

APPLICATIONS OF CR GEOMETRY TO REPRESENTATIONS OF $SU(p, q)$

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Conformal geometry on pseudo-Riemannian manifolds can be applied to the representation theory of the group $SO(p, q)$ (cf. [3] [11] [12] [13] [14] and references therein). Kostant used the conformal invariance of the vanishing of scalar curvature on 6 dimensional manifolds to explore the minimal representation of $SO(4, 4)$ in [14]. Recently, T. Kobayashi and B. Orsted [11] [12] [13] gave a geometric and intrinsic model of the minimal irreducible unitary representation $\varpi^{p,q}$ of $SO(p, q)$ on $S^{p-1} \times S^{q-1}$ and on various pseudo-Riemannian manifolds which are conformally equivalent, by using the Yamabe operator. They also gave branching formulae and unitarization of various models. Here we use CR geometry to realize representations of $SU(p, q)$.

1. Preliminaries on CR Geometry

Let M be a real $(2n + 1)$ -dimensional orientable C^∞ manifold. A *CR structure* on M is a n -dimensional complex subbundle $T_{1,0}M$ of the complexified tangent bundle CTM satisfying $T_{1,0}M \cap T_{0,1}M = \{0\}$, where $T_{0,1}M = \overline{T_{1,0}M}$, and the integrability condition: $[Z_1, Z_2] \in C^\infty(M, T_{1,0}M)$ whenever $Z_1, Z_2 \in C^\infty(M, T_{1,0}M)$. $T_{1,0}M$ is usually called the *complex tangential space*. Set

$$(1.1) \quad H = \operatorname{Re}\{T_{1,0}M \oplus T_{0,1}M\},$$

the $2n$ -dimensional *real horizontal subbundle* of TM . H carries a *complex structure* $J : H \longrightarrow H$ satisfying $J^2 = -\operatorname{id}_H$ and $T_{1,0}M = \ker(J - i \cdot \operatorname{id}_{CH})$, $T_{0,1}M = \ker(J + i \cdot \operatorname{id}_{CH})$. When M is the boundary of a domain in a complex manifold W , it has an *induced CR structure* from the complex structure of W defined by

$$(1.2) \quad T_{1,0}M = CTM \cap T_{1,0}W,$$

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if $\dim(T_{1,0}M)_x = \text{const.}$ for each $x \in M$, where $T_{1,0}W$ is the holomorphic tangential space of complex manifold W .

A mapping $f : (M_1, T_{1,0}M_1) \longrightarrow (M_2, T_{1,0}M_2)$ is called a *Cauchy-Riemann mapping* (or *CR mapping*) if

$$(1.3) \quad f_* T_{1,0}M_1 \subset T_{1,0}M_2,$$

where f_* is the tangential mapping of f . If f is invertible, f and f^{-1} are both CR mappings, f is called a *CR diffeomorphism*.

Let θ be a 1-form on M such that

$$(1.4) \quad \ker \theta = H.$$

We require θ to be a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is non-vanishing on M . Such θ is called a *pseudohermitian structure* on $(M, T_{1,0}M)$. We call the triple $(M, T_{1,0}M, \theta)$ a *pseudohermitian manifold*. θ plays the role of metric g in pseudo-Riemannian geometry.

We say $\tilde{\theta}$ is *conformal* to θ if

$$(1.5) \quad \tilde{\theta} = \phi^2 \theta$$

for some non-vanishing smooth function ϕ on M . A CR mapping between two pseudohermitian manifolds, $f : (M_1, T_{1,0}M_1, \theta_1) \longrightarrow (M_2, T_{1,0}M_2, \theta_2)$, is called *conformal* if $f^* \theta_2 = \phi^2 \theta_1$ for some non-vanishing smooth function ϕ on M_1 .

We can define a Hermitian form on $T_{1,0}M$ associated to a pseudohermitian structure θ by

$$(1.6) \quad L_\theta(V, \bar{W}) = -id\theta(V \wedge \bar{W}),$$

which is called *the Levi form* of θ .

