

On classes of operators generalizing class A and paranormality and related results

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This report is based on the following papers:

- [I] M.Ito, *On classes of operators generalizing class A and paranormality*, Sci. Math. Jpn., **57** (2003), 287–297, (online version, **7** (2002), 353–363). (§1–4)
- [IYY] M.Ito, T.Yamazaki and M.Yanagida, *Generalizations of results on relations between Furuta-type inequalities*, to appear in Acta Sci. Math. (Szeged). (§5)

Abstract

Recently, we introduced class A defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. As generalizations of class A and paranormality, Fujii-Nakamoto introduced class $F(p, r, q)$ and (p, r, q) -paranormality respectively. These classes are related to p -hyponormality and log-hyponormality.

In this report, we shall remove the assumption of invertibility from some results on invertible class $F(p, r, q)$ operators, and also we shall show that the families of class $F(p, r, \frac{p+r}{q+r})$ and $(p, r, \frac{p+r}{q+r})$ -paranormality are proper on p . Moreover, we shall obtain the relations between Furuta-type inequalities as a generalization of the key theorem in the proofs of our main results.

1 Introduction

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that p -hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$ and invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator T , and also an operator T is said to be p -quasihyponormal for $p > 0$ if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. We remark that we treat only invertible log-hyponormal operators in this paper (see also [26]). It is easily obtained that every p -hyponormal operator is q -hyponormal for

$p > q > 0$ by Löwner-Heinz theorem " $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$," and every invertible p -hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \rightarrow \log X$ as $p \rightarrow +0$ for $X > 0$. An operator T is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. Ando [2] showed that every p -hyponormal operator for $p > 0$ and (invertible) log-hyponormal operator is paranormal.

Recently, in [15], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms. And also Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [9] introduced class $A(p, r)$ and Yamazaki-Yanagida [28] introduced absolute- (p, r) -paranormality as follows: An operator T belongs to class $A(p, r)$ for $p > 0$ and $r > 0$ if $(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$, and also an operator T is absolute- (p, r) -paranormal if $\| |T|^p |T^*|^r x \|^r \geq \| |T^*|^r x \|^{p+r}$ for every unit vector $x \in H$. We remark that class $A(1, 1)$ equals class A and also absolute- $(1, 1)$ -paranormality equals paranormality. These classes are generalizations of class $A(k)$ and absolute- k -paranormality introduced as two families of classes based on class A and paranormality in [15], and also absolute- (p, r) -paranormality is a generalization of p -paranormality in [7]. We should remark that the families of class $A(p, r)$ determined by operator inequalities and absolute- (p, r) -paranormality determined by norm inequalities constitute two increasing lines on $p > 0$ and $r > 0$ whose origin is (invertible) log-hyponormality.

Moreover, as a continuation of the discussion in [9], Fujii-Nakamoto [10] introduced the following classes of operators.

Definition ([10]). For each $p > 0$, $r \geq 0$ and $q > 0$,

(i) An operator T belongs to class $F(p, r, q)$ if

$$(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \quad (1.1)$$

(ii) An operator T is (p, r, q) -paranormal if

$$\| |T|^p U |T|^r x \|^{\frac{1}{q}} \geq \| |T|^{\frac{p+r}{q}} x \| \quad (1.2)$$

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T . In particular, if $r > 0$ and $q \geq 1$, then (1.2) is equivalent to

$$\| |T|^p |T^*|^r x \|^{\frac{1}{q}} \geq \| |T^*|^{\frac{p+r}{q}} x \| \quad (1.3)$$

for every unit vector $x \in H$ ([18]).

We remark that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$ and also $(p, r, \frac{p+r}{r})$ -paranormality equals absolute- (p, r) -paranormality. In [18], we obtained the parallel result to that of class $A(p, r)$ and absolute- (p, r) -paranormality that invertible class $F(p, r, q)$ and (p, r, q) -paranormality constitute two increasing lines on $p \geq \delta > 0$ and $r \geq r_0 > 0$ whose origin is δ -quasihyponormality. And also we showed the result on powers of invertible class $F(p, r, q)$ operators. Thus many reseachers have been discussed parallel families of classes of operators which are generalizations of class A and paranormality.

