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Some remarks on grand Furuta inequality

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1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. For convenience, we denote $A\geq 0$ (resp. $A>0$) if A is a positive (resp. invertible) operator. We begin from Furuta inequality ([6],[7],[9]).

Furuta inequality: *If $A\geq B\geq 0$, then for each $r\geq 0$,*

(F) $$A^{\frac{p+r}{q}}\geq(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{and} \quad (B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}}\geq B^{\frac{p+r}{q}}$$

holds for p and q such that $p\geq 0$ and $q\geq 1$ with $(1+r)q\geq p+r$.

This yields the Löwner-Heinz inequality;

(LH) $$A\geq B\geq 0 \text{ implies } A^{\alpha}\geq B^{\alpha} \text{ for any } \alpha\in[0,\ 1].$$

We had reformed (F) in terms of the α-power mean (or generalized geometric operator mean) of A and B which is introduced by Kubo-Ando as follows [16]:

$$A\ \sharp_{\alpha}\ B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for } \alpha\in[0,\ 1],$$

the case $\alpha\notin[0,\ 1]$, we use the notation $\natural$ to distinguish the operator mean.

By using the α-power mean, Furuta inequality is given as follows:

(F) $$A\geq B\geq 0 \text{ implies } A^{-r}\ \sharp_{\frac{1+r}{p+r}}\ B^{p}\leq A \text{ for } p\geq 1 \text{ and } r\geq 0.$$

Based on this reformulation, we had proposed a satellite form of (F) [12],[13];

(SF) $$A\geq B\geq 0 \text{ implies } A^{-r}\ \sharp_{\frac{1+r}{p+r}}\ B^{p}\leq B\leq A \text{ for } p\geq 1 \text{ and } r\geq 0.$$

On the other hand, Ando and Hiai showed the next inequality [1],[11].

Ando-Hiai inequality: Ando-Hiai had shown the following inequality:

(AH) $$\textit{If } A\ \sharp_{\alpha}\ B\leq I \textit{ for } A,\ B>0, \textit{ then } A^{r}\ \sharp_{\alpha}\ B^{r}\leq 1 \textit{ holds for } r\geq 1.$$

From this relation, they had shown the following inequality (AH_0). It is equivalent to the main result of log majorization and can be given as the following form:

(AH_0) $$A^{-1}\ \sharp_{\frac{1}{p}}\ A^{-\frac{1}{2}}B^{p}A^{-\frac{1}{2}}\leq I \ \Rightarrow\ A^{-r}\ \sharp_{\frac{1}{p}}\ (A^{-\frac{1}{2}}B^{p}A^{-\frac{1}{2}})^{r}\leq I,\ \ p\geq 1,\ r\geq 1.$$

Furuta had constructed the following inequality which interpolats (AH_0) and (F), we call this grand Furuta inequality ([2],[4],[8],[9]).

Grand Furuta inequality: *If $A \geq B \geq 0$ and $A > 0$, then for each $1 \leq p$ and $t \in [0,\ 1]$,*

$$\text{(GF)} \qquad A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s \leq A^{1-t}$$

holds for $t \leq r$ and $1 \leq s$.

The satellite form of (GF) is given also as follows ([2],[14]):

$$\text{(SGF)} \qquad A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq B\ (\leq A).$$

We pointed out that (F) and (AH) are obtained from each other and gave a generarized form of (AH) ([3],[5]).

For $\alpha \in (0,\ 1)$ fixed,

$$\text{(GAH)} \qquad A \natural_\alpha B \leq I \implies A^r \natural_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \leq I \quad \text{for} \quad r,\ s \geq 1.$$

Using (GAF), we modified (GF) as follows [15]:

Theorem A. *If $A \geq B \geq 0$ and $A > 0$, then for each $1 \leq p$ and $t \in [0,\ 1]$,*

$$A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \natural_{\frac{1-t}{p-t}} B^p$$

holds for $t \leq r$ and $1 \leq s$.

Recently, Furuta has shown the following theorem concerning to the above theorem [10].