If the Levi form has k positive eigenvalues and $n - k$ negative eigenvalues, $(M, T_{1,0}M, \theta)$ is said to be *strictly k -pseudoconvex*. The inner product $L_\theta(\cdot, \cdot)$ determines a dual form $L_\theta^*(\cdot, \cdot)$ on H^* . $L_\theta^*(\cdot, \cdot)$ can be naturally extended to T^*M .

In [19], Webster showed that there exists a natural connection on the bundle $T_{1,0}M$ adapted to a pseudohermitian structure θ . For a pseudohermitian structure θ on a strictly

k -pseudoconvex CR manifold $(M, T_{1,0}M, \theta)$, there is a unique vector field T , which is transversal to H , defined by

$$(1.7) \quad \theta(T) = 1, \quad d\theta(T \wedge \cdot) = 0.$$

Let θ^α be an *admissible coframe*, i.e. $(1, 0)$ -forms θ^α form a basis for $T_{1,0}^*M$ such that $\theta^\alpha(T) = 0$ for all $\alpha = 1, \dots, n$. The integrability condition implies

$$(1.8) \quad d\theta = ig_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta$$

for some Hermitian matrix of functions $(g_{\alpha\bar{\beta}})$, which is nondegenerate and has k positive eigenvalues and $n - k$ negative eigenvalues if $(M, T_{1,0}M, \theta)$ is strictly k -pseudoconvex. Webster showed that there are uniquely determined 1-forms ω_α^β and τ^β on M satisfying

$$(1.9) \quad \begin{cases} d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta \\ \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} = dg_{\alpha\bar{\beta}} \\ \tau_\alpha \wedge \theta^\alpha = 0, \end{cases}$$

where we use $(g_{\alpha\bar{\beta}})$ to raise and lower indices, e.g. $\omega_{\alpha\bar{\beta}} = \omega_\alpha^\gamma g_{\gamma\bar{\beta}}$. Let

$$(1.10) \quad \Omega_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha.$$

Webster showed that Ω_β^α could be written as

$$(1.11) \quad \Omega_\beta^\alpha = R_{\beta\bar{\rho}\bar{\sigma}}^\alpha \theta^\rho \wedge \bar{\theta}^\sigma + W_{\beta\bar{\rho}}^\alpha \theta^\rho \wedge \theta - W_{\beta\bar{\rho}}^\alpha \bar{\theta}^\rho \wedge \bar{\theta} + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha$$

The *Webster-Ricci tensor* of $(M, T_{1,0}M, \theta)$ has components $R_{\alpha\bar{\beta}} = R_{\rho\alpha\bar{\beta}}^\rho$. The *Webster scalar curvature* is

$$(1.12) \quad R_\theta = g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}.$$

The *CR Yamabe problem* is to find a contact form $\tilde{\theta} = u^2\theta, u > 0$, which is conformal to the given contact form θ , such that $R_{\tilde{\theta}} \equiv \text{constant}$. This problem is considered by Lee and Jerison [9] for strictly pseudoconvex CR manifolds and completely solved recently by N. Gamara and R. Yacoub [6] [7].

A pseudohermitian manifold $(M, T_{1,0}M, \theta)$ has a natural volume form

$$(1.13) \quad \psi_\theta = (-1)^{n-k} \theta \wedge (d\theta)^n,$$

which is nowhere vanishing because M is strictly k -pseudoconvex. It induces an L^2 inner product on functions

$$(1.14) \quad \langle u, v \rangle_\theta = \int_M u \bar{v} \psi_\theta,$$

and an L^2 inner product on sections of H^* ,

$$(1.15) \quad \langle \omega, \eta \rangle_\theta = \int_M L_\theta^*(\omega, \eta) \psi_\theta.$$

For $u \in C^\infty(M)$, we define a section $d_b u$ of H^* by

$$(1.16) \quad d_b u = pr \circ du,$$

where $pr : T^*M \rightarrow H^*$ is the restriction map. We can define the *SubLaplacian* $\square_\theta$ associated to a strictly k -pseudoconvex contact form θ by

$$(1.17) \quad \langle \square_\theta u, v \rangle_\theta = \frac{1}{2} \langle d_b u, d_b v \rangle_\theta.$$

Since evidently, $|\theta|_\theta = 0$, $L_\theta^*(\cdot, \cdot)$ is degenerate on T^*M and so the operator $\square_\theta$ is a degenerate ultrahyperbolic operator.

