

Fixed Point Theorems and Convergence Theorems for Non-self Mappings in Hilbert Spaces (ヒルベルト空間における非自己写像の不動点定理と収束定理)

新潟大学大学院自然科学研究科 北條 真弓 (Mayumi Hojo)
Graduate School of Science and Technology, Niigata University, Japan

Abstract. In this article, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H . Kocourek, Takahashi and Yao [19] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [21] and hybrid mappings [30]. A mapping $T : C \rightarrow H$ is said to be *generalized hybrid* [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (1.1)$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such T an (α, β) -generalized hybrid mapping. An (α, β) -generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx - Ty\| \leq \|x - Ty\|$ for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$ for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|y - x\|^2$ for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [3] for generalized hybrid mappings in a Hilbert space; see also Kohsaka and Takahashi [20] and Iemoto and Takahashi [15]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1 - \alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1 - \beta)\|u - y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an (α, β) -generalized hybrid mapping with a fixed point is quasi-nonexpansive. Kocourek, Takahashi and Yao [19] also introduced a more broad class of nonlinear mappings which covers generalized hybrid mappings. A mapping

$S : C \rightarrow H$ is called *super hybrid* [19, 34] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha \|Sx - Sy\|^2 + (1 - \alpha + \gamma) \|x - Sy\|^2 \\ & \leq (\beta + (\beta - \alpha)\gamma) \|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma) \|x - y\|^2 \\ & \quad + (\alpha - \beta)\gamma \|x - Sx\|^2 + \gamma \|y - Sy\|^2 \end{aligned} \quad (1.2)$$

for all $x, y \in C$. We call such a mapping an (α, β, γ) -*super hybrid* mapping. An $(\alpha, \beta, 0)$ -super hybrid mapping is (α, β) -generalized hybrid. So, the class of super hybrid mappings contains generalized hybrid mappings. On the other hand, Hojo, Takahashi and Yao [12] defined the following class of nonlinear mappings which contains generalized hybrid mappings. A mapping $U : C \rightarrow H$ is called *extended hybrid* if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha(1 + \gamma) \|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma)) \|x - Uy\|^2 \\ & \leq (\beta + \alpha\gamma) \|Ux - y\|^2 + (1 - (\beta + \alpha\gamma)) \|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma \|x - Ux\|^2 - \gamma \|y - Uy\|^2 \end{aligned} \quad (1.3)$$

for all $x, y \in C$. We note that super hybrid mappings and extended hybrid mappings are not quasi-nonexpansive generally. We also know the following relation between generalized hybrid mappings and extended hybrid mappings

Theorem 1.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \rightarrow H$ is an (α, β) -generalized hybrid mapping if and only if $U : C \rightarrow H$ is an (α, β, γ) -extended hybrid mapping.*

In this article, motivated by these mappings and results, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalize Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. From [29], we know the following basic equality: For any $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2. \quad (2.1)$$

Furthermore, we know that for any $x, y, u, v \in H$

$$2 \langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \quad (2.2)$$

Let C be a nonempty closed convex subset of H and let T be a mapping from C into itself. Then, we denote by $F(T)$ the set of fixed points of T . A mapping $T : C \rightarrow H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \rightarrow H$ with $F(T) \neq \emptyset$

is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the *metric projection* of H onto C . It is known that P_C is nonexpansive and $\langle x - P_C x, P_C x - u \rangle \geq 0$ for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \quad (2.3)$$

for all $x, y \in H$; see [29] for more details. For proving main results in this paper, we also need the following lemmas proved in [31] and [2].

Lemma 2.1 ([31]). *Let D be a nonempty closed convex subset of H . Let P be the metric projection from H onto D . Let $\{u_n\}$ be a sequence in H . If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{P u_n\}$ converges strongly to some $u_0 \in D$.*

Lemma 2.2 ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞ . If μ is a Banach limit on l^∞ , then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [27] for the proof of existence of a Banach limit and its other elementary properties. Using Banach limits, Kocourek, Takahashi and Yao [19] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 2.3 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

3 Fixed Point Theorem for Non-Self Mappings

In this section, we first prove a fixed point theorem for generalized hybrid non-self mappings in a Hilbert space. For proving it, we need the following lemmas.

Lemma 3.1. *Let H be a Hilbert space and let C be a nonempty subset of H . Let α and β be in $\mathbb{R}$. Then, a non-self mapping $T : C \rightarrow H$ is (α, β) -generalized hybrid if and only if it satisfies that*

$$\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)\langle x - Tx, y - Ty \rangle - (\alpha - \beta - 1)\|y - Tx\|^2$$

for all $x, y \in C$.

Using Lemma 3.1, we have the following result.

Lemma 3.2. *Let H be a Hilbert space and let C be a nonempty bounded subset of H . If a non-self mapping $T : C \rightarrow H$ is generalized hybrid, then TC is bounded.*

The following is a fixed point theorem for non-self generalized hybrid mappings in a Hilbert space.