In this report, we shall remove the assumption of invertibility from some results on invertible class $F(p, r, q)$ operators in [18], and also we shall show that the families of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$ -paranormality are proper on p . Moreover, we shall obtain the relations between Furuta-type inequalities as a generalization of the result shown in [19] which is the key theorem in the proofs of our main results.

2 Preliminaries

Fujii-Nakamoto [10] observed that class $F(p, r, q)$ derives from the following Theorem 2.A shown in [11] and (p, r, q) -paranormality corresponds to class $F(p, r, q)$.

We remark that alternative proofs of Theorem 2.A were given in [5] and [21] and also an elementary one page proof in [12]. Tanahashi [23] showed that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem 2.A.

Theorem 2.A (Furuta inequality [11]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

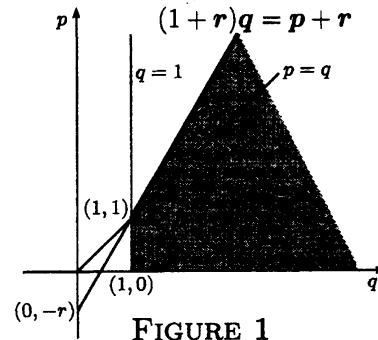


FIGURE 1

Fujii-Nakamoto [10] and the author [18] obtained the results on inclusion relations among the families of class $F(p, r, q)$ and (p, r, q) -paranormality.

Theorem 2.B ([10]).

- (i) *For a fixed $k > 0$, T is k -hyponormal if and only if T belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., T belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$.*

- (ii) If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.
- (iii) If T is (p_0, r_0, q_0) -paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then T is (p_0, r_0, q) -paranormal for any $q \geq q_0$.
- (iv) If T belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then T is (p, r, q) -paranormal.

Theorem 2.C ([18]).

- (i) For each $p > 0$ and $r > 0$,
 - (i-1) T is p -quasihyponormal if and only if T belongs to class $F(p, r, 1)$ if and only if T is $(p, r, 1)$ -paranormal.
 - (i-2) T is p -quasihyponormal if and only if T is $(p, 0, 1)$ -paranormal.
- (ii) Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.
 - (ii-1) If T is invertible and $0 \leq \delta \leq p_0$, then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.
 - (ii-2) If $-r_0 < \delta \leq p_0$, then T belongs to class $F(p_0, r, \frac{p_0+r}{\delta+r})$ for any $r \geq r_0$.
- (iii) Let T be a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ -paranormal operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.
 - (iii-1) If $0 \leq \delta \leq p_0$, then T is $(p, r, \frac{p+r}{\delta+r})$ -paranormal for any $p \geq p_0$ and $r \geq r_0$.
 - (iii-2) If $-r_0 < \delta \leq p_0$, then T is $(p_0, r, \frac{p_0+r}{\delta+r})$ -paranormal for any $r \geq r_0$.
 - (iii-3) If $0 \leq \delta$, then T is $(p, r_0, \frac{p+r_0}{\delta+r_0})$ -paranormal for any $p \geq p_0$.

We remark that only (ii-1) of Theorem 2.C requires invertibility of T , and also we obtained in [19] that every class $A(p_0, r_0)$ operator for $p_0 > 0$ and $r_0 > 0$ belongs to class $A(p, r)$ for any $p \geq p_0$ and $r \geq r_0$ (without assumption of invertibility).

Figure 2 on the following page represents the inclusion relations among the families of class $F(p, r, q)$ and (p, r, q) -paranormality.

On the other hand, we obtained the results on powers of p -hyponormal, class $A(p, r)$ and invertible class $F(p, r, q)$ operators.