Proposition 1.1. (Proposition 4.10 in [15]) *If $u \in C_0^\infty$, then,*

$$(1.18) \quad \square_\theta u = -u_\alpha{}^\alpha - u_{\bar{\alpha}}{}^{\bar{\alpha}}.$$

Define a product on $\mathbb{C}^{n+2}$ by

$$(1.19) \quad (\zeta, \xi)_{p,q} = \sum_{j=0}^{n+1} \varepsilon_j \zeta_j \bar{\xi}_j,$$

where $n+2 = p+q$, and

$$(1.20) \quad \varepsilon_j = \begin{cases} 1, & \text{for } j = 0, 1, \dots, p-1, \\ -1, & \text{for } j = p, \dots, p+q-1. \end{cases}$$

We denote $(\zeta, \zeta)_{p,q}$ by $|\zeta|_{p,q}^2$ for $\zeta \in \mathbb{C}^{n+2}$. Similarly, we define a product on $\mathbb{C}^n$ by

$$(1.21) \quad (z, w)_{p-1,q-1} = \sum_{\alpha=1}^n \varepsilon_\alpha z_\alpha \bar{w}_\alpha.$$

We also denote $(z, z)_{p-1,q-1}$ by $|z|_{p-1,q-1}^2$ for $z \in \mathbb{C}^n$.

APPLICATIONS OF CR GEOMETRY TO REPRESENTATIONS OF $SU(p, q)$

The simplest CR manifold is the *Heisenberg group* $\mathbb{H}^{p-1, q-1}$, whose underlying manifold is $\mathbb{C}^{p+q-2} \times \mathbb{R}$, with coordinates (z, t) . Its multiplication is given by

$$(1.22) \quad (z, t) \cdot (z', t') = (z + z', t + t' + 2\text{Im}(z, z')_{p-1, q-1}).$$

The vector fields

$$(1.23) \quad Z_\alpha = \frac{\partial}{\partial z_\alpha} + i\varepsilon_\alpha \bar{z}_\alpha \frac{\partial}{\partial t},$$

$\alpha = 1, \dots, n$, are left invariant vector fields on $\mathbb{H}^{p-1, q-1}$. The *standard CR structure* on the Heisenberg group $\mathbb{H}^{p-1, q-1}$ is given by the subbundle

$$(1.24) \quad T_{1,0}\mathbb{H}^{p-1, q-1} = \text{span}_{\mathbb{C}}\{Z_1, \dots, Z_n\}.$$

Let

$$(1.25) \quad \theta_{\mathbb{H}^{p-1, q-1}} = dt + \sum_{\alpha=1}^n i\varepsilon_\alpha (z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha)$$

be the *standard contact form* on $\mathbb{H}^{p-1, q-1}$.

$$(1.26) \quad \square_{\mathbb{H}^{p-1, q-1}} = -\frac{1}{2} \sum_{\alpha=1}^{p-1} (Z_\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z_\alpha) + \frac{1}{2} \sum_{\alpha=p}^{p+q-2} (Z_\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z_\alpha).$$

Let us consider a real hypersurface $Q'_{p,q}$ in $\mathbb{C}^{n+1}$ defined by equation

$$(1.27) \quad \text{Im} z_0 = |z|_{p-1, q-1}^2, \quad z \in \mathbb{C}^n, \quad z_0 \in \mathbb{C},$$

which is the boundary of the *Siegel upper half space*

$$(1.28) \quad \mathcal{S} = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^n; \text{Im } z_0 > |z|_{p-1, q-1}^2\}.$$

The *Cayley transformation* C is defined by

$$(1.29) \quad w_0 = \frac{z_0 - i}{z_0 + i}, \quad w_\alpha = \frac{2z_\alpha}{z_0 + i},$$

which transforms the hypersurface $Q'_{p,q}$ into the hyperquadric $Q_{p,q}$,

$$(1.30) \quad Q_{p,q} = \{w = (w_0, w'); w_0 \in \mathbb{C}, w' \in \mathbb{C}^n, |w_0|^2 + |w'|_{p-1, q-1}^2 = 1\}.$$