Theorem 3.3 ([12]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α and β be real numbers. Let T be an (α, β) -generalized hybrid mapping with $\alpha - \beta \geq 0$ of C into H . Suppose that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $1 \leq t \leq m$. Then, T has a fixed point in C .*

Recently, Hojo, Suzuki and Takahashi [10] also proved a more general fixed point theorem for nonlinear non-self mappings in a Hilbert space.

Theorem 3.4 ([10]). *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Let $T : C \rightarrow H$ be an $(\alpha, \beta, \gamma, \delta)$ -normal generalized hybrid mapping, i.e., there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\alpha + \beta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\alpha + \gamma \geq 0$.

Assume that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $0 < t \leq m$. Then T has a fixed point in C . In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

For proving this result, Hojo, Suzuki and Takahashi [10] used the following fixed point theorem obtained by Kawasaki and Takahashi [18].

Theorem 3.5 ([18]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ and $\varepsilon + \eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

Let us give an example of mappings which is related to the conditions in Theorem 3.4. In the case of $H = \mathbb{R}$, consider a mapping $T : [0, 1] \rightarrow \mathbb{R}$:

$$Tx = (1 + 2x) \cos x - 2x^2, \quad \forall x \in [0, 1].$$

Then, we have

$$Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, 1].$$

Take $m = 3$. For any $x \in [0, 1]$, take $t = 1 + 2x$ and $y = \cos x$. Then, we have that $Tx = t(y - x) + x$, $y = \cos x \in [0, 1]$ and $0 < t = 1 + 2x \leq 3$.

4 Weak convergence theorems

In this section, using the technique developed by Takahashi [26], we first prove a mean convergence theorem of Baillon's type [3] for super hybrid mappings in a Hilbert space. For proving it, we need the following lemma.

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be a generalized hybrid mapping from C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n} \sum_{k=1}^n T^k x$. Then, $\lim_{n \rightarrow \infty} \|S_n x - TS_n x\| = 0$. In particular, if C is bounded, then*

$$\lim_{n \rightarrow \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0.$$

Using Lemma 4.1, we obtain the the following mean convergence theorem.

Theorem 4.2 ([12]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let α, β and γ be real numbers with $\gamma \geq 0$ and let $S : C \rightarrow C$ be an (α, β, γ) -super hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=1}^n \left(\frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n \rightarrow \infty} PT^n x$ and $T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I$.

Next, we prove a weak convergence theorem of Mann's type [23] for nonlinear non-self mappings in a Hilbert space. For proving the result, we need the following two lemmas.

Lemma 4.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.

Then T is quasi-nonexpansive.

We remark that if $T : C \rightarrow H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [16]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \rightarrow z$. Since C is weakly closed, we have $z \in C$. Furthermore, from $\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \rightarrow 0$, we have that z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex.

For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.1) that

$$\begin{aligned}
 \|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\
 &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
 &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
 &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\
 &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 = 0
 \end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex.

Lemma 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping. Suppose that it satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \rightarrow z$ and $x_n - Tx_n \rightarrow 0$, then $z \in F(T)$.

Using Lemmas 4.3, 4.4 and the technique developed by Ibaraki and Takahashi [13, 14], we can prove the following weak convergence theorem.

Theorem 4.5 ([10]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow H$ be a widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$.

Let P be the metric projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)Tx_n), \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n \rightarrow \infty} Px_n$.

Using Theorem 4.5, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

Theorem 4.6 ([19]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

Proof. Since $T : C \rightarrow C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1 - \beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that this mapping is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta), 0, 0, 0)$ -widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.5. Therefore, we have the desired result from Theorem 4.5. $\square$

5 Strong Convergence Theorem

In this section, using an idea of mean convergence by Shimizu and Takahashi [24] and [25], we prove a strong convergence theorem of Halpern's type for super hybrid mappings in a Hilbert space.

Theorem 5.1 ([12]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \rightarrow C$ be a (α, β, γ) -super hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(S)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C, u \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=1}^n \left(\frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1, \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to Pu .

Recently, Hojo, Suzuki and Takahashi [10] also proved the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 5.2 ([10]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let T be a widely more generalized hybrid mapping of C into itself which satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0, \varepsilon + \eta \geq 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0, \zeta + \eta \geq 0$ and $\varepsilon + \eta \geq 0$.

Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1, \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu , where P is the metric projection of H onto $F(T)$.

Using Theorem 5.2, we can show the following result obtained by Hojo and Takahashi [11].

Theorem 5.3 ([11]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \\ z_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1, \alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where P is the metric projection of H onto $F(T)$.

Proof. As in the proof of Theorem 4.6, a generalized hybrid mapping is a widely more generalized hybrid mapping. Therefore, we have the desired result from Theorem 5.2. $\square$

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