

CYCLE CLASS MAPS FOR ARITHMETIC SCHEMES

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X/k : a proper smooth variety over a field k of characteristic zero.

Let

$$\mathrm{CH}^r(X) = \left(\bigoplus_{\substack{V \subset X \\ \text{irred. subvar.}}} \mathbb{Z} \right) / \sim_{\text{rat. equiv.}}$$

be the group of cycles of codimension r in X modulo rational equivalence, called the Chow group of cycles of codimension r .

For $r = 1$ we have

$$\mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$$

where $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X .

If X has a k -rational point, we have the exact sequence

$$0 \rightarrow \mathrm{Pic}_{X/k}^0(k) \rightarrow \mathrm{Pic}(X) \rightarrow \mathrm{NS}(X) \rightarrow 0$$

where $\mathrm{NS}(X)$ is the Neron-Severi group of X and $\mathrm{Pic}_{X/k}^0$ is the Picard variety of X/k . It is known that:

- (1) $\mathrm{NS}(X)$ is finitely generated (for an arbitrary k).
- (2) $\mathrm{Pic}_{X/k}^0(k)$ (=the group of the k -rational points of $\mathrm{Pic}_{X/k}^0$) is finitely generated if $[k : \mathbb{Q}] < \infty$. (the Mordell-Weil theorem).

Hence $\mathrm{CH}^1(X)$ is finitely generated if $[k : \mathbb{Q}] < \infty$.

Question: Is $\mathrm{CH}^r(X)$ finitely generated if $[k : \mathbb{Q}] < \infty$?

Remark: The rank of $\mathrm{CH}^r(X)$ and the order of $\mathrm{CH}^r(X)_{\mathrm{tors}}$ are expected to be related to special values of L -function of X (Tate, Birch-Swinnerton-Dyer, Beilinson, Bloch-Kato,...).

Only little is known about the above question. Difficulty comes from the fact that $\mathrm{CH}^r(X)$ for $r \geq 2$ is in general “not representable” so that over $\mathbb{C}$ it is as large as $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C} \cdots \otimes_{\mathbb{Z}} \mathbb{C}$ (Mumford theorem).

We now assume $[k : \mathbb{Q}] < \infty$ or $[k : \mathbb{Q}_\ell] < \infty$. We fix a prime p and are concerned with the finiteness of:

$$\mathrm{CH}^2(X)_{p\text{-tors}} \quad \text{and} \quad \mathrm{CH}^2(X)/p^n$$

where for an abelian group M , $M_{p\text{-tors}}$ denotes the p -primary torsion part. One way to approach to the fundamental question is to look at the cycle class map from Chow group to (continuous) étale cohomology of X :

$$\rho_{X, \mathbb{Z}/p^n\mathbb{Z}}^r : \mathrm{CH}^r(X)/p^n \rightarrow H_{\mathrm{ét}}^{2r}(X, \mathbb{Z}/p^n\mathbb{Z}(r))$$

$$\rho_{X, \mathbb{Z}_p}^r : \mathrm{CH}^r(X) \otimes \mathbb{Z}_p \rightarrow H_{\mathrm{cont}}^{2r}(X, \mathbb{Z}_p(r))$$

where $\mathbb{Z}/p^n\mathbb{Z}(r) = \mu_{p^n}^{\otimes r}$ is the r th tensor power of the sheaf of p^n th roots of unity and $\mathbb{Z}_p(r) = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}(r)$. Note that $H_{\text{ét}}^{2r}(X, \mathbb{Z}/p^n\mathbb{Z}(r))$ is not in general finite if $[k : \mathbb{Q}] < \infty$. But one can show that $\text{Im}(\rho_{X, \mathbb{Z}/p^n\mathbb{Z}}^r)$ is finite and $\text{Im}(\rho_{X, \mathbb{Z}_p}^r)$ is a finitely generated $\mathbb{Z}_p$ -module. Hence the injectivity of the above maps would imply the desired finiteness.