Now introduce homogeneous coordinates ζ_j , $j = 0, \dots, n+1$. By equations

$$(1.31) \quad z_j = \frac{\zeta_j}{\zeta_{n+1}}, \quad j = 0, \dots, n+1,$$

$\mathbf{C}^{n+1}$ is embedded as an open subset of the complex projective space $\mathbf{CP}^{n+1}$ of dimension $n + 1$. In the homogeneous coordinates, $Q_{p,q}$ is embedded as an open subset of the *projective hyperquadric*

$$(1.32) \quad \overline{Q}_{p,q} = \{\zeta = (\zeta_0, \dots, \zeta_{n+1}) \in \mathbf{CP}^{n+1}; |\zeta|_{p,q}^2 = 0\}.$$

Projective hyperquadric $\overline{Q}_{p,q}$ is the compactification of $Q_{p,q}$ in $\mathbf{CP}^{n+1}$. The hypersurface $Q'_{p,q}$ and the projective hyperquadric $\overline{Q}_{p,q}$ have induced CR structures by (1.2) from complex manifolds $\mathbf{C}^{n+1}$ and $\mathbf{CP}^{n+1}$, respectively.

$SU(p, q)$ is the group of unimodular transformations preserving the Hermitian form (1.19). Its center K consists of $n + 2$ transformations. Then $SU(p, q)/K$ acts on $\overline{Q}_{p,q}$ effectively and $PU(p, q) = SU(p, q)/K$. It is well known that $\text{Aut}_{CR} \overline{Q}_{p,q} = PU(p, q)$ [4].

Pseudo-Riemannian geometry	CR geometry
<i>A metric</i> g	<i>A contact form</i> θ
<i>conformal</i> $\tilde{g} = \phi^2 g$	$\tilde{\theta} = \phi^2 \theta$
<i>pseudo-Riemannian connection</i>	<i>Webster connection</i>
<i>the Laplacian</i> $\square_g$	<i>the SubLaplacians</i> $\square_\theta$
$SO(p, q)$	$SU(p, q)$
<i>the flat model</i> $\mathbb{R}^{p-1, q-1}$	$\mathbb{H}^{p-1, q-1}$
$S^{p-1} \times S^{q-1}$	<i>the projective hyperquadric</i> $\overline{Q}_{p,q}$
<i>the Yamabe operator</i>	<i>the CR Yamabe operator</i>
$\vdots$	$\vdots$

2. Representations realized as conformal CR diffeomorphisms

Let $Q = \dim M + 1 = 2n + 2$, the *homogeneous dimension* of M . The following transformation formula is due to Lee.

Proposition 2.1. *Let $(M, T_{1,0}M, \theta)$ be a pseudohermitian manifold with $\dim M = 2n + 1$. The Webster scalar curvature $R_{\tilde{\theta}}$ associated with the pseudohermitian structure $\tilde{\theta} = u^{\frac{4}{Q-2}} \theta$ satisfies*

$$(2.1) \quad b_n \square_\theta u + R_\theta u = R_{\tilde{\theta}} u^{\frac{Q+2}{Q-2}},$$

where $b_n = 2 + \frac{2}{n}$.

The following is a transformation formula for the SubLaplacians under a conformal CR transformation.

Proposition 2.2. *Let $(M_1, T_{1,0}M_1)$ and $(M_2, T_{1,0}M_2)$ be two CR manifolds with strictly k -pseudoconvex pseudohermitian structure θ_1 and θ_2 , respectively. Suppose $\Phi : (M_1, T_{1,0}M_1) \longrightarrow (M_2, T_{1,0}M_2)$ is a CR diffeomorphism with $\Phi^*\theta_2 = u^{\frac{4}{Q-2}}\theta_1$ for some positive smooth function u on M_1 . Then*

$$(2.2) \quad \square_{\theta_1}(u \cdot \Phi^* f) - u^{\frac{Q+2}{Q-2}} \Phi^* (\square_{\theta_2} f) = \square_{\theta_1} u \cdot \Phi^* f,$$

for any smooth real function f on M_2 .

