbb{Q} := ro(\mathbb{P}) * ((\mathbb{D} \restriction \dot{g}) \cap \mathbf{V})$ such that $\Vdash_{\mathbb{Q}} \dot{f}_{\mathbb{D}} := \bigcup_{p \in G} s^p \leq \dot{g}$.

Proof. We show that the embedding i from $\mathbb{D}$ into $\mathbb{Q}$, defined by

$$i(r) := \langle \|\text{body}(\check{r}) \leq \dot{g}\|_{ro(\mathbb{P})}, \check{r} \rangle$$

for each $r \in \mathbb{D}$, is a complete embedding.

To prove this, we show that for any dense subset D in $\mathbb{D}$ (in the ground model), the set $\{i(r); r \in D\}$ is predense in $\mathbb{Q}$. Let $\langle p, \check{q} \rangle \in \mathbb{Q}$, i.e.,

$$p \Vdash_{\mathbb{P}} \check{q} \in (\mathbb{D} \restriction \dot{g}) \cap \mathbf{V}, \text{ i.e. } p \leq_{\mathbb{P}} \|\text{body}(\check{q}) \leq \dot{g}\|_{ro(\mathbb{P})}.$$

Since, by the previous proposition,

$$\Vdash_{\mathbb{P}} \check{D} \cap (\mathbb{D} \restriction \dot{g}) \text{ is dense in } (\mathbb{D} \restriction \dot{g}) \cap \mathbf{V},$$

we can find $p' \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{D}} q$ such that $q' \in D$ and

$$p' \Vdash_{\mathbb{P}} \check{q}' \in \check{D} \cap (\mathbb{D} \restriction \dot{g}).$$

Then

$$\langle p', \check{q}' \rangle \leq_{\mathbb{Q}} \langle \|\text{body}(\check{q}') \leq \dot{g}\|_{ro(\mathbb{P})}, \check{q}' \rangle = i(q').$$

Assume that i is not a complete embedding, i.e. there exists $\langle p, \check{q} \rangle \in \mathbb{Q}$ such that the set

$$D := \{r \in \mathbb{D}; i(r) \text{ and } \langle p, \check{q} \rangle \text{ are incompatible in } \mathbb{Q}\}$$

is dense in $\mathbb{D}$. Then the set $\{i(r); r \in D\}$ is predense in $\mathbb{Q}$. However then, there exists $r \in D$ so that $i(r)$ and $\langle p, \check{q} \rangle$ are incompatible in $\mathbb{Q}$, which is a contradiction. $\square$

Lemma 2.5. *Suppose that M is a countable model of (a large enough fragment of) ZFC, $\mathbb{P}$ and $\dot{g}$ satisfy the hypothesis of Corollary 2.4 in M , $p \in \mathbb{P} \cap M$ and $f \in \omega^\omega$. (We may not assume $f \in M$.) Then there exists a $(M, \mathbb{P})$ -generic filter G containing p such that f is $\leq^*$ -dominated by $\dot{g}[G]$.*

Proof. We fix a complete embedding from $ro(\mathbb{D})$ into $\mathbb{Q} := ro(\mathbb{P}) * ((\mathbb{D} \restriction \dot{g}) \cap \mathbf{V})$ as in the previous corollary, and let p' be a projection of p via this embedding.

Let N be a countable model of a large enough fragment of ZFC containing $M \cup \{S\}$. Since N is a countable model, there exists a $(N, \mathbb{D})$ -generic filter F' containing p' . We let $F := F' \cap M$. Since $\mathbb{D}$ is a Suslin ccc forcing notion, all maximal antichains on $\mathbb{D}$ belonging to M are still maximal in N . Thus F

is $(M, \mathbb{D})$ -generic and $f \leq^* f_F$. We take a $(M, \mathbb{Q})$ -generic filter H extending F (via the fixed embedding) with $p \in H$ and let $G := ro(\mathbb{P}) \cap H$. We note that G is $(M, ro(\mathbb{P}))$ -generic. Then

$$f \leq^* f_F \sqsubseteq \dot{g}[H] = \dot{g}[G].$$

□

Theorem 2.6 (The optimal iteration lemma for the existence of a good scale). (ZFC) *If*

- (M, I) is an iterable pair,
- $a \in H(\omega_2)^M$ and $H(\omega_2)^M \models$ “ a is an I -good scale”,
- J is a normal uniform ideal on ω_1 , and
- $\text{cof}(\mathcal{N}) = \aleph_1$,

then there exists an iteration $j : (M, I) \rightarrow (M^*, I^*)$ of length ω_1 such that

- $I^* = J \cap M^*$, and
- $j(a)$ is a J -good scale.

Proof. Suppose that (M, I) is an iterable pair, i.e.

- M is a countable transitive model of ZFC, and
- $I \in M$ and $M \models$ “ I is a normal uniform ideal on ω_1^M ”.

Let $\langle f_\xi; \xi \in \omega_1^M \rangle$ be in M such that

$$M \models \langle f_\xi; \xi \in \omega_1^M \rangle \text{ is an } I\text{-good scale},$$

and $\langle g_\xi; \xi \in \omega_1 \rangle$ be a (J -good) scale. (We don't need J -goodness of the sequence $\langle g_\xi; \xi \in \omega_1 \rangle$.) Let $\langle X_{n,\alpha}; n \in \omega \ \& \ \alpha \in \omega_1 \rangle$ be a sequence of J -positive subsets of ω_1 which are pairwise disjoint.