For $r = 1$ one can show the injectivity of these maps by using the Kummer sequence

$$0 \rightarrow \mathbb{Z}/p^n\mathbb{Z}(1) \rightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \rightarrow 0$$

and the isomorphism

$$\text{CH}^1(X) \simeq \text{Pic}(X) \simeq H_{\text{ét}}^1(X, \mathbb{G}_m).$$

It is conjectured in case $[k : \mathbb{Q}] < \infty$ that the kernel of $\rho_{X, \mathbb{Z}_p}^r$ is torsion. On the other hand, using the theory of quadratic forms, Parimala and Suresh proved the following:

Theorem: There exists a smooth projective surface X over k with $H^2(X, \mathcal{O}_X) = 0$ (in fact X is a rational surface) such that $\text{Ker}(\rho_{X, \mathbb{Z}_2}^2)$ is a nonzero finite group.

In this talk we present a new viewpoint on the injectivity problem of cycle class maps by investigating cycle maps for models of X over the ring of integers of k . We fix the following setup:

k : $[k : \mathbb{Q}] < \infty$ or $[k : \mathbb{Q}_\ell] < \infty$.

$\mathcal{O}_k$: the integer ring of k and put $S := \text{Spec}(\mathcal{O}_k)$,

$\mathcal{X}$: a regular scheme which is proper flat of finite type over S .

$X = \mathcal{X} \times_S \text{Spec}(k)$: the generic fiber of $\mathcal{X}$.

We fix a prime p and assume the following condition:

If p is not invertible on $\mathcal{X}$, then $\mathcal{X}$ has good or semistable reduction at each prime ideal of $\mathcal{O}_k$ dividing (p) .

If p is not invertible on $\mathcal{X}$, étale cohomology of $\mathcal{X}$ with $\mu_{p^n}^{\otimes r}$ -coefficient does not work well. Instead the p -adic étale Tate twist

$$\mathfrak{T}_n(r)_{\mathcal{X}} \in D^b(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z})$$

defined by K.Sato plays an important role. Here $D^b(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z})$ denotes the derived category of bounded complexes of étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$ -modules on $\mathcal{X}$.

Remark

- (1) Letting $\mathcal{X}[\frac{1}{p}] \subset \mathcal{X}$ be the open subscheme obtained by removing the fibers over the points of characteristic p of S ,

$$\mathfrak{T}_n(r)_{\mathcal{X}[\frac{1}{p}]} = \mu_{p^n, \mathcal{X}[\frac{1}{p}]}^{\otimes r}.$$

- (2) Sato proved the finiteness of $H_{\text{ét}}^i(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}})$.
- (3) It is expected that:

$$\mathfrak{T}_n(r)_{\mathcal{X}} = \mathbb{Z}(r)_{\mathcal{X}}^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z},$$

where $\mathbb{Z}(r)_{\mathcal{X}}^{\text{ét}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum for $\mathcal{X}$.

By the semi-purity property of $\mathfrak{T}_n(r)_{\mathcal{X}}$ shown by Sato, we can define the cycle map

$$\boxed{\rho_{\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z}}^r : \text{CH}^r(\mathcal{X})/p^n \rightarrow H_{\text{ét}}^{2r}(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}})}$$

We are now concerned with the induced maps

$$\begin{aligned}\rho_{\mathcal{X}, p\text{-tors}}^r &: \mathrm{CH}^r(\mathcal{X})_{p\text{-tors}} \rightarrow H_{\mathrm{\acute{e}t}}^{2r}(\mathcal{X}, \mathfrak{T}_{\mathbb{Z}_p}(r)_{\mathcal{X}}) \\ \rho_{\mathcal{X}, \mathbb{Z}_p}^r &: \mathrm{CH}^r(\mathcal{X}) \otimes \mathbb{Z}_p \rightarrow H_{\mathrm{\acute{e}t}}^{2r}(\mathcal{X}, \mathfrak{T}_{\mathbb{Z}_p}(r)_{\mathcal{X}}),\end{aligned}$$

where

$$H_{\mathrm{\acute{e}t}}^*(\mathcal{X}, \mathfrak{T}_{\mathbb{Z}_p}(r)_{\mathcal{X}}) = \varprojlim_{n \geq 1} H_{\mathrm{\acute{e}t}}^*(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}}).$$

