

Some subordination criteria concerning Sălăgean operator

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Abstract

Applying Sălăgean operator, for the class $\mathcal{A}$ of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ which are normalized by $f(0) = f'(0) - 1 = 0$, the generalization of an analytic function to discuss the starlikeness is considered. Furthermore, from the subordination criteria for Janowski functions generalized by some complex parameters, some interesting subordination criteria for $f(z) \in \mathcal{A}$ are given.

1 Introduction, definition and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

$$(1.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in $\mathbb{U}$. If $p(z) \in \mathcal{P}$ satisfies $\operatorname{Re}(p(z)) > 0$ ($z \in \mathbb{U}$), then we say that $p(z)$ is the Carathéodory function (cf. [1]).

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$(1.3) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$). We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are starlike of order α in $\mathbb{U}$.

Similarly, if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$(1.4) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$), then $f(z)$ is said to be convex of order α in $\mathbb{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are convex of order α in $\mathbb{U}$. As usual, in the present investigation, we write

$$\mathcal{S}^*(0) \equiv \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) \equiv \mathcal{K}.$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [7].

By the familiar principle of differential subordination between analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $p(z) \in \mathcal{P}$, we introduce the following function

$$(1.5) \quad p(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1)$$

which has been investigated by Janowski [3]. Thus, the function $p(z)$ given by (1.5) is said to be the Janowski function.

Here, for some A and B ($-1 < B < A \leq 1$), the function $p(z)$ given by (1.5) is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by

$$\left| p(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}.$$

Thus, it is clear that

$$(1.6) \quad \operatorname{Re}(p(z)) > \frac{1 - A}{1 - B} \geq 0 \quad (z \in \mathbb{U}).$$

Also, if we take $B = -1$ in (1.5), then we see that

$$(1.7) \quad p(z) = \frac{1 + Az}{1 - z} \quad (-1 < A \leq 1)$$

is analytic and univalent in $\mathbb{U}$ and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$(1.8) \quad \operatorname{Re}(p(z)) > \frac{1}{2}(1 - A) \geq 0.$$

Hence, we see that the Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

And, as the generalization of Janowski function, Kuroki, Owa and Srivastava [2] have discussed the function

$$p(z) = \frac{1 + Az}{1 + Bz}$$

for some complex parameters A and B which satisfy one of following conditions

$$\begin{cases} (i) & |B| < 1, A \neq B, \text{ and } \operatorname{Re}(1 - A\bar{B}) \geq |A - B| \\ (ii) & |B| = 1, A \neq B, |A| \leq 1, \text{ and } 1 - A\bar{B} > 0. \end{cases}$$

First, for some complex numbers A and B which satisfy the following condition

$$(i) \quad |B| < 1, A \neq B, \text{ and } \operatorname{Re}(1 - A\bar{B}) \geq |A - B|,$$

the function $p(z) = \frac{1 + Az}{1 + Bz}$ is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by

$$\left| p(z) - \frac{1 - A\bar{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2}.$$

Thus, it is clear that

$$(1.9) \quad \operatorname{Re}(p(z)) > \frac{\operatorname{Re}(1 - A\bar{B}) - |A - B|}{1 - |B|^2} \geq 0 \quad (z \in \mathbb{U}).$$

Also, for some complex numbers A and B which satisfy the following condition

$$(ii) \quad |B| = 1, A \neq B, |A| \leq 1, \text{ and } 1 - A\bar{B} > 0,$$

the function $p(z) = \frac{1 + Az}{1 + Bz}$ is analytic and univalent in $\mathbb{U}$ and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$(1.10) \quad \operatorname{Re}(p(z)) > \frac{1 - |A|^2}{2(1 - A\bar{B})} \geq 0.$$

Hence, we see that the generalized Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

We define the following differential operator due to Sălăgean [8].
For a function $f(z)$ and $j = 1, 2, 3, \dots$,

$$(1.11) \quad D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$(1.12) \quad D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n,$$

$$(1.13) \quad D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n.$$

Also, we meditate the following integral operator

$$(1.14) \quad D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n,$$

$$(1.15) \quad D^{-j} f(z) = D^{-1}(D^{-(j-1)} f(z)) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n$$

for any negative integers.