Now define the CR Yamabe operator to be

$$(2.3) \quad \tilde{\square}_{\theta} = \square_{\theta} + \frac{1}{b_n} R_{\theta},$$

where $b_n = 2 + \frac{2}{n}$, R_{θ} is the Webster scalar curvature (1.12). The transformation formula for the CR Yamabe operator is a consequence of Corollary 2.1 and Proposition 2.2 as follows.

Proposition 2.3. *Under the same assumption as in proposition 2.2, we have that*

$$(2.4) \quad \tilde{\square}_{\theta_1}(u \cdot \Phi^* f) = u^{\frac{Q+2}{Q-2}} \Phi^* (\tilde{\square}_{\theta_2} f),$$

for any smooth function f on M_2 .

Suppose $(M_1, T_{1,0}M_1, \theta_1)$ and $(M_2, T_{1,0}M_2, \theta_2)$ are two pseudohermitian manifolds of homogeneous dimension Q . Let conformal CR mapping $\Phi : (M_1, T_{1,0}M_1, \theta_1) \longrightarrow (M_2, T_{1,0}M_2, \theta_2)$ be a local diffeomorphism such that

$$(2.5) \quad \Phi^*\theta_2 = \Omega^2\theta_1,$$

for some positive function Ω on M_1 . We can define *twisted pull back*

$$(2.6) \quad \Phi_{\lambda}^* : C^{\infty}(M_2) \longrightarrow C^{\infty}(M_1), \quad f \longmapsto \Omega^{\lambda}(\Phi^* f).$$

Let G be a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$. We write the action of $h \in G$ as $L_h : (M, T_{1,0}M, \theta) \longrightarrow$

$(M, T_{1,0}M, \theta)$, $x \mapsto L_h x$. There exists a positive valued function $\Omega(h, x)$ for $h \in G$ and $x \in M$ such that

$$(2.7) \quad L_h^* \theta = \Omega(h, \cdot)^2 \theta.$$

We have the *cocycle formula* for $\Omega(\cdot, \cdot)$.

Proposition 2.4. *For $h_1, h_2 \in G$ and $x \in M$, we have*

$$(2.8) \quad \Omega(h_1 h_2, x) = \Omega(h_1, L_{h_2} x) \Omega(h_2, x).$$

Now for $\lambda \in \mathbf{C}$, we can define a representation ϖ_λ of the group G on $C^\infty(M)$ as follows. For $h \in G$, $f \in C^\infty(M)$ and $x \in M$, let

$$(2.9) \quad (\varpi_\lambda(h^{-1})f)(x) = \Omega(h, x)^\lambda f(L_h x).$$

Proposition 2.4 assures that $\varpi_\lambda(h_1 h_2) = \varpi_\lambda(h_1) \varpi_\lambda(h_2)$, i.e., ϖ_λ is a representation of G . Thus, $\tilde{\square}_\theta f = 0$ if and only if $\tilde{\square}_\theta \left(\Omega^{\frac{Q-2}{2}} \tilde{\Phi}^* f \right) = 0$. In summary, we have the following theorem.

Theorem 2.5. *Suppose G is a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$ of homogeneous dimension Q . Then,*

- (1) *the CR Yamabe operator $\tilde{\square}_\theta$ is an intertwining operator from $\varpi_{\frac{Q-2}{2}}$ to $\varpi_{\frac{Q+2}{2}}$.*
- (2) *The kernel $\ker \tilde{\square}_\theta$ is a subrepresentation of G through $\varpi_{\frac{Q-2}{2}}$.*

3. The CR Yamabe operator on the hypersurface $Q'_{p,q}$

Let $\xi \mapsto [\xi]$ denote the canonical projection of $\mathbf{C}^{n+2} \setminus \{0\}$ into the complex projective space $\mathbf{C}P^{n+1}$. It is easy to see that the transformation

$$(3.1) \quad I(z_0, z_1, \dots, z_n) = \left[\frac{z_0 - i}{2}, z, \frac{z_0 + i}{2} \right],$$

maps the hypersurface $Q'_{p,q}$ defined by (1.27) into the projective hyperquadric $\bar{Q}_{p,q}$ (1.32).