Our main results on these maps concern the injectivity of these two maps in case $r = 2$. Roughly speaking, the injectivity of $\rho_{\mathcal{X}, p\text{-tors}}^2$ and $\rho_{\mathcal{X}, \mathbb{Z}_p}^2$ follows from a list of assumptions, each of which is a consequence of a well-known conjecture in arithmetic geometry. As a corollary we will get the following result: (Recall $X = \mathcal{X} \times_S \mathrm{Spec}(k)$)

Theorem 0.1. *Assume $H^2(X, \mathcal{O}_X) = 0$. Then:*

- (1) $\rho_{\mathcal{X}, p\text{-tors}}^2$ is injective.
- (2) Suppose that $[k : \mathbb{Q}_\ell] < \infty$ with $\ell \neq p$ and $\dim(X) = 2$. Then $\mathrm{Ker}(\rho_{\mathcal{X}, \mathbb{Z}_p}^2)$ is uniquely p -divisible.
- (3) Suppose that $[k : \mathbb{Q}_p] < \infty$ and $\dim(X) = 2$ with $\kappa_X \leq 1$. Then $\rho_{\mathcal{X}, \mathbb{Z}_p}^2$ is injective.
- (4) Suppose that $[k : \mathbb{Q}] < \infty$ and $\dim(X) = 2$ with $\kappa_X \leq 1$. Then $\rho_{\mathcal{X}, \mathbb{Z}_p}^2$ is injective.

Unramified cohomology:

Let $\mathcal{X}/\mathcal{O}_k$ be as before and let K be its function field.

The *unramified cohomology* of K (here we write $\mathbb{Q}_p/\mathbb{Z}_p(n) = \mu_{p^\infty}^{\otimes n}$)

$$H_{\mathrm{ur}}^{n+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) \subset H_{\mathrm{\acute{e}t}}^{n+1}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n))$$

is defined to be the subgroup of those elements which are *unramified along every point of codimension one* on $\mathcal{X}$. More precisely it is the kernel of the boundary map

$$H_{\mathrm{\acute{e}t}}^{n+1}(\mathrm{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(n)) \rightarrow \bigoplus_{y \in \mathcal{X}^1} H_{y, \mathrm{\acute{e}t}}^{n+2}(\mathcal{X}, \mathfrak{T}_\infty(r)_{\mathcal{X}})$$

in the localization sequence, where $\mathfrak{T}_\infty(r)_{\mathcal{X}} = \varprojlim_{n \geq 1} \mathfrak{T}_n(r)_{\mathcal{X}}$ and $\mathcal{X}^1$ is the set of the points of codimension one in $\mathcal{X}$.

The following isomorphisms hold true:

$$\begin{aligned}H_{\mathrm{ur}}^1(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) &\simeq H_{\mathrm{\acute{e}t}}^1(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{ab}(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p), \\ H_{\mathrm{ur}}^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) &\simeq \mathrm{Br}(\mathcal{X})_{p\text{-tors}},\end{aligned}$$

where $\pi_1^{ab}(\mathcal{X})$ denotes the abelian fundamental group of $\mathcal{X}$ and $\mathrm{Br}(\mathcal{X})$ denotes the Grothendieck-Brauer group $H_{\mathrm{\acute{e}t}}^2(\mathcal{X}, \mathbb{G}_m)$.

In case $[k : \mathbb{Q}] < \infty$, $\mathrm{Br}(\mathcal{X})$ is isomorphic (up to finite groups) to the Tate-Shafarevich group of $\mathrm{Pic}_{X/k}^0$, the Picard variety of the generic fiber X of $\mathcal{X}$.