Theorem F. *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,\ 1]$ and $p \geq 1$. Then*

$$F(\lambda,\ \mu) = A^{-\frac{\lambda}{2}}\{A^{\frac{\lambda}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^\mu A^{\frac{\lambda}{2}}\}^{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} A^{-\frac{\lambda}{2}}$$

satisfies the following properties:

$$\text{(i)} \qquad F(r,\ w) \geq F(r,\ 1) \geq F(r,\ s) \geq F(r,\ s')$$

holds for any $s' \geq s \geq 1$, $r \geq t$ and $1 - t \leq (p-t)w \leq p - t$.

$$\text{(ii)} \qquad F(q,\ s) \geq F(t,\ s) \geq F(r,\ s) \geq F(r',\ s)$$

holds for any $r' \geq r \geq t,\ s \geq 1$ *and* $t-1 \leq q \leq t$.

In this note, we observe this theorem from the $\alpha-$power mean.

2. Review of Theorem F. We rewrite Theorem F by the form of α-power mean. Then

$$F(\lambda,\ \mu) = A^{-\lambda} \natural_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\mu}$$

and by putting $B_1 = (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{1}{p-t}}$, (i) and (ii) of Theorem F are written as follows:

(i)
$$A^{-r} \natural_{\frac{1-t+r}{(p-t)w+r}} B_1^{(p-t)w} \geq A^{-r} \natural_{\frac{1-t+r}{p-t+r}} B_1^{p-t}$$
$$\geq A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} B_1^{(p-t)s} \geq A^{-r} \natural_{\frac{1-t+r}{(p-t)s'+r}} B_1^{(p-t)s'}$$

and

(ii)
$$A^{-q} \natural_{\frac{1-t+q}{(p-t)s+q}} B_1^{(p-t)s} \geq A^{-t} \natural_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s}$$
$$\geq A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} B_1^{(p-t)s} \geq A^{-r'} \natural_{\frac{1-t+r'}{(p-t)s+r'}} B_1^{(p-t)s}.$$

We point out that Theorem A can be written more precisely,

$$A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leq B \leq A^t \natural_{\frac{1-t}{p-t}} B^p.$$

Because $A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leq B$ is already shown in our proof of (SGF). So the result of Theorem A has shown the folloing inequality.

$$A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-t} \natural_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s} \leq B_1^{1-t},$$

and Furuta improved on the second inequality of this form to

$$A^{-t} \natural_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s} \leq A^{-q} \natural_{\frac{1-t+q}{(p-t)s+q}} B_1^{(p-t)s},\quad t-1 \leq q \leq t.$$

Furuta's process is the following:
Since $0 \leq t-q \leq 1$, $(A^{\frac{t}{2}}B_1^{(p-t)s}A^{\frac{t}{2}})^{\frac{t-q}{(p-t)s+t}} \leq A^{t-q}$ holds by (LH), and we can obtain the result as follows:

$$\begin{aligned} & A^{-t} \natural_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s} \\ = \ & B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+t}} A^{-t} \\ = \ & B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} (B_1^{(p-t)s} \natural_{\frac{(p-t)s+q}{(p-t)s+t}} A^{-t}) \end{aligned}$$

$$
\begin{aligned}
&= B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} (A^{-t} \natural_{\frac{t-q}{(p-t)s+t}} B_1^{(p-t)s}) \\
&= B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} A^{-\frac{t}{2}}(A^{\frac{t}{2}} B_1^{(p-t)s} A^{\frac{t}{2}})^{\frac{t-q}{(p-t)s+t}} A^{-\frac{t}{2}} \\
&\leq B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} A^{-\frac{t}{2}} A^{t-q} A^{-\frac{t}{2}} \\
&= A^{-q} \natural_{\frac{1-t+q}{(p-t)s+q}} B_1^{(p-t)s}.
\end{aligned}
$$

3. Modification of Theorem F. Furuta's results (i) and (ii) are holds suppose $A \geq B_1$, but in Theorem F this order does not hold. We search a suitable relation between A and B_1 by the help of (GAH).

$A \geq B \geq 0$ implies $A^t \geq B^t \geq 0$ for $t \in [0,\ 1]$ by (LH). This is equivalent to $A^{-t} \natural_{\frac{t}{p}} B_1^{p-t} \leq I$. By (GAH), we have

$$A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} = B_1^{(p-t)} \natural_{\frac{p-t}{p-t+r}} A^{-r} \leq I.$$

That is,

$$A \geq B \geq 0 \;\Rightarrow\; A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I \;\Rightarrow\; A^{-r'} \natural_{\frac{r'}{(p-t)s+r'}} B_1^{(p-t)s} \leq I \text{ for } r' \geq r,\ s \geq 1.$$

So we begin from the assumption $A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I$.