Define a 1-form

$$(3.2) \quad \theta = \frac{\sum_{j=0}^{n+1} i \varepsilon_j (\xi_j d\bar{\xi}_j - \bar{\xi}_j d\xi_j)}{\sum_{j=0}^{p-1} |\xi_j|^2},$$

on $\mathbb{C}^{n+2} \setminus \{\xi \in \mathbb{C}^{n+2}; \xi_0 = \dots = \xi_{p-1} = 0\}$. It induces a 1-form on the projective hyperquadric $\bar{Q}_{p,q}$ in (1.32). We denote it by $\theta_{\bar{Q}_{p,q}}$. The hyperquadric $Q_{p,q}$ in (1.30) has a contact form

$$(3.3) \quad \theta_{Q_{p,q}} = \sum_{\alpha=0}^n i\varepsilon_\alpha (z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha),$$

(here we use variables z_α instead of w_α , $\alpha = 0, \dots, n$, in the definition of $Q_{p,q}$ in (1.30)) and the hypersurface $Q'_{p,q}$ in (1.27) has a contact form

$$(3.4) \quad \theta_{Q'_{p,q}} = \sum_{\alpha=1}^n i\varepsilon_\alpha (z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha) + \frac{1}{2}(d\bar{z}_0 + dz_0).$$

Contact forms (3.3) and (3.4) are actually

$$(3.5) \quad i(\bar{\partial} - \partial)r$$

for corresponding defining functions r of $Q_{p,q}$ and $Q'_{p,q}$, respectively.

Proposition 3.1.

$$(3.6) \quad I^* \theta_{\bar{Q}_{p,q}} = \frac{1}{\frac{1}{4}|z_0 - i|^2 + \sum_{j=1}^{p-1} |z_j|^2} \theta_{Q'_{p,q}},$$

on the hypersurface $Q'_{p,q}$.

Proposition 3.2. Let $S_0 = \sum_{j=0}^{n+1} a_j |\xi_j|^2$ with $a_j = \varepsilon_j$ or 0, but $a_0 = 1$ and $a_{n+1} = 0$. Then the function

$$(3.7) \quad S(z_0, z) = S_0 \left(\frac{z_0 - i}{2}, z, \frac{z_0 + i}{2} \right)$$

on hypersurface $Q'_{p,q}$ satisfies where it is positive

$$(3.8) \quad \tilde{\square}_{\theta_{Q'_{p,q}}} S^{-\frac{Q-2}{4}} = \frac{n+1}{2} \left(\sum_{j=1}^n 2a_j \varepsilon_j - n \right) S^{-\frac{Q+2}{4}},$$

where $Q = 2n + 2$.

Corollary 3.3. The scalar curvature of the projective hyperquadric $\bar{Q}_{p,q}$ with contact form $\theta_{\bar{Q}_{p,q}}$ is $\frac{n+1}{2}(p - q)$.

4. Representations on the projective hyperquadric $\overline{Q}_{p,q}$

Proposition 4.1. For $g \in \mathrm{SU}(p, q)$ and $z \in Q_{p,q}$, we have

$$(4.1) \quad g^* \theta_{Q_{p,q}}(z) = \frac{1}{|g(z, 1)_{n+1}|^2} \theta_{Q_{p,q}}(z).$$

Define the *light cone* to be

$$(4.2) \quad \Xi := \{\xi \in \mathbf{C}^{n+2}; |\xi|_{p,q} = 0\} \setminus \{0\},$$

and

$$(4.3) \quad \Sigma := \left\{ \xi \in \mathbf{C}^{n+2}; \sum_{j=0}^{p-1} |\xi_j|^2 = \sum_{j=p}^{p+q-1} |\xi_j|^2 = 1 \right\} \simeq S^{2p-1} \times S^{2q-1}.$$

The multiplicative group $\mathbf{R}_+^\times$ acts on Ξ as a dilation and the quotient space $\Xi/\mathbf{R}_+^\times$ is identified with Σ . By definition, $\Xi/\mathbf{C}^\times \simeq \Sigma/S^1 \simeq \overline{Q}_{p,q}$. Because the action of $\mathrm{SU}(p, q)$ on $\mathbf{C}^{n+2}$ commutes with that of $\mathbf{C}^\times$, we can define the action of $\mathrm{SU}(p, q)$ on the quotient space $\Xi/\mathbf{C}^\times$, and also on $\overline{Q}_{p,q}$ through the above diffeomorphism. This action will be denoted by

$$(4.4) \quad L_h : \overline{Q}_{p,q} \longrightarrow \overline{Q}_{p,q}, \quad \xi \mapsto L_h \xi,$$

for $h \in \mathrm{SU}(p, q)$, $\xi \in \overline{Q}_{p,q}$.