For $n = 0$, the quotient $H_{\mathrm{\acute{e}t}}^1(\mathcal{X}, \mathbb{Q}/\mathbb{Z})/H_{\mathrm{\acute{e}t}}^1(S, \mathbb{Q}/\mathbb{Z})$ is finite by a theorem of Katz-Lang and in case $[k : \mathbb{Q}] < \infty$, $H_{\mathrm{\acute{e}t}}^1(\mathcal{X}, \mathbb{Q}/\mathbb{Z})$ is finite as well, because $H_{\mathrm{\acute{e}t}}^1(S, \mathbb{Q}/\mathbb{Z})$ is finite.

In case $[k : \mathbb{Q}] < \infty$, $H_{\mathrm{ur}}^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is expected to be finite due to the finiteness conjecture of the Tate-Shafarevich group of the Picard variety of X .

In case $n = d := \dim(\mathcal{X})$, $H_{\mathrm{ur}}^{d+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(d))$ has been considered by K. Kato who conjectured $H_{\mathrm{ur}}^{d+1}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = 0$ if $p \neq 2$ or k has no embedding into $\mathbb{R}$ (The last conjecture is proved by Kato in case $d = 2$ and by Jannsen-Saito in case $d = 3$).

Motivated by the above facts we propose the following:

Conjecture 0.2. $H_{\mathrm{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.

The conjecture plays a central role in the proof of our main result. Indeed we have the following result.

Proposition 0.3. *Let*

$$H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

be the intersection of $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ with

$$\text{Im}(H_{\text{et}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow H_{\text{et}}^3(\text{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

- (1) *If $H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite, then $\text{Ker}(\rho_{\mathcal{X}, p\text{-tors}}^2)$ coincides with the maximal divisible subgroup of $\text{CH}^2(\mathcal{X})_{p\text{-tors}}$.*
- (2) *If $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite then $\text{Ker}(\rho_{\mathcal{X}, \mathbb{Z}_p}^2)$ coincides with the maximal divisible subgroup of $\text{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_p$.*

The proposition is deduced from the exact sequence

$$H_{\text{ur}}^3(K, \mathbb{Z}/p^n\mathbb{Z}(2)) \rightarrow \text{CH}^2(\mathcal{X})/p^n \xrightarrow{\rho_{\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z}}^2} H_{\text{et}}^4(\mathcal{X}, \mathfrak{I}_n(r)_{\mathcal{X}})$$

which is constructed by using the semi-purity property of the Sato complex.

By the proposition the injectivity problem of our cycle class maps is reduced to the finiteness problem of the unramified cohomology $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$. We next relate it to other well-known conjectures in arithmetic geometry.

Bloch-Kato conjecture:

Let $\mathcal{X}/S = \text{Spec}(\mathfrak{O}_k)$ and $X = \mathcal{X} \times_S \text{Spec}(k)$ be as before. The conjecture concerns the p -adic regulator map from Bloch's higher Chow group to continuous Galois cohomology:

$$\text{reg}_X^{r,q} : \text{CH}^r(X, q) \otimes \mathbb{Q}_p \rightarrow H_{\text{cont}}^1(G_k, H_{\text{et}}^{2r-q-1}(\overline{X}, \mathbb{Q}_p(r))) \quad (r, q \geq 1)$$

where $G_k = \text{Gal}(\overline{k}/k)$ and $\overline{X} = X \times_k \overline{k}$.

Conjecture (Bloch-Kato):

$$\text{Im}(\text{reg}_X^{r,q}) = H_g^1(G_k, H_{\text{et}}^{2r-q-1}(\overline{X}, \mathbb{Q}_p(r)))$$

where the right hand side is the subspace defined by Bloch-Kato by using the p -adic Hodge theory. In case $[k : \mathbb{Q}_\ell] < \infty$,

$$H_g^1(G_k, V) = \begin{cases} H_{\text{cont}}^1(G_k, V) & (p \neq \ell) \\ \text{Ker}(H_{\text{cont}}^1(G_k, V) \rightarrow H_{\text{cont}}^1(G_k, V \otimes B_{DR})) & (p = \ell) \end{cases}$$

where $V = H_{\text{et}}^*(\overline{X}, \mathbb{Q}_p(r))$.