We build an iteration $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \omega_1 \rangle$ of (M, I) of length ω_1 such that

- for each $\alpha \in \omega_1$, we fix a sequence $\langle Y_{n,\alpha}; n \in \omega \rangle$ of all $j_{0,\alpha}(I)$ -positive subsets of $\omega_1^{M_\alpha}$,

- if $\alpha \leq \gamma$ in ω_1 , $n \in \omega$ and $\omega_1^{M_\gamma} \in X_{n,\alpha}$, then $j_{\alpha,\gamma}(Y_{n,\alpha}) \in G_\gamma$, and
- for every $\alpha \in \omega_1$, $g_\alpha \leq^* f_{\omega_1^{M_\alpha}}^{\alpha+1}$ ($= f_{\omega_1^{M_\alpha}}^{\omega_1}$), where for each $\alpha \leq \omega_1$, we write

$$j_{0,\alpha}(\langle f_\xi; \xi \in \omega_1^M \rangle) = \langle f_\xi^\alpha; \xi \in \omega_1^{M_\alpha} \rangle.$$

(We note that if $\alpha \leq \beta$ in $\omega_1 + 1$ and $\xi \in \omega_1^{M_\alpha}$, then $f_\xi^\alpha = f_\xi^\beta$.)

This can be done by the following claim:

Claim Assume that we have constructed $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \alpha \rangle$ and $Z \in (\mathcal{P}(\omega_1^{M_\alpha}) \setminus j_{0,\alpha}(I))^{M_\alpha}$. Then there is a $(\mathcal{P}(\omega_1^{M_\alpha}) \setminus j_{0,\alpha}(I))^{M_\alpha}$ -generic filter G_α with $Z \in G_\alpha$ such that $g_\alpha \leq^* f_{\omega_1^{M_\alpha}}^{\alpha+1}$.

Proof of Claim. We have to notice that

- in a generic extension of M_α with $(\mathcal{P}(\omega_1^{M_\alpha}) \setminus j_{0,\alpha}(I))^{M_\alpha}$, $f_\xi^\alpha \leq^* f_{\omega_1^{M_\alpha}}^{\alpha+1}$ holds, hence $f_{\omega_1^{M_\alpha}}^{\alpha+1} \leq^*$ -dominates all slaloms in $\mathcal{S} \cap M_\alpha$, and
- for each $p \in \mathbb{D} \cap M_\alpha$, the set

$$\{\xi \in \omega_1^{M_\alpha}; \text{body}(p) \leq f_\xi^\alpha\}$$

is $j_{0,\alpha}(I)$ -positive.

(We note that $f_{\omega_1^{M_\alpha}}^{\alpha+1}$ is in $M_{\alpha+1}$ which is a subuniverse of $M_\alpha[G]$ and it is not changed by the transitive collapse and the relation $\leq^*$ is absolute.) So by Lemma 2.5, we can find a desired G_α . $\dashv$

By the construction (and the standard argument, e.g. [11, Lemma 4.36.] or [7, 2.8 Lemma]), $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ and $j_{0,\omega_1}(\langle f_\xi; \xi \in \omega_1^M \rangle)$ is a scale. What we need to check is J -goodness of the scale.

To see J -goodness, take any $p \in \mathbb{D}$. Then there is $\alpha \in \omega_1$ such that $\text{body}(p) \leq^* g_\alpha$, so we can find $n \in \omega$ such that $\text{body}(p) \leq^n f_{\omega_1^{M_\alpha}}^{\alpha+1}$. Let $g \in \mathcal{S}$ be such that $g := (\text{body}(p) \upharpoonright n) \frown f_{\omega_1^{M_\alpha}}^{\alpha+1} \upharpoonright [n, \infty)$. We note that g is in $M_{\alpha+1}$. Since

$$M \models \langle f_\xi; \xi \in \omega_1^M \rangle \text{ is an } I\text{-good scale},$$

by elementarity of $j_{0,\alpha+1}$

$$M_{\alpha+1} \models j_{0,\alpha+1}(\langle f_\xi; \xi \in \omega_1^M \rangle) \text{ is an } j_{0,\alpha+1}(I)\text{-good scale}.$$

Therefore the set

$$\{\xi \in \omega_1^{M_{\alpha+1}}; g \leq f_\xi^{\alpha+1}\}$$

belongs to $M_{\alpha+1}$ and is $j_{0,\alpha+1}(I)$ -positive. Since $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ and

$$\begin{aligned} j_{\alpha+1,\omega_1}(\{\xi \in \omega_1^{M_{\alpha+1}}; g \leq f_\xi^{\alpha+1}\}) &= \{\xi \in \omega_1; g \leq f_\xi^{\omega_1}\} \\ &\subseteq \{\xi \in \omega_1; \text{body}(p) \leq f_\xi^{\omega_1}\}, \end{aligned}$$

the set $\{\xi \in \omega_1; \text{body}(p) \leq f_\xi^{\omega_1}\}$ is J -positive. □

We can show the strategic iteration lemma for the existence of a good scale using arguments of the previous proof and [10, Lemma 2.8.]. So we can conclude Shelah–Zapletal’s theorem that $\mathfrak{d} = \aleph_1$ is Π_2 -compact.

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