

Location Game with Degressive Weighted Voronoi Region

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Abstract: This paper investigates a facility location model with n weighted demand points on a line segment and a plane under a competitive environment. Customers at a demand point patronize one facility according to the attractiveness and the transportation cost. The smaller the difference of the location of the facilities, the less the customers distinguish the facilities. Two companies, the leader and the follower, establish their facilities by turns in this market to get as much buying power as possible. We formulate the problems to find the optimal location for the follower and for the leader, as a medianoid problem and a centroid problem respectively, and propose a solution procedure to solve them.

Keywords: continuous location, noncooperative games

1 Introduction

Competitive facility location problem was introduced by H.Hotelling [1], who studied the Nash equilibrium problem of two sellers on a linear market. S.L.Hakimi considered the Stackelberg equilibrium problem on a network [2], that is, two companies “leader” and “follower” establish their facilities on nodes in order to capture as much buying power as possible. He showed that the problem is NP-hard. Z.Drezner studied the same kind of a competitive problem on a plane [3].

This paper investigates an alternative game by two players on a linear market and on a plane. The leader company A locate his facility on the market first, and then the follower company B locate his facility. The aim of each player is to maximize their gain, i.e., to capture as much buying power as possible.

Various models has been proposed in this field [4], considering how demands are allocated, how is the customer's preference in facilities. Most commonly used assumption is that customers utilize only the nearest facility. This is expected to derive a proper approximation for fast food restaurants, coffee chain stores, video rental shops, etc. However, in this assumption, when two facilities are mutually located in near, the property which cannot necessarily be referred to as realistic will appear. We describe the details of this aspect below.

On a simple and typical case, the optimal strategy for B is locating adjacent to A, since B can expand his domain of influence by approaching to A. Suppose the demands are distributed continuously on a linear market, and the amount of demand at x is given by a non-negative integrable function $f(x)$. For the sake of simplicity, we assume that the gain of each player is equal to the amount of the capturing demand. In this case when A exists at a , the gain of B at

b is represented by $G(b|a) = \int_{-\infty}^{\frac{a+b}{2}} f(x) dx$ in the case where $b < a$, and $G(b|a) = \int_{\frac{a+b}{2}}^{\infty} f(x) dx$ in the case where $a < b$. It is natural to assume that $G(b|a) = \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx$ in the case where $b = a$. Since $f(x)$ is non-negative, $G(b|a)$ becomes greater as b approaches a , so the optimal strategy for B is locating his facility adjacent to A. This property holds not only continuous model but also discrete model. But the undesirable part of this model is that generally $G(b|a)$ is “too sensitive” around $b = a$. For example, if the demands are uniformly distributed on the interval $[0, 1]$ and the leader x exists at $\frac{1}{3}$, then the left-hand limit of $G(b|a)$ as b approaches a is equal to $\frac{1}{3}$, while the right-hand limit of $G(b|a)$ as b approaches a is equal to $\frac{2}{3}$. This means that the location of B exerts an influence on his gain too sensitively on the neighborhood of A. It must be noted that B gets or loses double gain by moving slightly around A with this case.

Similar sensitivity arises in the case where n competitive facilities exist on a plane. When a certain facility approaches to the neighborhood facility and pass through it to the other side, the Voronoi regions of them change places, which leads to the change of the amount of the capturing demand discontinuously at the point where the two facilities meet. These hypersensitive properties described above is not so realistic, so we propose an improved model based on two types of preferences for more reality.

It may be worth mentioning, in passing, that B's adjacency strategy is not optimal on the condition that A can locate his second facility in the same market [5]. In such a case, if B locate his facility close to A's first facility, then A locate the second one so as to narrow B's Voronoi region from both side, which results in making the gain of B close to 0.