The following special case is relevant to our problem.

$$\text{reg}_X = \text{reg}_X^{2,1} : \text{CH}^2(X, 1) \otimes \mathbb{Q}_p \rightarrow H_{\text{cont}}^1(G_k, H_{\text{et}}^2(\overline{X}, \mathbb{Q}_p(2)))$$

where $\text{CH}^2(X, 1)$ coincides with the cohomology of the following complex

$$K_2(K) \xrightarrow{\delta_1} \bigoplus_{x \in X^1} k(x)^\times \xrightarrow{\delta_1} \bigoplus_{x \in X^2} \mathbb{Z},$$

(recall K is the function field of X), where

$$K_2(K) = (K^\times \otimes_{\mathbb{Z}} K^\times) / \langle x \otimes y \mid x + y = 1 \ (x, y \in K^\times) \rangle,$$

and X^r denotes the set of the points of codimension r on X and $k(x)$ is the residue field of $x \in X^r$. The map δ_1 is the so-called tame symbol and δ_2 is the map taking the divisors of functions.

We now state the Bloch-Kato conjecture in the relevant case as a condition:

$$\boxed{(\mathbf{H1}) : \operatorname{Im}(reg_X) = H_g^1(G_k, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2)))}$$

where

$$reg_X : CH^2(X, 1) \otimes \mathbb{Q}_p \rightarrow H_{\text{cont}}^1(G_k, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2)))$$

(H1) is known to hold in the following cases:

- (1) $H^2(X, \mathcal{O}_X) = 0$,
- (2) $X = E \times E$ where E is a modular elliptic curve without CM over $\mathbb{Q}$ and $p \nmid (\text{level of } E)$, $p \geq 5$,
- (3) X is an elliptic modular surface of level 4 over $\mathbb{Q}$ and $p \geq 5$,
- (4) X is a Fermat quartic surface over $k = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$,

The first case is easy and the other cases follow from the works of Mildenhall, Flach, Langer-Saito, Langer, Otsubo.

We now consider the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$ -coefficient

$$reg_{X, \mathbb{Q}_p/\mathbb{Z}_p} : CH^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(G_k, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

Consider the following variant of H1:

$$\boxed{(\mathbf{H1}^*) : \operatorname{Im}(reg_{X, \mathbb{Q}_p/\mathbb{Z}_p}) = H_g^1(G_k, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2)))_{Div}}$$

where for an abelian group M , M_{Div} denotes its maximal divisible subgroup.

H1 always implies H1* and that the converse holds under some assumptions (for example in case $[k : \mathbb{Q}_\ell] < \infty$).

In what follows we assume $H_{\text{ét}}^3(X_{\overline{k}}, \mathbb{Q}_p(2))^{G_k} = 0$, which holds if $[k : \mathbb{Q}] < \infty$ by the Weil conjecture (Deligne). If $[k : \mathbb{Q}_\ell] < \infty$, it is a consequence of the monodromy-weight conjecture so that it holds if $\dim(X) = 2$ or $\mathcal{X}$ is proper smooth over S . We also assume $p \geq 5$ by a technical reason coming from p -adic Hodge theory.

Theorem 0.4. *Let the assumption be as above.*

- (1) H1* implies the following two finiteness conditions:

F1: $CH^2(X)_{p\text{-tors}}$ is finite.

F2: $H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.

- (2) Assume further

T: The reduced part of every fiber of $\mathcal{X}/\mathcal{O}_k$ has simple normal crossings on $\mathcal{X}$ and the Tate conjecture for divisors holds for the irreducible components of those fibers.

Then F1 and F2 imply H1*.

As for the finiteness of $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, we need another condition:

H2: Let

$$AJ_X^2 : CH^2(X) \otimes \mathbb{Z}_p \rightarrow H_{\text{cont}}^1(k, H_{\text{ét}}^3(\overline{X}, \mathbb{Z}_p(2)))$$

the p -adic Abel-Jacobi map for X . Then the quotient of $\operatorname{Ker}(AJ_X^2)$ by its torsion subgroup is divisible.