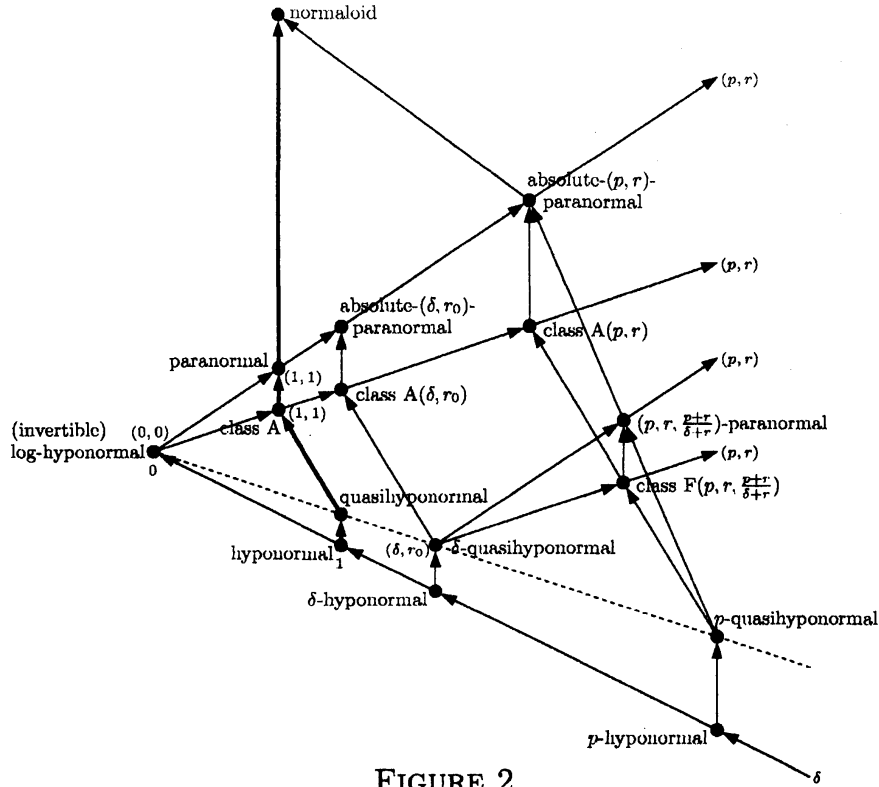


FIGURE 2

Theorem 2.D.

- (i) Let T be a p -hyponormal operator for $0 < p \leq 1$. Then T^n is $\frac{p}{n}$ -hyponormal for all positive integer n ([1]).
- (ii) Let T be a class $A(p, r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then T^n belongs to class $A(\frac{p}{n}, \frac{r}{n})$ for all positive integer n ([19]).
- (iii) Let T be an invertible class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n ([18]).

We remark that (iii) interpolates (i) and (ii) if T is invertible in Theorem 2.D. In fact, (iii) yields (i) by putting $q = 1$ and $r = 0$, and also (iii) yields (ii) by putting $q = \frac{p+r}{r}$.

Moreover we have another result on powers of class A operators by combining [29, Theorem 1] and [19, Theorem 3].

Theorem 2.1. *If T is a class A operator, then*

$$|T|^2 \leq |T^2| \leq \dots \leq |T^n|^{\frac{2}{n}} \quad \text{and} \quad |T^*|^2 \geq |T^{2*}| \geq \dots \geq |T^{n*}|^{\frac{2}{n}}$$

hold for all positive integer n .

We remark that (ii) of Theorem 2.D and Theorem 2.1 in case of invertible operators were shown in [27] and [17], respectively.

3 Main results

In this section, we shall show the results which remove the assumption of invertibility from (ii-1) of Theorem 2.C and (iii) of Theorem 2.D.

Theorem 3.1. *Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $0 \leq \delta \leq p_0$. Then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

Theorem 3.2. *Let T be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n .*

We need the following two results in order to prove Theorem 3.1.

Theorem 3.A ([19, Theorem 1]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$,*

$$(i) \text{ If } (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r, \text{ then } A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

$$(ii) \text{ If } A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}} \text{ and } N(A) \subseteq N(B), \text{ then } (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r.$$

Theorem 3.B ([29]). *If $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for positive operators A and B and fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then*

$$A^\alpha \geq (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta_0}}$$

holds for any $\alpha \geq \alpha_0$. Moreover, for each fixed $\gamma \geq -\beta_0$,

$$g_{\beta_0, \delta}(\alpha) = (B^{\frac{\beta_0}{2}} A^\alpha B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha+\beta_0}}$$

is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence $(B^{\frac{\beta_0}{2}} A^{\alpha_2} B^{\frac{\beta_0}{2}})^{\frac{\alpha_1+\beta_0}{\alpha_2+\beta_0}} \geq B^{\frac{\beta_0}{2}} A^{\alpha_1} B^{\frac{\beta_0}{2}}$ holds for any α_1 and α_2 such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