Then, for $f(z) \in \mathcal{A}$ given by (1.1), we know that

$$(1.16) \quad D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 0, \pm 1, \pm 2, \dots).$$

Using the above operator $D^j f(z)$, we consider the subclass $\mathcal{S}_j^k(\alpha)$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_j^k(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left(\frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}.$$

Remark 1.1 Noting

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)}, \quad \frac{D^2 f(z)}{D^1 f(z)} = \frac{z(zf'(z))'}{zf'(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

we see that

$$\mathcal{S}_0^1(\alpha) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{S}_1^2(\alpha) \equiv \mathcal{K}(\alpha) \quad (0 \leq \alpha < 1).$$

Remark 1.2 For some α ($0 \leq \alpha < 1$), we find

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \iff \operatorname{Re} \left(\frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}).$$

In our investigation here, we need the following lemma concerning the differential subordination given by Miller and Mocanu [5] (see also [6, p. 132]).

Lemma 1.3 *Let the function $q(z)$ be analytic and univalent in $\mathbb{U}$. Also let $\phi(\omega)$ and $\psi(\omega)$ be analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with*

$$\psi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

Set

$$Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

and suppose that

$$(i) \quad Q(z) \text{ is starlike and univalent in } \mathbb{U};$$

and

$$(ii) \quad \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) = \operatorname{Re} \left(\frac{\phi'(q(z))}{\psi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$, with

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset \mathcal{C},$$

and

$$\phi(p(z)) + zp'(z)\psi(p(z)) \prec \phi(q(z)) + zq'(z)\psi(q(z)) =: h(z) \quad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant of this subordination.

By making use of lemma 1.3, Kuroki, Owa and Srivastava [2] have investigated some subordination criteria for the generalized Janowski functions and deduced the following lemma.

Lemma 1.4 Let the function $f(z) \in \mathcal{A}$ be so chosen that $\frac{f(z)}{z} \neq 0$ ($z \in \mathbb{U}$).

Also, let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1 - |A|^2)}{2(1 - A\bar{B})} + \frac{(1-\beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If

$$(1.17) \quad \left(\frac{zf'(z)}{f(z)} \right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^{\beta-1} \left\{ (1-\alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

2 Subordinations for the class defined by Sălăgean operator

Applying Sălăgean operator for $f(z) \in \mathcal{A}$, we deduced the following subordination criterion for the generalized Janowski function.

Theorem 2.1 *Let the function $f(z) \in \mathcal{A}$ be so chosen that $\frac{D^j f(z)}{z} \neq 0$ ($z \in \mathbb{U}$). Also, let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B which satisfy one of following conditions*

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1 - |A|^2)}{2(1 - A\bar{B})} + \frac{(1-\beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If

$$(2.1) \quad \left(\frac{D^k f(z)}{D^j f(z)} \right)^\beta \left\{ (1-\alpha) + \alpha \left(\frac{D^k f(z)}{D^j f(z)} + \frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)} \right) \right\} \prec h(z),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^{\beta-1} \left\{ (1-\alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^k f(z)}{D^j f(z)} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Further, since

$$zp'(z) = \left(\frac{D^k f(z)}{D^j f(z)} \right) \left(\frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)} \right),$$

the condition (2.1) can be written as follows:

$$\{p(z)\}^\beta \{(1-\alpha) + \alpha p(z)\} + \alpha zp'(z) \{p(z)\}^{\beta-1} \prec h(z) \quad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

and

$$\phi(\omega) = \omega^\beta(1 - \alpha + \alpha\omega), \quad \text{and} \quad \psi(\omega) = \alpha\omega^{\beta-1}$$

for $\omega \in q(\mathbb{U})$. Then, it is clear that the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and have a positive real part in $\mathbb{U}$ for the conditions (i) and (ii).

Therefore, ϕ and ψ are analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with

$$\psi(\omega) = \alpha\omega^{\beta-1} \neq 0 \quad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

Also, for the function $Q(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}},$$

we obtain

$$(2.2) \quad \frac{zQ'(z)}{Q(z)} = \frac{1 - \beta}{1 + Az} + \frac{1 + \beta}{1 + Bz} - 1.$$

Furthermore, we have

$$\begin{aligned} h(z) &= \phi(q(z)) + Q(z) \\ &= \left(\frac{1 + Az}{1 + Bz} \right)^\beta \left(1 - \alpha + \alpha \frac{1 + Az}{1 + Bz} \right) + \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}} \end{aligned}$$

and

$$(2.3) \quad \frac{zh'(z)}{Q(z)} = \frac{\beta(1 - \alpha)}{\alpha} + (1 + \beta)q(z) + \frac{zQ'(z)}{Q(z)}.$$