2 Our Model

We consider continuous facility location problem with discrete demand. Two companies, A and B, establish their facilities by turns in the same market to get as much weight of demand point as possible. In the later part, we formulate the problems to find the optimal location for the follower and for the leader, as a medianoid problem and a centroid problem respectively, and propose a solution procedure to solve them.

p_i	location of demand point P_i ($i = 1 \cdots n$)
w_i	weight of P_i ($W = \sum_i w_i$)
a	location of the leader A
b	location of the follower B
α, β	weight of A, B
$d(a, b)$	Euclidean distance between A and B
G_A, G_B	gain of A, B

We introduce two types of preferences for customers to choose the facility. It will be appropriate to consider that attractiveness of the facility affects a customer's preference. So we define preference type 1 as follows.

Preference Type 1 : Customers choose one facility not only by the distance to them but also by the weight (attractiveness) of them. B's domain of influence is defined by

$$D_B = \{p = (x, y) \mid \alpha d(b, p) \leq \beta d(a, p)\} .$$

According to this preference, customers in domain D_B choose facility B, shown in figure 1. A's domain of influence D_A can be defined by complementary set of D_B . These forms weighted Voronoi regions.

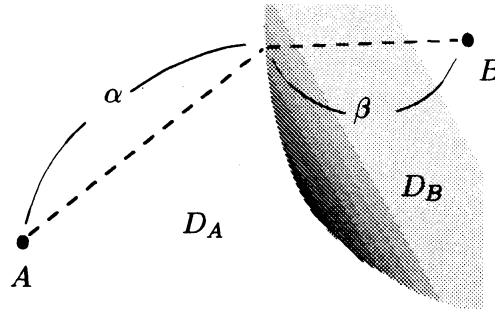


Figure 1: Proportional distribution model using distance and weight

D_B is a bounded set where $\alpha > \beta$. Obviously, in the case where $\alpha \leq \beta$, B can make the area of D_A close to 0 by approaching A, consequently the optimal strategy for B is always adjacent to A. So we consider only the case where $\alpha > \beta$ in the following part of this paper. Note that the area of D_B becomes larger as B go away from A, but at the same time D_B becomes out of the convex hull of demand points and B loses weights of demand point, so differentiation strategy is not always advantageous to B.

This preference avoids the hypersensitive properties mentioned in the introduction, but new problem arises. Even in the case where B is only slightly inferior to A, G_B becomes 0 at the same location as the competitor, because the area of D_B becomes 0 at that point. This property cannot be always acceptable, so we introduce following preference to avoid this.

Preference Type 2 : Not all customers are so sensitive about the difference of the distance between A and B. Let λ denotes the ratio of the customers who cannot ignore the magnitude of the distance between A and B. We define λ as follows.

$$\lambda = \begin{cases} 0, & 0 \leq d(a, b) \leq d_1 \\ 1, & d_2 \leq d(a, b) \\ \frac{1}{d_2 - d_1} (d(a, b) - d_1) & \text{otherwise} \end{cases}$$

d_1 and d_2 are given constants. Since λ is a function of $d(a, b)$, we use the notation $\lambda(d(a, b))$ in some cases.

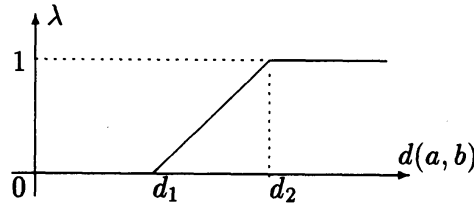


Figure 2: Ratio of the customers who cannot ignore the distance between A and B

If A and B are close to each other, no customers mind the difference of their location, which means that the influence of weighted Voronoi region is reduced close to zero. We call such Voronoi regions as the **degressive weighted Voronoi regions**.

