

Brauer twists of K3 surfaces admitting van Geemen-Sarti involution

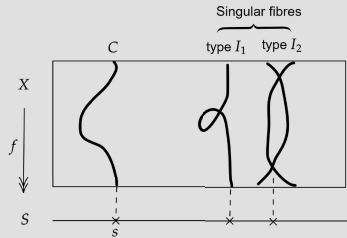
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1) Elliptic surfaces

Let X : smooth projective surface/ $\mathbb{C}$,
 S : smooth projective curve/ $\mathbb{C}$.

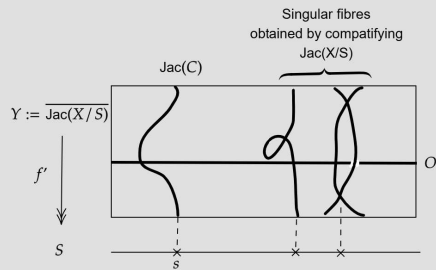
Then X is an **elliptic surface** if $\exists f : X \rightarrow S$ where the generic fibre is smooth and of genus 1.



- May or may not have sections.
- Singular fibres are classified by Kodaira.

↓ Take relative Jacobian and compactify

$\overline{\text{Jac}(X/S)}$ of X is another elliptic surface Y singular fibres of the same types and a **section** O .



★ The relative Jacobian construction is not one-to-one!

2) Geometric Tate-Shafarevich group

$$\text{III}_S(Y) = \left\{ \text{elliptic fibrations } f : X \rightarrow S \text{ such that: } \begin{array}{l} \text{Jac}(X/S) = Y, \text{ generic fibre is reduced} \end{array} \right\} / \simeq$$

where $\simeq$ denotes birational equivalence.

3) Question:

If we consider some Y whose geometry is well-understood, can we (geometrically) realise the elliptic surfaces parametrised by $\text{III}_S(Y)$?

4) Lattice polarised K3 surfaces

A **K3 surface** X is a simply connected compact complex projective surface with $K_X = \mathcal{O}_X$ and $h^{1,0}(X) = 0$.

Examples of (birational models of) K3 surfaces:

- Smooth quartic in $\mathbb{P}^3$.
- Double sextic = double cover of $\mathbb{P}^2$ ramified at a sextic curve.
- Double quartic = double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified at a $(4, 4)$ -curve.

A **lattice polarisation** of X is roughly a primitive embedding $j : P \hookrightarrow \text{Pic}(X)$ where

- P is a lattice called the **polarisation lattice**,
- $j(P)$ contains a big and nef line bundle.

The **Picard rank** of X is $\rho := \text{rk}(P)$.

The **transcendental lattice** T is the orthogonal complement of P in the K3 lattice Λ_{K3}

6) Objects of interest

Consider Y : a very general $(U \oplus N)$ -polarised K3 surface (has elliptic fib. with sing. fibres $8I_2 + 8I_1$ and one 2-torsion section T),
 $\text{Br}(Y)[2]$: the 2-torsion elements in the Brauer group of Y .

What are the elliptic K3 surfaces that correspond to the elements in $\text{Br}(Y)[2]$?

7) van Geemen's construction

Let Y be a K3 surface.

For any $\alpha \in \text{Br}(Y)[d]$, define

$$T_{(\alpha)} := \ker(\alpha : T \rightarrow \mathbb{Z}/d\mathbb{Z}) < T.$$

The “**twist** of Y by α ”

= K3 surface $X_{(\alpha)}$ with transcendental lattice $T_{(\alpha)}$.

Properties of twists of an elliptic Y :

- $X_{(\alpha)}$ has no section but a d -multisection if α is non-trivial.
 e.g. S on the right is a 2-multisection.
- $\overline{\text{Jac}(X_{(\alpha)})} = Y$.
- $X_{(\alpha)} \dashrightarrow Y$ generically 2:1.

5) Elliptic K3 surfaces

If X is a K3 surface with an elliptic fibration $f : X \rightarrow S$, then

- the base S is a $\mathbb{P}^1$;
- the irr. components of singular fibres, the generic fibre F and the section O generate a sublattice in $\text{NS}(X)$.

X has an elliptic fibration **with a 2-torsion section** if there is a primitive embedding

$$U \oplus N \hookrightarrow \text{NS}(X),$$

where

$N = \langle N_1, \dots, N_8, (\sum N_i)/2 \rangle$ is the rank 8 Nikulin lattice,

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic lattice with basis $\{F, F + O\}$.

For an elliptic K3 surface $X \rightarrow S$, the Tate-Shafarevich group coincides with the **Brauer group**:

$$\text{III}_S(X) \simeq \text{Br}(X) \simeq T^* \otimes \mathbb{Q}/\mathbb{Z} \simeq (\mathbb{Q}/\mathbb{Z})^{22-\rho}.$$

9) Removing sections from the trivial twist X

Note:

- zero section O' in Y gives rise to sections O and T in X .
- 2-torsion section T' in Y gives rise to 2-multisection S in X .
 $\Rightarrow$ The sections O, T can be killed by contracting some N'_i 's instead of the N_i 's in Y .

Considering **Morrison's even eight**:

If $\Sigma(\text{exceptional curves}) \in 2\text{NS}(Y)$,

then the corresponding double cover is a twist of Y .

WLOG let

$$\text{exceptional curves} = \{N_1, \dots, N_d, N'_{d+1}, \dots, N'_8\}.$$

Computations show:

the double cover arises this way is a twist of Y iff d is even.

10) Results

We list all the twists of Y due to different d 's.

d	#combinations	birational model
0 or 8	1	trivial twist
2 or 6	$C_2^8 = 28$	double sextic
4	$C_4^8/2 = 35$	double quartic

11) Numerology and further questions

- $|\text{Br}(Y)[2]| = 2^{12}$.
- The corresponding 2^{12} twists of Y form 5 isomorphism classes, each occur with multiplicity

1, 28, 35, 1792 and 2240.

The first 3 iso. classes are described by our constructions

Question: How to geometrically realise the last two classes?

References

- [1] B. van Geemen. Some remarks on Brauer groups of K3 surfaces, *Advances in Mathematics*, 197(1), 222–247, 2005.
- [2] D. R. Morrison. On K3 surfaces with large Picard number, *Invent. Math.*, 75, 105–121, 1984.

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