For $a \in \mathbf{C}$, denote by $S^a(\Xi)$ the space of smooth function on Ξ homogeneous of degree a , i.e.

$$(4.5) \quad S^a(\Xi) = \{f \in C^\infty(\Xi); f(t\xi) = t^a f(\xi), \xi \in \Xi, t \in \mathbf{R}_+^\times\}.$$

A character ψ of $\mathbf{C}^\times$ has the form

$$(4.6) \quad \psi(t) = |t|^a \left(\frac{t}{|t|} \right)^m,$$

for some $a \in \mathbf{C}^\times$, $m \in \mathbf{Z}$, which can be formally written as

$$(4.7) \quad \psi(t) = \psi^{\alpha,\beta}(t) = t^\alpha \bar{t}^\beta,$$

with $\alpha + \beta = a$ and $\alpha - \beta = m$. We see that a pair (α, β) can occur if and only if $\alpha - \beta$ is an integer. For such a pair, we define $S^{\alpha,\beta}(\Xi) \subset S^a(\Xi)$ to be the $\psi^{\alpha,\beta}$ eigenspace for $\mathbf{C}^\times$.

Then, we have a decomposition

$$(4.8) \quad S^a(\Xi) = \sum_{\substack{\alpha+\beta=a, \\ \alpha-\beta \in \mathbb{Z}}} S^{\alpha, \beta}(\Xi).$$

Let $\nu : \Xi \longrightarrow \mathbb{R}_+$ be defined by

$$(4.9) \quad \nu(\xi) = \left(\sum_{j=0}^{p-1} |\xi_j|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=p}^{p+q-1} |\xi_j|^2 \right)^{\frac{1}{2}}.$$

Proposition 4.2. *For $g \in SU(p, q)$ and $\xi \in \overline{Q}_{p, q}$, we have*

$$(4.10) \quad g^* \theta_{\overline{Q}_{p, q}}(\xi) = \frac{1}{\nu(g(\xi))^2} \theta_{\overline{Q}_{p, q}}(\xi),$$

if we require the coordinates of ξ satisfying $\sum_{j=0}^{p-1} |\xi_j|^2 = 1$.

Proposition 4.3. *$S^{-\frac{1}{2}, -\frac{1}{2}}(\Xi)$ is isomorphic to $(\varpi_\lambda, C^\infty(\overline{Q}_{p, q}))$ as $U(p, q)$ modules.*

Define the representation $(\varpi^{p, q}, V^{p, q})$ to be $(\varpi_{\frac{Q-2}{2}}, \ker \tilde{\square}_{\theta_{\overline{Q}_{p, q}}})$.

We can identify $S^{\alpha, \beta}(\Xi)$ with degenerate principal series representations in standard notation (cf. [5]).

Corollary 4.4. *$(\varpi^{p, q}, V^{p, q})$ is a subrepresentation of $S^{-\frac{p-1}{2}, -\frac{p-1}{2}}(\Xi)$, or equivalently, of $C^\infty - \text{Ind}_{P^G_{\max}}(\chi_0 \otimes \mathbb{C}_{-1})$*

5. Basic properties of $(\varpi^{p, q}, V^{p, q})$

There is a natural action of S^1 on Σ defined by

$$(5.1) \quad \mu_\sigma : \Sigma \longrightarrow \Sigma, \quad (\xi_1, \dots, \xi_{n+1}) \mapsto (e^{i\sigma} \xi_1, \dots, e^{i\sigma} \xi_{n+1}),$$

for $\sigma \in [0, 2\pi)$. We can define the projection

$$(5.2) \quad \Pi : \Sigma \simeq S^{2p-1} \times S^{2q-1} \longrightarrow \overline{Q}_{p, q},$$

by $\Pi(\xi_1, \dots, \xi_{n+1}) = [\xi_1, \dots, \xi_{n+1}] \in \mathbb{CP}^{n+1}$. Namely, Σ is a S^1 fiber bundle over the projective hyperquadric $\overline{Q}_{p, q}$. Let

$$(5.3) \quad \theta_{S^{2p-1}} = i \sum_{j=0}^{p-1} \xi_j d\bar{\xi}_j - \bar{\xi}_j d\xi_j, \quad \theta_{S^{2q-1}} = i \sum_{j=p}^{n+1} \xi_j d\bar{\xi}_j - \bar{\xi}_j d\xi_j,$$