In case $[k : \mathbb{Q}] < \infty$, Beilinson conjectured that $\operatorname{Ker}(AJ_X^2)$ is torsion.

In case $\dim(X) = 2$, H2 holds true in the following cases:

- $[k : \mathbb{Q}_\ell] < \infty$ with $\ell \neq p$ (Saito-Sujatha).

◦ $H^2(X, \mathcal{O}_X) = 0$ and $\kappa_X \leq 1$ (Bloch-Kas-Lieberman).

Theorem 0.5. *Let the assumption be as before. Then **H1*** and **H2** imply that $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.*

Summing up these, we get the following result which implies the first main result on the injectivity of cycle class map.

Corollary 0.6. *Assume $H^2(X, \mathcal{O}_X) = 0$. Then:*

- (1) $H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.
- (2) $H_{\text{ur}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite under one of the following:
 - (i) $[k : \mathbb{Q}_\ell] < \infty$ with $\ell \neq p$ and $\dim(X) = 2$,
 - (ii) $[k : \mathbb{Q}_p] < \infty$ and $\dim(X) = 2$ and $\kappa_X \leq 1$.
 - (iii) $[k : \mathbb{Q}] < \infty$ and $\dim(X) = 2$ and $\kappa_X \leq 1$.

Idea of Proof: We now explain the idea to show that **H1*** implies the finiteness of $\text{CH}^2(X)_{p\text{-tors}}$ and $H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$. We only treat the case $[k : \mathbb{Q}_p] < \infty$. We consider the following groups

$$\text{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset N^1 H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset U \subset H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

where $N^1 H_{\text{ét}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is the kernel of the natural map

$$\begin{aligned} H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) &\xrightarrow{\iota} H^3(\text{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2)), \\ U &= \iota^{-1}(H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))), \end{aligned}$$

The first inclusion comes from Bloch's exact sequence

$$0 \rightarrow \text{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow N^1 H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow \text{CH}^2(X)_{p\text{-tors}} \rightarrow 0$$

which is obtained by using the theorem of Mercuriev-Suslin on the surjectivity of the Galois symbol map for K_2 . Thus it suffices to show

$$[U : \text{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p] < \infty.$$

The assumption implies that $H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_k}$ is finite and the Hochschild-Serre spectral sequence

$$E_2^{u,v} := H^u(G_k, H^v(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \implies H^{u+v}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

induces the edge homomorphism

$$\nu : H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{Div}} \rightarrow H^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

where for an abelian group M , M_{Div} denotes its maximal divisible subgroup. We note that the composition

$$\text{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\nu} H^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

is the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$ -coefficient. Thus **H1*** implies

$$\nu(\text{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = H_g^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{\text{Div}}.$$

Hence we are reduced to show the following:

Claim A: $\nu(U_{\text{Div}}) \subset H_g^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$.

Claim B: $U \cap \text{Ker}(\nu)$ is finite.

To show Claim A, we first prove the inclusions

$$U \hookrightarrow W = H^3(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p/\mathbb{Z}_p(2)) \hookrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)),$$

where $j : X \hookrightarrow \mathcal{X}$ is the natural immersion. It is derived from the following purity theorem. Let $Y \subset \mathcal{X}$ be the special fiber of $\mathcal{X}/\mathcal{O}_k$.

Theorem (Hagihara): Let n, r and c be integers with $n \geq 0$ and $r, c \geq 1$. Then for any integer $q \leq n + c$ and any closed subscheme $Z \subset Y$ with $\text{codim}_{\mathcal{X}}(Z) \geq c$, we have

$$H_Z^q(\mathcal{X}, \tau_{\leq n} Rj_* \mu_{p^r}^{\otimes n}) = 0 = H_Z^{q+1}(\mathcal{X}, \tau_{\geq n+1} Rj_* \mu_{p^r}^{\otimes n}).$$

By the above inclusions the proof of Claim A is reduced to show

$$(*) \quad \nu(W_{Div}) \subset H_g^1(G_k, H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

The first step is to relate W with syntomic cohomology of $\mathcal{X}/\mathfrak{O}_k$.