The customers of the rate λ of the whole market distinguish the distance between A and B, on the other hand the customers of the rate $1 - \lambda$ do not distinguish the distance but

utilize a facility in proportion to the attractiveness of the facility. So we define G_B at location b with given a as follows.

$$G_B(b|a) = (1-\lambda)\frac{\beta}{\alpha+\beta}W + \lambda \sum_{p_i \in D_B} w_i$$

provided that $D_B = \{p = (x, y) \mid \alpha d(b, p) \leq \beta d(a, p)\}$

Since D_B is determined by the location of a and b , we use the notation $D_B(a, b)$ in some cases. The first term represents the amount of weight which is obtained from the customers who feel the distance to each facility is same and utilize it according to the attractiveness. The second term represents the amount of weight which is gained from the customers who cannot ignore the distance between A and B, and utilize a facility according to the distance and the attractiveness. The ratio of these customer segments change with the distance between A and B.

In this model, G_B becomes continuous function and has no hypersensitive properties, whereas $G(b|a)$ in the introduction is not continuous and too sensitive around $b = a$.

The medianoid problem is the problem to find the optimal location for the follower B which maximizes

$$\text{MP : } \max_b G_B(b|a) .$$

Let $b^*(a)$ denote the solution for MP with given a . The centroid problem is the problem to find the optimal location for the leader A which maximizes

$$\text{CP : } \max_a G_A(b^*(a)|a) .$$

This can be transformed into

$$\text{CP : } \min_a G_B(b^*(a)|a)$$

by using $G_A = W - G_B$.

In general, solving centroid problems are much harder than the case of medianoid problems, since the leader must take it into account that the follower locate his facility afterward with the aim of maximization of his own profit.

If B establish his facility at the location where $d(a, b) \leq d_1$, then G_B takes the constant value $\frac{\beta}{\alpha+\beta}W$ irrespective of the distribution of demand points, so we can use this value as the lower bound LB in searching for the optimal solution for medianoid problem.

3 Linear Market

In this section, we consider the case where demands are distributed discretely on the interval $[0, 1]$. At first, we consider MP with given location a of A.

Let D_{P_i} denote the existence region of B where B can include P_i in his domain of influence D_B . This domain is determined by the location a , so it is represented by

$$D_{P_i}(a) = \left[\frac{\alpha - \beta}{\alpha} p_i + \frac{\beta}{\alpha} a, \frac{\alpha + \beta}{\alpha} p_i - \frac{\beta}{\alpha} a \right] .$$

It is equivalent that P_i is under the influence of B, and B is inside D_{P_i} . So the next duality relation holds.

$$p_i \in D_B(a, b) \iff b \in D_{P_i}(a)$$

When the set of demand points and α, β are given, $D_{P_i}(a)$ of all demand points are fixed with arbitrary a . So we can draw a diagram with a as horizontal axis and the extreme points of $D_{P_i}(a)$ as vertical axis. An example with five demand points are shown in figure 3.

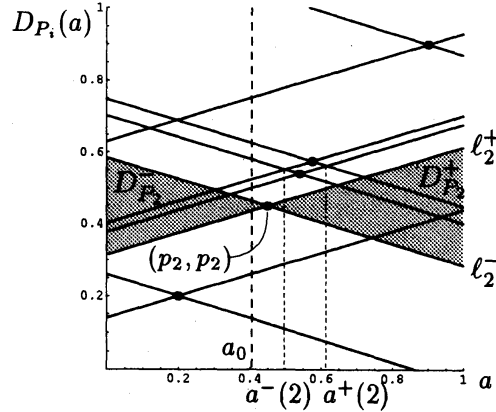


Figure 3: Diagram with 5 demand points (Shading region represents D_{P_2})

In this diagram, D_{P_i} is the domain that lies between the two lines passing through the point $P_i = (p_i, p_i)$ with slopes of $\frac{\beta}{\alpha}$ and $-\frac{\beta}{\alpha}$. Let ℓ_i^+, ℓ_i^- denote these two lines respectively. For the later part, we separate D_{P_i} into the left side and the right side of P_i , with labels of $D_{P_i}^-$ and $D_{P_i}^+$ respectively.

When A is at a certain point a_0 , $D_{P_i}(a_0)$ becomes the interval which is the intersection of the line $a = a_0$ and D_{P_i} . The values of the extreme points of $D_{P_i}(a_0)$ are read from the vertical axis; from the example, if A locates his facility at $a = 0.4$, B can take P_2 in his domain of influence locating at $b \in D_{P_2}(0.4) \approx [0.42, 0.44]$.

