

Nonstandard theory of functional and arithmetical divisors

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Let $\mathbb{Q}$ be the field of rational numbers and ${}^*\mathbb{Q}$ an ultrapower of $\mathbb{Q}$. Let Γ be an irreducible plane curve defined by

$$f(X, Y) = 0$$

where f is an irreducible polynomial whose coefficients are contained in $\mathbb{Q}$.

Assume that there are infinitely many rational points on Γ , then there exists a nonstandard rational point (x, y) on Γ .

Since $\mathbb{Q}$ is relatively algebraically closed in ${}^*\mathbb{Q}$, x is transcendental over $\mathbb{Q}$. Hence the field $F = \mathbb{Q}(x, y)$ is the function field of one variable of Γ . Now we have the situation

$$\begin{array}{c} {}^*\mathbb{Q} \\ | \\ F = \mathbb{Q}(x, y) \\ | \\ \mathbb{Q} \end{array} \quad (1)$$

In this situation we have two kinds of prime divisors, arithmetic prime divisors and functional prime divisors. By arithmetical prime divisors, we mean prime divisors of $^*\mathbb{Q}$, namely archimedean absolute value or nonarchimedean valuation. Nonarchimedean valuations are correspond to prime numbers of $^*\mathbb{Q}$. By functional prime divisors we mean nontrivial valuations of F over $\mathbb{Q}$.

In their paper, A. Robinson and P. Roquette gave a relation between arithmetical and functional prime divisors.

Lemma 1. Every functional prime divisor is induced by an arithmetical prime divisor.

Using this lemma, they gave an another proof of so called the first fundamental inequality of Siegel;

Let u, v be any nonconstants of F . For any $\epsilon > 0$, there exists a constant C such that for any rational point P on Γ , if $H(P) > C$, then

$$\left| \frac{\log H(u(P))}{\log H(v(P))} - \frac{[F; \mathbb{Q}(u)]}{[F; \mathbb{Q}(v)]} \right| < \epsilon$$

where H is the height function.

Our aim is to generalize the siegels inequality to be

applicable to algebraic surfaces.

Theorem. *Let V be an irreducible algebraic surface defined over $\mathbb{Q}$ and G the function field of V over $\mathbb{Q}$. Let $t \in G$ be transcendental over $\mathbb{Q}$ and u, v transcendental over $\mathbb{Q}(t)$. For any $\epsilon > 0$, there exists a constant C such that for any rational point P on V if*

$$\min(H(u(P)), H(v(P))) > H(t(P))^C > C^C,$$

then

$$\left| \frac{\log H(u(P))}{\log H(v(P))} - \frac{[G; \mathbb{Q}(t, u)]}{[G; \mathbb{Q}(t, v)]} \right| < \epsilon.$$

To prove Theorem, we introduce H -convex subfields of ${}^*\mathbb{Q}$.

A subfield K of ${}^*\mathbb{Q}$ is called H -convex if $H(a) \leq H(b)$ and $b \in K$ imply $a \in K$.

Let Q_1 be a H -convex subfield of ${}^*\mathbb{Q}$ and F is a function field of one variable over Q_1 , embedded into ${}^*\mathbb{Q}$. Now we have the same situation as before, but unfortunately Lemma 1 does not hold in this case. Hence we must slightly modify Lemma 1 as follows.

Let $R_\infty = \{\beta/\alpha \in {}^*\mathbb{Q} \mid |\beta/\alpha| < \gamma \text{ for some } \gamma \in \mathbb{Z}_1\}$, then R_∞ is a valuation ring whose maximal ideal is $\{\beta/\alpha \in {}^*\mathbb{Q} \mid |\beta/\alpha| < 1/\gamma \text{ for all } \gamma \in \mathbb{Z}_1\}$

where $Z_1 = {}^*\mathbb{Z} \cap Q_1$.

Let $R = \{\beta/\alpha \in {}^*\mathbb{Q} \mid \alpha \in Z_1, \beta \in {}^*\mathbb{Z}\}$ and I a maximal ideal of R . Then the local ring of I is a valuation ring which we denote by R_I .

If $F \cap R_\infty$ is not trivial, namely $F \not\subseteq R_\infty$, then $F \cap R_\infty$ is a valuation ring. Since $F \cap R_\infty \supset Q_1$, this valuation ring yields a functional prime P of F . We say that P is induced by the archimedean prime. If $R_I \cap F$ is not trivial, then $R_I \cap F$ also yields a functional prime P of F . We say that P is induced by I .

Lemma 2. Every functional prime P is induced by the archimedean prime or a maximal ideal of R .

Proof. By the theorem of Riemann-Roch, there exists β/α which admits P as its only pole. If $|\beta/\alpha| > \gamma$ for all $\gamma \in Z_1$, then $\beta/\alpha \notin R_\infty$. Hence $\beta/\alpha \notin R_\infty \cap F$. Then the functional prime induced by the archimedean prime is a pole of β/α . Since P is the only functional prime which is a pole of β/α , P is induced by the archimedean prime. If $|\beta/\alpha| < \gamma$ for some $\gamma \in Z_1$, then $\alpha \in {}^*\mathbb{Z} - Z_1$ because β/α is a nonconstant. Let I_α be a maximal ideal of which contains α . Then the local ring of I_α does not contain β/α .

Then $\beta/\alpha \notin R_{I_\alpha} \cap F$. By the same arguments as above, P is induced by I_α . Lemma 2 is proved.

For details of proof of Theorem, Please refer to [2].

Refernces

- [1] A. Robinson and P. Roquette, On the finiteness theorem of Siegel and Mahler concerning diophantine equations, J.Number Theory 7 (1975), 121-176.
- [2] M. Yasumoto, Nonstandard arithmetic of function fields over H-convex subfields of $^*\mathbb{Q}$, Journal fur die reine und angewandte Mathematik, 342 (1983), 1-11.