Hence,

(i) For the complex numbers A and B such that

$$|B| < 1, \quad A \neq B, \quad \text{and} \quad \operatorname{Re}(1 - A\bar{B}) \geq |A - B|,$$

it follows from (2.2) and (2.3) that

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

and

$$\begin{aligned} \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) &> \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} \\ &\quad + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$

(ii) For the complex numbers A and B such that

$$|B| = 1, |A| \leq 1, A \neq B, \text{ and } 1 - A\bar{B} > 0,$$

from (2.2) and (2.3), we get

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > \frac{1-\beta}{1+|A|} + \frac{1}{2}(1+\beta) - 1 = \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geq 0,$$

and

$$\operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) > \frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\bar{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geq 0 \quad (z \in \mathbb{U}).$$

Since all conditions of Lemma 1.3 are satisfied, we conclude that

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.1. $\square$

Letting $k = j + 1$ in Theorem 2.1, we obtain

Corollary 2.2 *Let the function $f(z) \in \mathcal{A}$ be so chosen that $\frac{D^j f(z)}{z} \neq 0$ ($z \in \mathbb{U}$).*

Also, let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\bar{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geq 0.$$

If

$$(2.2) \quad \left(\frac{D^{j+1} f(z)}{D^j f(z)} \right)^\beta \left(1 - \alpha + \alpha \frac{D^{j+2} f(z)}{D^{j+1} f(z)} \right) \prec h(z),$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz} \right)^{\beta-1} \left\{ (1-\alpha) \frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2} \right\},$$

then

$$\frac{D^{j+1} f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Remark 2.3 Setting $j = 0$ in Corollary 2.2, we obtain Lemma 1.4 proven by Kuroki, Owa and Srivastava [2].

Also, if we assume that $\alpha = 1$, $\beta = A = 0$, and $B = \frac{1-\mu}{1+\mu}e^{i\theta}$ ($0 \leq \mu < 1$, $0 \leq \theta < 2\pi$), Corollary 2.2 becomes the following corollary.

Corollary 2.4 *If $f(z) \in \mathcal{A}$ $\left(\frac{D^j f(z)}{z} \neq 0 \text{ in } \mathbb{U}\right)$ satisfies*

$$\frac{D^{j+2}f(z)}{D^{j+1}f(z)} \prec \frac{1+\mu-(1-\mu)e^{i\theta}z}{1+\mu+(1-\mu)e^{i\theta}z} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$

for some μ ($0 \leq \mu < 1$), then

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1+\mu}{1+\mu+(1-\mu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$

From the above corollary, we have

$$\operatorname{Re} \left(\frac{D^{j+2}f(z)}{D^{j+1}f(z)} \right) > \mu \implies \operatorname{Re} \left(\frac{D^{j+1}f(z)}{D^j f(z)} \right) > \frac{1+\mu}{2} \quad (z \in \mathbb{U}; 0 \leq \mu < 1).$$

Thus, we see that

$$\begin{aligned} f(z) \in \mathcal{S}_{j+1}^{j+2}(\mu) &\implies f(z) \in \mathcal{S}_j^{j+1} \left(\frac{1+\mu}{2} \right) \implies f(z) \in \mathcal{S}_{j-1}^j \left(\frac{3+\mu}{4} \right) \\ &\implies \dots \implies f(z) \in \mathcal{S}_1^2 \left(\frac{2^j - 1 + \mu}{2^j} \right) \\ &\implies f(z) \in \mathcal{S}_0^1 \left(\frac{2^{j+1} - 1 + \mu}{2^{j+1}} \right) \quad (z \in \mathbb{U}; 0 \leq \mu < 1). \end{aligned}$$

In particular, we find

$$\begin{aligned} f(z) \in \mathcal{S}_{j+1}^{j+2}(\mu) &\implies f(z) \in \mathcal{K} \left(\frac{2^j - 1 + \mu}{2^j} \right) \\ &\implies f(z) \in \mathcal{S}^* \left(\frac{2^{j+1} - 1 + \mu}{2^{j+1}} \right) \quad (z \in \mathbb{U}; 0 \leq \mu < 1). \end{aligned}$$

And, taking $j = 0$ and $\mu = 0$, we find the fact that every convex function is starlike of order $\frac{1}{2}$. This fact is well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

3 Subordination criteria for other analytic functions

In this section, by making use of Lemma 1.3, we consider some subordination criteria concerning analytic function $\frac{D^j f(z)}{z}$ for $f(z) \in \mathcal{A}$.