Suppose $D_{P_i}(a_0)$ and $D_{P_j}(a_0)$ have an common interval and b exists on the interval, and $b \notin D_{P_k}(a_0)$ for all k ($k \neq i, j$), rearranging terms with G_B yields

$$G_B(b|a_0) = (1 - \lambda) \frac{\beta}{\alpha + \beta} W + \lambda(w_i + w_j) = LB + \lambda(w_i + w_j - LB) .$$

Since LB is a constant and λ is an increasing function in the wider sense with $d(a_0, b)$, there is advantage to B in going away from A in the case where $w_i + w_j > LB$, meanwhile there is advantage to B in approaching A in the case where $w_j + w_j < LB$.

Let R_x denote a piece of region on a line $a = a_0$ divided by the boundaries of $D_{P_i}(a_0)$. As long as B is in a certain R_x , wherever B moves, the member of the capturing demand points remains unchanged. So in general, next property holds.

Property 1 With $b \in R_x$, if $\sum_{p_i \in D_B} w_i > LB$ then G_B increases in the wider sense as B goes away from A, else if $\sum_{p_i \in D_B} w_i < LB$ then G_B increases in the wider sense as B approaches to A.

Therefore in either case the candidate solutions for MP with given a_0 are obtained by enumerating the extreme points of R_x , such points are easily calculated as the intersection points of line $a = a_0$ and lines ℓ_i^+, ℓ_i^- . The optimal solution b^* for MP can be searched linearly in the order of $O(n)$ among the candidate solutions, for the point which maximizes G_B .

CP is the problem to find the location of a which minimizes $G_B(b^*(a) | a)$. At first, we consider the example in figure 3. The interval between two thin broken lines is the region where no pairing of D_{P_i} has an intersection. In such a region, no D_{P_i} overlaps with each other, which means that if A locates his facility in this region then B cannot take more than 1 demand point in B's domain of influence. For details, if all w_i are less than LB then the optimal location for B is the point which satisfies $d(a, b) \leq d_1$ and $\max G_B$ becomes LB . If w_m which has the maximum value among w_i is greater than LB then the optimal location for B is the farthest point from A in the range of $D_{P_m}(a)$ and $\max G_B$ becomes $LB + \lambda(w_m - LB)$.

But this example is a special case, since such an interval of a as no D_{P_i} overlaps with each other does not always exist. Let $a^-(k), a^+(k)$ denote the maximum and minimum value of a -coordinate where k domains of $D_{P_i}^-$ and $D_{P_i}^+$ overlap each other. For example, in figure 3, a -coordinate values of thin broken lines are $a^-(2), a^+(2)$ from left to right. If $a^-(2) > a^+(2)$ then there is no interval of a where less than or equal to two D_{P_i} overlap with each other.

Let k^* denote minimum k which satisfied $a^-(k) < a^+(k)$. The solution for CP exists in the interval $[a^-(k^*), a^+(k^*)]$, since A can reduce $\sum_{p_i \in D_B} w_i$ as small as possible by locating his facility in this interval. Simultaneously A must reduce the maximum value of λ with b^* , by minimizing the maximum distance from a to the extreme points of $D_{P_i}(a)$ ($p_i \in [a^-(k^*), a^+(k^*)]$). Such a point a^* is calculated by

$$\begin{aligned} a^* &= \frac{1}{2} \left(\left(\frac{\alpha + \beta}{\alpha} p_i - \frac{\beta}{\alpha} a^+(k^*) \right) + \left(\frac{\alpha + \beta}{\alpha} p_j - \frac{\beta}{\alpha} a^-(k^*) \right) \right) \\ &= \left(\frac{\alpha + \beta}{2\alpha} \right) (p_i + p_j) - \frac{\beta}{2\alpha} (a^+(k^*) + a^-(k^*)) \end{aligned}$$

where $p_i, p_j \in [a^-(k^*), a^+(k^*)]$ are the nearest demand points to $a^-(k^*), a^+(k^*)$ respectively. Then a^* is the solution for CP.