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the standard contact forms on spheres S^{p-1} and S^{q-1} , respectively. Then,

$$(5.4) \quad \Pi^* \theta_{\overline{Q}_{p,q}} = \theta_{S^{2p-1}} - \theta_{S^{2q-1}}.$$

Let $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^p)$ denote the space of harmonic polynomials of bi-degree (α, β) in $\mathbb{C}^p$, i.e., harmonic polynomials which are homogeneous of degree α in the z_j 's and of degree β in the $\bar{z}_j$'s

$$(5.5) \quad L^2(S^{2p-1}) \simeq \sum_{\alpha,\beta=0}^{\infty} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^p).$$

For a function $f \in L^2(\overline{Q}_{p,q})$, $\Pi^* f$ is an L^2 function in Σ invariant under the action of S^1 . Thus,

$$(5.6) \quad L^2(\overline{Q}_{p,q}) \simeq \sum_{m=0}^{\infty} \sum_{\substack{m_1+m_2=m, \\ n_1+n_2=m}} \mathcal{H}^{m_1,n_1}(\mathbb{C}^p) \boxtimes \mathcal{H}^{m_2,n_2}(\mathbb{C}^q)$$

as Hilbert direct sum. We denote by $C^\infty(S^{2p-1} \times S^{2q-1})_0$ the space of S^1 -invariant functions in $C^\infty(S^{2p-1} \times S^{2q-1})$. We can identify $C^\infty(\overline{Q}_{p,q})$ with the subspace $C^\infty(S^{2p-1} \times S^{2q-1})_0$ by the mapping Π .

Proposition 5.1. *For $u \in C^\infty(S^{2p-1} \times S^{2q-1})_0$, we have*

$$(5.7) \quad \tilde{\square}_{\theta_{\overline{Q}_{p,q}}} (u \circ \Pi^{-1}) = \tilde{\square}_{\theta_{S^{2p-1}}} u - \tilde{\square}_{\theta_{S^{2q-1}}} u.$$

The Yamabe operator on the projective hyperquadric $\overline{Q}_{p,q}$ is

$$(5.8) \quad \tilde{\square}_{\theta_{\overline{Q}_{p,q}}} = \square_{\theta_{\overline{Q}_{p,q}}} + \frac{n}{4}(p-q).$$

Proposition 5.2. $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^p)$ is an eigenspace of $\square_{\theta_{S^{2p-1}}}$ on S^{2p-1} with eigenvalue $\frac{1}{4}(\alpha + \beta)(2p - 2 + \alpha + \beta) - \frac{1}{4}(\alpha - \beta)^2$.

Theorem 5.3. *The underlying $(\mathfrak{g}, K)$ -module $(\varpi^{p,q})_K$ has the following K -type formula*

$$(5.9) \quad (\varpi^{p,q})_K \simeq \bigoplus_{\substack{m_1+n_1+p=m_2+n_2+q, \\ m_1+m_2=n_1+n_2}} \mathcal{H}^{m_1,n_1}(\mathbb{C}^p) \boxtimes \mathcal{H}^{m_2,n_2}(\mathbb{C}^q)$$

APPLICATIONS OF CR GEOMETRY TO REPRESENTATIONS OF $SU(p, q)$

Remark 5.4. For other rank-1 Lie groups $Sp(1)Sp(n+1, 1)$ and F_4^{-20} , there exist quaternionic and octanionic CR geometries. For example, we have corresponding Webster connections, corresponding conformal geometry, corresponding Yamabe operators, etc. (cf. [3]). It is interesting to study the representation theories of $Sp(1)Sp(n+1, 1)$ (more generally, of $Sp(p, q)$) and F_4^{-20} by using corresponding conformal geometries.

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