Proof of Theorem 3.1. In case $r_0 = 0$, it is already shown in (i) of Theorem 2.B since class $F(p_0, 0, \frac{p_0}{\delta})$ for $0 < \delta \leq p_0$ equals δ -hyponormality. So we may assume $r_0 > 0$. Suppose that T belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 > 0$ and $0 \leq \delta \leq p_0$, i.e.,

$$(|T^*|^{r_0} |T|^{2p_0} |T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \geq |T^*|^{2(\delta+r_0)}. \quad (3.1)$$

Applying Löwner-Heinz theorem to (3.1), we have

$$(|T^*|^{r_0}|T|^{2p_0}|T^*|^{r_0})^{\frac{r_0}{p_0+r_0}} \geq |T^*|^{2r_0},$$

and also we have

$$|T|^{2p_0} \geq (|T|^{p_0}|T^*|^{2r_0}|T|^{p_0})^{\frac{p_0}{p_0+r_0}} \quad (3.2)$$

by (i) of Theorem 3.A. By applying Theorem 3.B to (3.2), we obtain that

$$g_{r_0,\delta}(p) = (|T^*|^{r_0}|T|^{2p}|T^*|^{r_0})^{\frac{\delta+r_0}{p+r_0}} \quad (3.3)$$

is an increasing function for $p \geq \max\{p_0, \delta\} = p_0$.

Therefore we have

$$\begin{aligned} (|T^*|^{r_0}|T|^{2p}|T^*|^{r_0})^{\frac{\delta+r_0}{p+r_0}} &= g_{r_0,\delta}(p) \\ &\geq g_{r_0,\delta}(p_0) && \text{by (3.3)} \\ &= (|T^*|^{r_0}|T|^{2p_0}|T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \\ &\geq |T^*|^{2(\delta+r_0)} && \text{by (3.1)} \end{aligned}$$

for any $p \geq p_0$, i.e., T belongs to class $F(p, r_0, \frac{p+r_0}{\delta+r_0})$ for any $p \geq p_0$. Hence T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$ by (ii-2) of Theorem 2.C. $\square$

To prove Theorem 3.2, we prepare the following result which is a slight modification of [29, Lemma 5].

Lemma 3.3. *Let A , B and C be positive operators, $p > 0$, $0 < r \leq 1$ and $q \geq 1$ with $rq \leq p + r \leq (1 + r)q$. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$ and $B \geq C$, then $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{1}{q}} \geq C^{\frac{p+r}{q}}$.*

Proof. The hypothesis $B \geq C$ ensures $B^r \geq C^r$ for $r \in (0, 1]$ by Löwner-Heinz theorem. By Douglas' theorem [4], there exists an operator X such that

$$B^{\frac{r}{2}}X = X^*B^{\frac{r}{2}} = C^{\frac{r}{2}} \quad (3.4)$$

and $\|X\| \leq 1$. Then we have

$$\begin{aligned} (C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{1}{q}} &= (X^*B^{\frac{r}{2}}A^pB^{\frac{r}{2}}X)^{\frac{1}{q}} \\ &\geq X^*(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}}X && \text{by Hansen's inequality [16]} \\ &\geq X^*B^{\frac{p+r}{q}}X && \text{by the hypothesis} \\ &= C^{\frac{r}{2}}B^{\frac{p+r}{q}-r}C^{\frac{r}{2}} && \text{by (3.4) since } \frac{p+r}{q} - r \in [0, 1] \\ &\geq C^{\frac{p+r}{q}} && \text{by Löwner-Heinz theorem.} \end{aligned}$$

Hence the proof is complete. $\square$

Proof of Theorem 3.2. Let T be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$, i.e.,

$$(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \quad (1.1)$$

Class $F(p, r, q)$ operator T for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ belongs to class $F(1, 1, 2)$, i.e., class A by (ii) of Theorem 2.B and Theorem 3.1, and also

$$|T^n|^{\frac{2}{n}} \geq |T|^2 \quad (3.5)$$

and

$$|T^*|^2 \geq |T^{n*}|^{\frac{2}{n}} \quad (3.6)$$

hold for all positive integer n by Theorem 2.1. By applying Lemma 3.3 to (1.1) and (3.6), we have

$$(|T^{n*}|^{\frac{r}{n}} |T|^{2p} |T^{n*}|^{\frac{r}{n}})^{\frac{1}{q}} \geq |T^{n*}|^{\frac{2}{n} \frac{p+r}{q}} \quad (3.7)$$