Now we must search for k^* and the interval $[a^-(k^*), a^+(k^*)]$. The candidate solutions for $a^-(k^*), a^+(k^*)$ are the points where the number of the overlap of D_{P_i} changes, i.e., the lattice points composed by ℓ_i^+ and ℓ_i^- . Let $P_{i,j}^+$ denote the intersection of ℓ_i^+ and ℓ_j^- where $i < j$, $P_{i,j}^-$ denote the intersection of ℓ_i^- and ℓ_j^+ . Then $a^+(k)$ is the point which has the minimum value of a -coordinate among $P_{i,i+k-1}^+$, $a^-(k)$ is the maximum a -coordinate point among $P_{i,i+k-1}^-$.

When searching for $a^+(k)$, we can utilize the following property. If inequality

$$a_{i+k} - a_{i+k-1} < \frac{\alpha - \beta}{\alpha + \beta} (a_{i+1} - a_i)$$

holds then a -coordinate value of $P_{i,i+k-1}^+$ is greater than that of $P_{i+1,i+k}^+$. We omit the similar inequality for $a^-(k)$. If $a^-(k) < a^+(k)$ then $k, a^-(k), a^+(k)$ are the candidate solutions. When searching for k^* , binary search is available. This method changes the value of k from 1 to $n-1, 2, n-2, 3, \dots, \frac{n}{2}$ to find smallest k which satisfies $a^-(k) < a^+(k)$. By this means we can find $k^*, a^-(k^*), a^+(k^*)$ and a^* which minimizes $\max G_B$ as the solution for CP.

4 Plane Market

This section considers MP on a plane. We use following notations for two dimensional model.

$$\begin{aligned} p_i &= (p_{i1}, p_{i2}) && \text{location of demand point } P_i \ (i = 1 \cdots n) \\ a &= (a_1, a_2) && \text{location of A} \\ b &= (b_1, b_2) && \text{location of B} \end{aligned}$$

The other notations are same as previous section.

The shape of B's domain of influence D_B becomes a circle represented by

$$D_B = \left\{ (x, y) \mid \left(x - \frac{b_1 - k^2 a_1}{1 - k^2} \right)^2 + \left(y - \frac{b_2 - k^2 a_2}{1 - k^2} \right)^2 \leq \frac{k^2}{(1 - k^2)^2} ((a_1 - b_1)^2 + (a_2 - b_2)^2) \right\}$$

provided that $k = \frac{\beta}{\alpha}$.

Using similar way of thinking in the previous section, we start with formulating the domain D_{P_i} , which represent the existence region of B where B can include P_i in his domain of influence D_B .

Let O denote the center of D_B , r denote the radius of D_B . Calculating the length of $\overline{AO}$ and r , we obtain the relation

$$r = \frac{\beta}{\alpha} \overline{AO}.$$

If we draw two tangent lines to the circles D_B through A, then the angle between the lines which contain D_A remains constant wherever B moves. Let θ denote the angle, then we obtain

$$\sin \frac{\theta}{2} = \frac{\beta}{\alpha}.$$

Since the radius of D_B is in proportion to the distance between A and O, and the included angle of two tangent lines is constant, it is natural to use polar coordinate in this part. Without loss of generality, A is assumed to be given at origin (0,0). We use polar coordinate $b = (\gamma, d)$, $p_i = (\delta_i, \ell_i)$, provided that $d = d(a, b)$ and γ, δ_i, ℓ_i satisfy the relations $(b_1, b_2) = (d \cos \gamma, d \sin \gamma)$, $(p_{i1}, p_{i2}) = (\ell_i \cos \delta_i, \ell_i \sin \delta_i)$, $0 \leq \gamma, \delta_i < 2\pi$.