Lemma 1. *Let $A,\ B \geq 0$ and $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for $p,\ r \geq 0$. Then the following hold:*

(i) $$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq B^\delta \qquad 0 \leq \delta \leq p$$

and

(ii) $$A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p \leq A^\lambda \qquad -r \leq \lambda \leq 0.$$

These results are already known, but these play essential roles in our following discussions. We can arrange Theorem F as follows except $F(q,\ s) \geq F(t,\ s)$ for $t-1 \leq q \leq 0$.

Theorem 1. *Let $A,\ B \geq$ and $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for $p,\ r \geq 0$. Then*

(1) $$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B^\mu$$

holds for $p \geq \mu \geq \delta \geq 0$ and

(ii) $$A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\lambda+t}{p+t}} B^p$$

holds for $r \geq t \geq 0, \quad -t \leq \lambda \leq p$.

Proof. (i) is obtained by the followin calculation:

$$A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p = A^{-r} \sharp_{\frac{\delta+r}{\mu+r}} (A^{-r} \sharp_{\frac{\mu+r}{p+r}} B^p) \leq A^{-r} \sharp_{\frac{\delta+r}{\mu+r}} B^\mu.$$

(ii) can be shown as follows:

$$\begin{aligned} A^{-r} \sharp_{\frac{\lambda+r}{p+r}} B^p &= B^p \sharp_{\frac{p-\lambda}{p+r}} A^{-r} = B^p \sharp_{\frac{p-\lambda}{p+t}} (B^p \sharp_{\frac{p+t}{p+r}} A^{-r}) \\ &= B^p \sharp_{\frac{p-\lambda}{p+t}} (A^{-r} \sharp_{\frac{-t+r}{p+r}} B^p) \leq B^p \sharp_{\frac{p-\lambda}{p+t}} A^{-t} = A^{-t} \sharp_{\frac{\lambda+t}{p+t}} B^p. \end{aligned}$$

4. Applications. Return to Theorem A, we summarize the above discussions.

Theorem A(1). *If* $A \geq B \geq 0$ *and* $t \in [0, \ 1]$, $p \geq t$, $r \geq t$, $0 \leq \delta \leq (p-t)s$, *then*

$$A^{-r+t} \sharp_{\frac{\delta+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{\delta+t}{(p-t)s+t}} \leq A^\alpha \sharp_{\frac{\delta+t-\alpha}{(p-t)s+t-\alpha}} (A^t \natural_s B^p)$$

holds for $\min\{\delta+t, \ 1\} \geq \alpha \geq 0$.

This is equivalent to

$$A^{-r} \sharp_{\frac{\delta+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-t} \sharp_{\frac{\delta+t}{(p-t)s+t}} B_1^{(p-t)s} \leq A^{\alpha-t} \sharp_{\frac{\delta+t-\alpha}{(p-t)s+t-\alpha}} B_1^{(p-t)s}.$$

If $p \geq 1$ and $\delta = 1-t$, $\alpha = t-q$, we have Furta's result (ii) containing the case $t-1 \leq q \leq 0$.

Under the assumption $A^{-r} \sharp_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I$, our Theorem A can be written as follows:

Theorem A(2). *Let* $A, \ B \geq 0$ *and put* $B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}}$ *for* $p \geq t \geq 0$. *If* $A^{-r} \sharp_{\frac{r}{p-t+r}} B_1^{(p-t)} \leq I$ *for* $r \geq t \geq 0$, *then for* $s \geq 1$

(i) $$A^{-r+t} \sharp_{\frac{\delta+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^{-r+t} \sharp_{\frac{\delta+r}{\mu+r}} (A^t \natural_{\frac{\mu}{p-t}} B^p)$$

holds for $0 \leq \delta \leq \mu \leq (p-t)s$ *and*

(ii) $$A^{-r+t} \sharp_{\frac{\lambda+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{\lambda+t}{(p-t)s+t}}$$

holds for $-t \leq \lambda \leq (p-t)s$.

But this case reduces to Theorem 1 because

$$\text{(i)} \iff A^{-r} \natural_{\frac{\delta+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B_1^{\mu}$$

and

$$\text{(ii)} \iff A^{-r} \natural_{\frac{\lambda+r}{(p-t)s+r}} B_1^{(p-t)s} \leq A^{-t} \natural_{\frac{\lambda+t}{(p-t)s+t}} B_1^{(p-t)s}.$$

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