for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ since $p + r \leq (1 + r)q$ always holds. Hence we obtain

$$\begin{aligned} (|T^{n*}|^{\frac{r}{n}} |T^n|^{\frac{2p}{n}} |T^{n*}|^{\frac{r}{n}})^{\frac{1}{q}} &\geq (|T^{n*}|^{\frac{r}{n}} |T|^{2p} |T^{n*}|^{\frac{r}{n}})^{\frac{1}{q}} \quad \text{by (3.5) and Löwner-Heinz theorem} \\ &\geq |T^{n*}|^{\frac{2}{q}(\frac{p}{n} + \frac{r}{n})} \quad \text{by (3.7)} \end{aligned}$$

for all positive integer n , that is, T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n . $\square$

4 Properness of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$ -paranormality

In this section, we shall show the results on inclusion relation among the families of p -quasihyponormality, class $F(p, r, q)$ and (p, r, q) -paranormality.

Theorem 4.1. *For each $p_0 > 0$, there exists a p_0 -quasihyponormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$ -paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.*

Theorem 4.2. *For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$,*

- (i) *There exists a p_0 -quasihyponormal operator T such that T is not p -quasihyponormal for any $p > 0$ such that $0 < p < p_0$.*

- (ii) *There exists a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.*
- (iii) *There exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ -paranormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$ -paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.*

In Theorem 4.2, (i) has been obtained in [24], and also (ii) and (iii) asserts that the families of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$ -paranormality are proper on p . Moreover we remark that these properness on p has no connection with r , and also we have the following corollary by putting $r = r_0$ in Theorem 4.2.

Corollary 4.3. *For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$,*

- (i) *There exists a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $F(p, r_0, \frac{p+r_0}{\delta+r_0})$ for any $p > 0$ such that $\delta \leq p < p_0$.*
- (ii) *There exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ -paranormal operator T such that T is not $(p, r_0, \frac{p+r_0}{\delta+r_0})$ -paranormal for any $p > 0$ such that $\delta \leq p < p_0$.*

Here we shall show two propositions as a preparation of the proof of Theorem 4.1. We remark that these propositions are similar arguments to [2], [15], [20] and so on.

Firstly we shall give a characterization of (p, r, q) -paranormal operators.

Proposition 4.4. *For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, an operator T is $(p, r, \frac{p+r}{\delta+r})$ -paranormal if and only if*

$$(\delta + r)|T^*|^r|T|^{2p}|T^*|^r - (p + r)\lambda^{p-\delta}|T^*|^{2(\delta+r)} + (p - \delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0.$$

Proof. Suppose that T is $(p, r, \frac{p+r}{\delta+r})$ -paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$, i.e.,

$$\| |T|^p |T^*|^r x \|_{\frac{\delta+r}{p+r}}^{\frac{\delta+r}{p+r}} \geq \| |T^*|^{\delta+r} x \| \quad \text{for every unit vector } x \in H. \quad (1.3)$$

(1.3) holds iff

$$\| |T|^p |T^*|^r x \|_{\frac{\delta+r}{p+r}}^{\frac{\delta+r}{p+r}} \| x \|_{\frac{p-\delta}{p+r}}^{\frac{p-\delta}{p+r}} \geq \| |T^*|^{\delta+r} x \| \quad \text{for all } x \in H$$

iff

$$(|T^*|^r |T|^{2p} |T^*|^r x, x)_{\frac{\delta+r}{p+r}}^{\frac{\delta+r}{p+r}}(x, x)_{\frac{p-\delta}{p+r}}^{\frac{p-\delta}{p+r}} \geq (|T^*|^{2(\delta+r)} x, x) \quad \text{for all } x \in H. \quad (4.1)$$

By arithmetic-geometric mean inequality,

$$\begin{aligned}
& (|T^*|^r |T|^{2p} |T^*|^r x, x)^{\frac{\delta+r}{p+r}} (x, x)^{\frac{p-\delta}{p+r}} \\
&= \left\{ \left(\frac{1}{\lambda} \right)^{p-\delta} (|T^*|^r |T|^{2p} |T^*|^r x, x) \right\}^{\frac{\delta+r}{p+r}} \cdot \left\{ \lambda^{\delta+r} (x, x) \right\}^{\frac{p-\delta}{p+r}} \\
&\leq \frac{\delta+r}{p+r} \frac{1}{\lambda^{p-\delta}} (|T^*|^r |T|^{2p} |T^*|^r x, x) + \frac{p-\delta}{p+r} \lambda^{\delta+r} (x, x)
\end{aligned} \tag{4.2}$$

for all $x \in H$ and all $\lambda > 0$, so (4.1) ensures the following (4.3) by (4.2).