When P_i is on the boundary of D_B , using cosine theorem, the following equation is derived on condition that $\delta_i - \frac{\theta}{2} \leq \gamma \leq \delta_i + \frac{\theta}{2}$.

$$2d^2 + (1 + \cos \theta) \ell_i^2 - 4d\ell_i \cos(\delta_i - \gamma) = 0$$

Conversely if this relation is satisfied, then P_i is on the boundary of D_B . Eliminating θ by using $\sin \frac{\theta}{2} = \frac{\beta}{\alpha} = k$, this equation becomes

$$d^2 + (1 - k^2) \ell_i^2 - 2d\ell_i \cos(\delta_i - \gamma) = 0.$$

When α, β and P_i are given, we can think left side member of this equation as an implicit function $h(\gamma, d)$. Now D_{P_i} can be formulated as $D_{P_i} = \{(\gamma, d) \mid h(\gamma, d) \leq 0\}$. If B is in this domain, then P_i is in B's domain of influence as follows.

$$b \in D_{P_i} \iff p_i \in D_B$$

Let B_{P_i} denote the boundary of D_{P_i} , then B_{P_i} is formulated as $B_{P_i} = \{(\gamma, d) \mid h(\gamma, d) = 0\}$ which shape of the graph becomes ovaloid shape illustrated in figure 4.

If $D_{P_i} \cap D_{P_j} \neq \emptyset$ and $b \in D_{P_i} \cap D_{P_j}$, then B can get at least $\max\{\lambda(w_i + w_j - LB), LB\}$. So if we can omit the value of λ and w_i , the candidate solution for MP is b^0 corresponding to D_B which covers the maximum number of P_i . Such b^0 is on the intersection region of D_{P_i} , which includes the intersection points of B_{P_i} . For the first step to find the solution for MP, we start to examine the intersection points of B_{P_i} , which are potential candidate solutions.

When searching for the intersection with a certain B_{P_i} , we can obviously exclude B_{P_j} which satisfies $|\delta_i - \delta_j| > \theta$ or $\ell_j(1 - k) > \ell_i(1 + k)$ or $\ell_j(1 + k) < \ell_i(1 - k)$. The existence region of demand points which boundary intersects B_{P_i} is expressed by $\{(x, y) \mid (x - \frac{p_{i1}}{1 - k^2})^2 +$

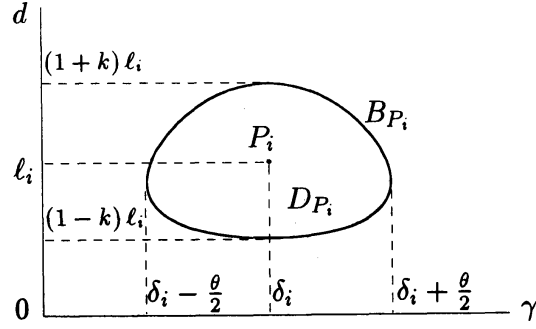


Figure 4: Boundary of existence region of B which satisfies $p_i \in D_B$

$(y - \frac{p_{i2}}{1-k^2})^2 \leq \frac{(2-k^2)k^2}{(1-k^2)^2} (p_{i1}^2 + p_{i2}^2)$. If P_j is not in this region, then B_{P_j} does not intersect B_{P_i} . The boundary is constrained in shape, two of them intersect at most 2 times each other, and one never enclosed by the other.

We can calculate the coordinates of intersection points between B_{P_i} and B_{P_j} , but the result becomes very complicated expression either in polar coordinates and in orthogonal coordinates. It becomes a little simpler in orthogonal coordinates, so we use orthogonal coordinates in this part. Solving the following simultaneous equations,

$$\begin{cases} (x - \frac{p_{i1}}{1-k^2})^2 + (y - \frac{p_{i2}}{1-k^2})^2 = \frac{k^2}{(1-k^2)^2} (p_{i1}^2 + p_{i2}^2) \\ (x - \frac{p_{j1}}{1-k^2})^2 + (y - \frac{p_{j2}}{1-k^2})^2 = \frac{k^2}{(1-k^2)^2} (p_{j1}^2 + p_{j2}^2) \end{cases}$$

we obtain the intersection points between B_{P_i} and B_{P_j} as follows.

