

Some remark on weak dividing

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Abstract

Weak dividing has been characterized variously in simple theory.
We try to argue about the restricted notions of it.

1. Weak dividing

We recall some definitions.

Definition 1 Let $\varphi(x_0, x_1, \dots, x_{n-1})$ be a formula and $p(x)$ be a type. We denote the type $\{\varphi(x_0, x_1, \dots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \dots \cup p(x_{n-1})$ by $[p]^\varphi$.

Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *divides over* A if there are a formula $\varphi(x, b) \in p(x)$ and an infinite sequence $\{b_i : i < \omega\}$ with $b \equiv b_i(A)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is k -inconsistent for some $k < \omega$.

$p(x)$ *weakly divides over* A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ such that $[p[A]]^\varphi$ is consistent, while $[p]^\varphi$ is inconsistent.

In this note, we call such formula " $\varphi(\bar{x})$ " in the definition the *witness formula* of weak dividing for the sake of convenience.

I introduce examples from [2], [3].

Example 2 Let T be the theory of dense linear order and $p(x) = "a < x < b"$. Then $p(x)$ does not weakly divide over $\emptyset$. And let $q(x, y) = "x < c < y"$. Then $q(x, y)$ weakly divides over $\emptyset$ by the formula $\varphi(x_1, y_1; x_2, y_2) = "y_1 < x_2"$.

Example 3 Let T be the theory of an equivalence relation with two infinite classes of the language $L = \{a \text{ binary relation } E(x, y)\}$. And let $\models \neg E(a, b)$. Then the type $\text{tp}(a/b)$ does not divide over $\emptyset$, while $\text{tp}(a/b)$ weakly divides over $\emptyset$ by the formula $\neg E(x, y)$.

Example 4 Let $(V, <, >)$ be a vector space V over a finite field equipped with an inner product giving orthogonality between two independent vectors. Let a, b, c be independent vectors in V such that $a \perp b$, while $b \not\perp c$ and $a \not\perp c$. Then $\text{tp}(a/bc)$ does not weakly divide over $\emptyset$. But $\text{tp}(a/bc)$ weakly divides over c by the formula $\varphi(x, y) = "x \text{ is a linear combination of } y \text{ and } c"$.

In various characterizations, one of the most important results is the next theorem.

Theorem 5 (Kim [3])

T is stable if and only if weak dividing is symmetric in T .

2. Restricted notions of weak dividing

In examples above, we can see that witness formulas have different properties in the sense of stability theory. So I considered that we can divide witness formulas into some classes according to the properties.

Definition 6 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *$\mathcal{M}$ -weakly divides over A* if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p|_A$ such that $\models \varphi(a_0, a_1, \dots, a_{n-1})$, while the type $[p]^\varphi$ is inconsistent.

$p(x)$ *M -weakly divides over A* if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p|_A$ such that $\models \varphi(a_0, a_1, \dots, a_{n-1})$, while there is no Morley sequence $J = \{b_i : i < n+1\}$ of p over A such that $\models \varphi(b_0, b_1, \dots, b_{n-1})$.

If we set the sequence I indiscernible over A in the definition above, we can define $\mathcal{I}$ -weak dividing and I -weak dividing in the same way.

I proved the next result.

Theorem 7 *Let T be simple.*

Then T is stable if and only if $\mathcal{M}$ -weak dividing is symmetric in T .

Example 3 is easy. But some of examples of $\mathcal{M}$ -weak dividing may be variants of it.

Fact 8 *Let T be simple. And let $A \subset B$ and $p(x) \in S(B)$.*

$p(x)$ does not divide over A , while $p(x)$ $\mathcal{M}$ -weakly divides over A by a formula $\varphi(x, y) \in L_2(A)$. Then for any realization ab of φ with $a \perp_A b$, $L\text{stp}(a/A) \neq L\text{stp}(b/A)$.

I showed Fact 8 for 2-variable witness formulas. But the same Fact holds for n -variable witness formulas under the assumption that T has n -amalgamation property. (see [4],[5]) Once I told about some weak dividing for n -dividing in n -simple theory.

In recent years another variant of dividing, "thorn"-dividing has been characterized in rosy theory. (see e.g. [6]) I tried to define weak notion of p -dividing (thorn-dividing). We recall some definitions first.

Definition 9 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *strongly divides over* A if there is a formula $\varphi(x, b) \in p(x)$ such that $b \notin \text{acl}(A)$ and $\{\varphi(x, b_i) : b_i \models \text{tp}(b/A)\}$ is k -inconsistent for some $k < \omega$.

$p(x)$ *p -divides over* A if $p(x)$ strongly divides over Ac for some parameter c .

Weak notions of p -dividing could be defined in many ways. By the definition, p -dividing implies dividing. So we expect that weak p -dividing implies weak dividing.

Definition 10 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *weakly p -divides over* A if there is a formula $\varphi(\bar{x}) = \exists y \bigwedge_{i < n} \psi(x_i, y) \in L_n(A)$ such that $[p \upharpoonright A]^\varphi$ is consistent, while $[p]^\varphi$ is inconsistent.

References

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