$$\frac{\delta+r}{p+r} \frac{1}{\lambda^{p-\delta}} (|T^*|^r |T|^{2p} |T^*|^r x, x) + \frac{p-\delta}{p+r} \lambda^{\delta+r} (x, x) \geq (|T^*|^{2(\delta+r)} x, x) \tag{4.3}$$

for all $x \in H$ and all $\lambda > 0$.

Conversely, (4.1) follows from (4.3) by putting $\lambda = \left\{ \frac{(|T^*|^r |T|^{2p} |T^*|^r x, x)}{(x, x)} \right\}^{\frac{1}{p+r}}$. (In case $(|T^*|^r |T|^{2p} |T^*|^r x, x) = 0$, let $\lambda \rightarrow +0$.) Hence (4.3) holds if and only if

$$(\delta+r)|T^*|^r |T|^{2p} |T^*|^r - (p+r)\lambda^{p-\delta} |T^*|^{2(\delta+r)} + (p-\delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0,$$

so that the proof is complete. $\square$

Secondly we shall give the following Proposition 4.5. But we omit to describe these calculation because it is obtained by easy calculation.

Proposition 4.5. Let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong H$. For given positive operators A, B on H , define the operator $T_{A,B}$ on K as follows:

$$T_{A,B} = \begin{pmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & B^{\frac{1}{2}} & 0 & & & \\ & & & B^{\frac{1}{2}} & \boxed{0} & & \\ & & & & A^{\frac{1}{2}} & 0 & \\ & & & & & A^{\frac{1}{2}} & 0 \\ & & & & & & \ddots & \ddots \end{pmatrix}, \tag{4.4}$$

where $\boxed{0}$ shows the place of the $(0,0)$ matrix element.

(i) For each $p > 0$, $T_{A,B}$ is p -quasihyponormal if and only if

$$B^{\frac{1}{2}} A^p B^{\frac{1}{2}} \geq B^{p+1}.$$

(ii) For each $p > 0$, $r \geq 0$ and $\delta \geq -r$, $T_{A,B}$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ if and only if

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}.$$

(iii) For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, $T_{A,B}$ is $(p, r, \frac{p+r}{\delta+r})$ -paranormal if and only if

$$(\delta + r)B^{\frac{r}{2}} A^p B^{\frac{r}{2}} - (p + r)\lambda^{p-\delta} B^{\delta+r} + (p - \delta)\lambda^{p+r} I \geq 0 \quad \text{for all } \lambda > 0.$$

Proof of Theorem 4.1. Let

$$A = U \Lambda U^* \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.5)$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} (2 - e^{-p_0})^{\frac{1}{p_0}} & 0 \\ 0 & e^{-2} \end{pmatrix}$,

and also let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong \mathbb{R}^2$. For positive matrices A, B on $\mathbb{R}^2$ given in (4.5), define the operator $T_{A,B}$ on K as (4.4) in Proposition 4.5. By (i) of Proposition 4.5, $T_{A,B}$ is p -quasihyponormal for $p > 0$ if and only if

$$B^{\frac{1}{2}} A^p B^{\frac{1}{2}} - B^{p+1} = \begin{pmatrix} \frac{1}{2} \{ (2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p} \} - 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

if and only if

$$f(p) \equiv \frac{1}{2} \{ (2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p} \} - 1 \geq 0.$$

On the other hand, let $X_p(\lambda)$ as

$$\begin{aligned} X_p(\lambda) &\equiv (\delta + r)B^{\frac{r}{2}} A^p B^{\frac{r}{2}} - (p + r)\lambda^{p-\delta} B^{\delta+r} + (p - \delta)\lambda^{p+r} I \\ &= \begin{pmatrix} \frac{1}{2}(\delta + r) \{ (2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p} \} - (p + r)\lambda^{p-\delta} + (p - \delta)\lambda^{p+r} & 0 \\ 0 & (p - \delta)\lambda^{p+r} \end{pmatrix}. \end{aligned}$$