$$\begin{aligned} x &= \frac{-2p_{i2}UV + 2p_{i1}V^2 + RU(US + VT) \pm \sqrt{Z}}{2R(U^2 + V^2)} \\ y &= \frac{V(2p_{i2}U^2 + V(-2p_{i1}U + RUS + RVT)) \mp U\sqrt{Z}}{2RV(U^2 + V^2)} \end{aligned}$$

provided that

$$\begin{aligned} Z &= (2V(p_{i1}V - p_{i2}U) + RU(US + VT))^2 \\ &\quad - R(U^2 + V^2)(R(US + VT)^2 + 4V(p_{i1}^2V + p_{i2}^2V - p_{i2}(US + VT))) \\ S &= p_{i1} + p_{j1}, T = p_{i2} + p_{j2}, U = p_{i1} - p_{j1}, V = p_{i2} - p_{j2}. \end{aligned}$$

These solutions can be translated into polar coordinates as $d = \sqrt{x^2 + y^2}$, $\gamma = \arccos(y/d)$ where $y \geq 0$, $\gamma = 2\pi - \arccos(y/d)$ where $y < 0$.

Let R_x denote a piece of region on γ - d plane divided by B_{P_i} . Then the same property discussed in the previous section holds, i.e., as long as b remains in a fixed region R_x , if $\sum_{p_i \in D_B} w_i > LB$ then G_B is expected to increase as B goes away from A, else if $\sum_{p_i \in D_B} w_i < LB$ then G_B is expected to increase as B approaches to A. So the candidate solution for MP is the farthest or nearest point from A in the region of R_x .

The shading region in figure 5 shows an example of the region of $D_{P_i} \cap \overline{D_{P_j}} \cap D_{P_k} \cap \overline{D_{P_l}}$. As long as b is included in this region, the member of demand points covered by D_B is fixed.

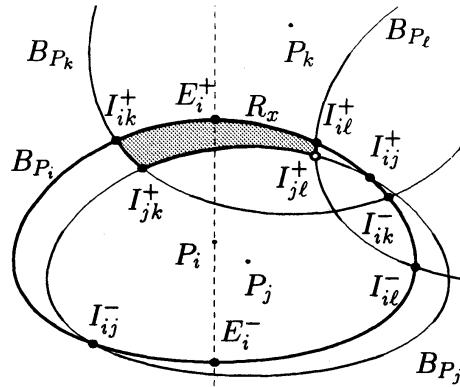


Figure 5: Labels for the extreme points of divided regions

In order to treat the extreme point of divided region R_x , we label each intersection of B_{P_i} as shown in figure 5. When $\delta_i < \delta_j$, we can uniquely label the intersection of B_{P_i} and B_{P_j} as I_{ij}^+ and I_{ij}^- , according to the descending order of the value of d -coordinate ($I_{ij}^+ = I_{ij}^-$ when touching). The points where B_{P_i} attains maximum and minimum d -coordinates value must be taken into consideration. These points are labeled E_i^+ and E_i^- respectively, which coordinates are $(\delta_i, (1+k)\ell_i)$ and $(\delta_i, (1-k)\ell_i)$.

In general, tree algorithm is useful for searching for this kind of intersections, but what we want is the maximum value of G_B in the region, so we must calculate it with λ and compare with the other candidate solution, so tree algorithm is not appropriate in this case.

On R_x in figure 5, E_i^+ is the farthest point from A, and I_{jk}^+ is the nearest. Therefore E_i^+ and I_{jk}^+ are listed as the temporary candidate solutions with R_x . Comparing $G_B(E_i^+)$ and $G_B(I_{jk}^+)$, the greater one is the candidate solution. Note that strictly I_{jl}^+ is not contained in R_x in this figure, but it is the extreme point of the other region, so it will be listed as a candidate solution in the other scan. In the same way, $I_{il}^+, I_{jl}^+, I_{ij}^+, \dots$ will be also scanned in our algorithm.