Theorem 3.1 Let α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and some complex parameters A and B which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\operatorname{Re}(1 - A\bar{B}) \geq |A - B|$ be so that

$$\frac{\beta}{\alpha} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - A\bar{B} > 0$ be so that

$$\frac{\beta}{\alpha} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If $f(z) \in \mathcal{A}$ satisfies

$$(3.1) \quad \left(\frac{D^j f(z)}{z} \right)^\beta \left(1 - \alpha + \alpha \frac{D^{j+1} f(z)}{D^j f(z)} \right) \prec \left(\frac{1 + Az}{1 + Bz} \right)^\beta + \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}},$$

then

$$\frac{D^j f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^j f(z)}{z} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and the condition (3.1) can be written as follows:

$$\{p(z)\}^\beta + \alpha z p'(z) \{p(z)\}^{\beta-1} \prec h(z) \quad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

and

$$\phi(\omega) = \omega^\beta, \quad \text{and} \quad \psi(\omega) = \alpha \omega^{\beta-1}$$

for $\omega \in q(\mathbb{U})$. Then, the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and satisfies

$$\operatorname{Re}(q(z)) > 0 \quad (z \in \mathbb{U})$$

for the condition (i) and (ii).

Thus, the functions ϕ and ψ satisfy the conditions required by Lemma 1.3.

Further, for the functions $Q(z)$ and $h(z)$ given by

$$Q(z) = z q'(z) \psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

we have

$$\frac{z Q'(z)}{Q(z)} = \frac{1 - \beta}{1 + Az} + \frac{1 + \beta}{1 + Bz} - 1 \quad \text{and} \quad \frac{z h'(z)}{Q(z)} = \frac{\beta}{\alpha} + \frac{z Q'(z)}{Q(z)}.$$

Then, similarly to proof of Theorem 2.1, we see that

$$\operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) > 0 \quad \text{and} \quad \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U})$$

for the conditions (i) and (ii).

Thus, by applying Lemma 1.3, we conclude that $p(z) \prec q(z)$ ($z \in \mathbb{U}$).

The proof of the theorem is completed. $\square$

In Theorem 3.1, taking $\alpha = 1$, $\beta = A = 0$, and $B = \frac{1-\nu}{\nu}e^{i\theta}$ ($\frac{1}{2} \leq \nu < 1$, $0 \leq \theta < 2\pi$), we obtain the following corollary.

Corollary 3.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$

for some ν ($\frac{1}{2} \leq \nu < 1$), then

$$\frac{D^j f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$

Also, making $\alpha = \beta = 1$, $A = 0$, and $B = \frac{1-\nu}{\nu}e^{i\theta}$ ($\frac{1}{2} \leq \nu < 1$, $0 \leq \theta < 2\pi$) in Theorem 3.1, we get

Corollary 3.3 *If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{D^{j+1}f(z)}{z} \prec \left(\frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \right)^2 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)$$

for some ν ($\frac{1}{2} \leq \nu < 1$), then

$$\frac{D^j f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}).$$

The above corollaries derive each of the facts that

$$\operatorname{Re} \left(\frac{D^{j+1}f(z)}{D^j f(z)} \right) > \nu \implies \operatorname{Re} \left(\frac{D^j f(z)}{z} \right) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right),$$

and

$$\operatorname{Re} \sqrt{\frac{D^{j+1}f(z)}{z}} > \nu \implies \operatorname{Re} \left(\frac{D^j f(z)}{z} \right) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right).$$

In particular, for $j = 0$, we see that

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \nu \implies \operatorname{Re} \left(\frac{f(z)}{z} \right) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right),$$

and

$$\operatorname{Re} \sqrt{f'(z)} > \nu \implies \operatorname{Re} \left(\frac{f(z)}{z} \right) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right).$$

Here, taking $\nu = \frac{1}{2}$, we find some results well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

Also, letting $j = 1$ in Corollary 3.2, we get the following fact:

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \nu \implies \operatorname{Re} (f'(z)) > \nu \quad \left(z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right).$$

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