Theorem (Kato-Kurihara-Tsuji) : There is a canonical isomorphism

$$\eta : s_n^{\log}(r)_{\mathcal{X}} \xrightarrow{\cong} i_* i^* \tau_{\leq r} Rj_* \mu_{p^n}^{\otimes r},$$

where the right hand side denotes the log-syntomic complex of Kato and $i : Y \rightarrow \mathcal{X}$ is the closed immersion of the closed fiber of $\mathcal{X}/S$.

Now put

$$H^*(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(r)_{\mathcal{X}}) := \left\{ \lim_{\leftarrow r \geq 1} H^*(\mathcal{X}, s_n^{\log}(r)_{\mathcal{X}}) \right\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

Assume $H_{\text{ét}}^{i+1}(\overline{X}, \mathbb{Q}_p(r))^{G_k} = 0$. Let ξ be the composite map:

$$H^{i+1}(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(n)_{\mathcal{X}}) \rightarrow H_{\text{ét}}^{i+1}(X, \mathbb{Q}_p(r)) \rightarrow H^1(G_k, H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_p(r)))$$

where the second map comes from the Hochschild-Serre spectral sequence. The desired assertion (*) follows from the following result shown via theory of log-syntomic and log-crystalline cohomology.

Theorem (Langer and Nekovář) : We have

$$\text{Im}(\xi) = H_g^1(G_k, H_{\text{ét}}^i(\overline{X}, \mathbb{Q}_p(r))).$$

Finally we explain the idea to show Claim B, namely the finiteness of $U \cap \text{Ker}(\nu)$. By definition of $H_{\text{ur}}^3(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$, U is the kernel of the localization map

$$\delta : H_{\text{ét}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow \bigoplus_{y \in \mathcal{X}^1} H_y^4(\mathcal{X}, \mathfrak{I}_{\infty}(2)_{\mathcal{X}})$$

where $\mathfrak{I}_{\infty}(2)_{\mathcal{X}} = \varinjlim_n \mathfrak{I}_n(2)_{\mathcal{X}}$.

On the other hand, $\text{Ker}(\nu) = F^2 H_{\text{ét}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ with F^2 denoting the filtration coming from the Hochschild-Serre spectral sequence. Hence we have a surjection

$$H^2(G_k, H_{\text{ét}}^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \rightarrow \text{Ker}(\nu).$$

Therefore we are reduced to show the finiteness of the kernel of the composite map

$$H^2(G_k, H_{\text{ét}}^1(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \rightarrow \bigoplus_{y \in \mathcal{X}^1} H_y^4(\mathcal{X}, \mathfrak{I}_{\infty}(2)_{\mathcal{X}}).$$

In order to show this, one is required to describe the above map explicitly in terms of geometry of the special fiber Y of $\mathcal{X}/\mathfrak{O}_k$. This is rather technical and complicate. Here we only point out one key ingredient.

Let $\bar{Y} = \mathcal{X} \times_{\mathcal{O}_k} \bar{F}$ where F is the residue field of k and $\bar{F}$ is an algebraic closure of F . Let $W_n \omega_{Y, \log}^q$ be the logarithmic part of the de Rham-Witt differential $W_n \omega_Y^q$ associated to the semi-stable scheme $\mathcal{X}/\mathcal{O}_k$ defined by Hyodo. Then one constructs a natural map

$$h : H^2(G_k, H_{\text{ét}}^1(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \rightarrow H^1(F, H^0(\bar{Y}, W_{\infty} \omega_{Y, \log}^1))$$

$$(W_{\infty} \omega_{Y, \log}^1 = \varinjlim_n W_n \omega_{Y, \log}^1)$$

and show that it has finite kernel and cokernel by using the Fontaine-Jannsen conjecture (the comparison isomorphism between p -adic étale cohomology and log-crystalline cohomology of $\mathcal{X}/\mathcal{O}_k$) proved by Hyodo-Kato and Tsuji.