Our algorithm for solving M_P is shown below.

Step 1. Sort P_i by ascending order of δ_i , and save the original order to S_k .

Step 2. $u \leftarrow 1$

Step 3. $G_B \leftarrow LB$, $v \leftarrow u + 1$, $L \leftarrow \phi$

Step 4. If $\delta_v - \delta_u > \theta$ then go to **step 6**.

Check whether B_{P_u} and B_{P_v} intersect each other. If intersection exists, calc the points I_{uv}^+ and I_{uv}^- . $L \leftarrow L + \{I_{uv}^+, I_{uv}^-\}$.

Step 5. $v \leftarrow v + 1$. Go to **step 4**.

Step 6. $L \leftarrow L + \{E^+, E^-\}$. Sort elements of L by ascending order of the clockwise angles between the element and $\overline{P_u T}$. (T is a contact point between B_{P_u} and the line $\gamma = \delta_u - \frac{\theta}{2}$)

Step 7. $G_u \leftarrow 0$, $B_u \leftarrow (0, 0)$, $t \leftarrow 1$, $V \leftarrow \{S_u\}$

Step 8. Choose t -th element of L as L_t . If $d(a, L_t) \leq d_1$ then $G_u = LB$ and go to **step 10**.

If $L_t = I_{um}^+$ then $V \leftarrow V \cup \{S_m\}$ and $V_t \leftarrow V$. If $L_t = I_{um}^-$ then $V_t \leftarrow V$ and

- $V \leftarrow V \setminus \{S_m\}$.
 If $L_t = E^+$ or $L_t = E^-$ then $V_t \leftarrow V$.
- step 9.** $m \leftarrow (1 - \lambda(L_t))LB + \lambda(L_t) \sum_{i \in V_t} w_i$. If $m > \max\{G_u, LB\}$ then $G_u \leftarrow m$ and $B_u \leftarrow L_t$.
- ep 10.** If $t < \text{the number of element of } L$ then $t \leftarrow t + 1$ and go to **step 8**.
- ep 11.** If $G_u > G_B$ then $G_B \leftarrow G_u$ and $b \leftarrow B_u$. If $u \geq n - 1$ then **stop** else $u \leftarrow u + 1$.
 go to **step 3**.

Basic idea is selecting one candidate solution for one corresponding region. S_k holds the original index i of P_k and δ_k . L holds the position of the temporary candidate solutions on B_{P_u} . For example, sorted L becomes $\{I_{ik}^+, E_i^+, I_{il}^+, I_{ij}^+, I_{ik}^-, I_{il}^-, E_i^-, I_{ij}^-\}$ with $u = i$ in figure 5. V holds a set of indices of demand points which are in B's domain of influence when B is at L_t . If $u = i$ and $L_t = I_{ij}^+$ then $V = \{i, k, \ell, j\}$ in figure 5. When algorithm stops, the solution for MP is b , with the maximum value of G_B .

On this algorithm, step 4-5 takes $O(n-1)$ times, step 6 takes $O(2(n-1) \log 2(n-1))$ times, step 8-10 takes $O(2(n-1))$ times, step 3-11 takes $O(n)$ times, so the total computational complexity becomes $O(n^2 \log n)$.

5 Conclusion and Further Research

We considered a competitive facility location problem on a linear and a plane market introducing two types of preferences to avoid hypersensitive property. One of the preferences is a model for a kind of psychological distance. We formulated the alternative game by two players and proposed a method to find the optimal location for the follower and the leader on a linear market. The solution procedure for the follower on a plane was also shown.

Our further research will be on finding the optimal location for the leader on a plane market. We used Euclidean distance in this paper, but weight proportional distribution model with ℓ_1 -distance seems to be another interesting problem. But it is conjectured much more difficult, since B's domain of influence generally becomes non-convex domain.

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