By (iii) of Proposition 4.5, $T_{A,B}$ is $(p, r, \frac{p+r}{\delta+r})$ -paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$ if and only if $X_p(\lambda) \geq 0$ for all $\lambda > 0$ if and only if

$$g_p(\lambda) \equiv \frac{1}{2}(\delta + r) \{ (2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p} \} - (p + r)\lambda^{p-\delta} + (p - \delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0 \quad (4.6)$$

since $(p - \delta)\lambda^{p+r} \geq 0$ for all $\lambda > 0$. Since $g'_p(\lambda) = (p + r)(p - \delta)\lambda^{p-\delta-1}(-1 + \lambda^{\delta+r})$, we get that

$$\min_{\lambda > 0} g_p(\lambda) = g_p(1) = \frac{1}{2}(\delta + r) \{ (2 - e^{-p_0})^{\frac{p}{p_0}} + e^{-2p} \} - (\delta + r) = (\delta + r)f(p),$$

so that (4.6) holds if and only if $f(p) \geq 0$.

$f(p)$ is a convex function for $p > 0$ since

$$f''(p) = \frac{1}{2}[(2 - e^{-p_0})^{\frac{p}{p_0}} \{\log(2 - e^{-p_0})^{\frac{1}{p_0}}\}^2 + 4e^{-2p}] > 0 \quad \text{for all } p > 0,$$

and also $f(p) = 0$ if $p = 0, p_0$. So we have $f(p_0) = 0$ but $f(p) < 0$ for $0 < p < p_0$. Therefore $g_p(1) < 0$, that is $X_p(1) \not\geq 0$ for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.

Hence $T_{A,B}$ is p_0 -quasihyponormal but non- $(p, r, \frac{p+r}{\delta+r})$ -paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$, so the proof is complete. $\square$

Proof of Theorem 4.2. Let $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$.

Proof of (i). By (i-1) of Theorem 2.C, T is p -quasihyponormal if and only if T is $(p, r, 1)$ -paranormal for some $p > 0$ and $r > 0$. Therefore there exists a p_0 -quasihyponormal operator T such that T is not p -quasihyponormal for any $0 < p < p_0$ by putting $\delta = p$ in Theorem 4.1.

Proof of (ii). By (i-1) of Theorem 2.C and (ii) of Theorem 2.B, every p_0 -quasihyponormal operator belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$. And also, by (iv) of Theorem 2.B, T does not belong to class $F(p, r, \frac{p+r}{\delta+r})$ if T is not $(p, r, \frac{p+r}{\delta+r})$ -paranormal for each $p > 0$, $r > 0$ and $-r < \delta \leq p$. Therefore there exists a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator T such that T does not belong to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$ by Theorem 4.1.

Proof of (iii). By (i-1) of Theorem 2.C and (iii) of Theorem 2.B, every p_0 -quasihyponormal operator is $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ -paranormal. Therefore there exists a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ -paranormal operator T such that T is not $(p, r, \frac{p+r}{\delta+r})$ -paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$ by Theorem 4.1. $\square$

Remark 1. In [15], we introduced two families of classes of operators based on class A and paranormality as follows: An operator T belongs to class $A(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, and also an operator T is absolute- k -paranormal for $k > 0$ if $\| |T|^k T x \| \geq \| T x \|^{k+1}$ for every unit vector $x \in H$. In [7], Fujii-Izumino-Nakamoto introduced p -paranormality for $p > 0$ defined by $\| |T|^p U |T|^p x \| \geq \| |T|^p x \|^2$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T . It was pointed out in [27] that class $A(k)$ equals class $A(k, 1)$, and also it was shown in [28] that absolute- k -paranormality equals absolute- $(k, 1)$ -paranormality and p -paranormality equals absolute- (p, p) -paranormality. We remark that p -paranormality corresponds to class $A(p, p)$. We shall also get the results on inclusion relation among the families of these classes.

Corollary 4.6.

- (i) For each $k_0 > 0$, there exists a class $A(k_0)$ operator T such that T does not belong to class $A(k)$ for any $0 < k < k_0$.
- (ii) For each $k_0 > 0$, there exists an absolute- k_0 -paranormal operator T such that T is not absolute- k -paranormal for any $0 < k < k_0$.
- (iii) For each $p_0 > 0$, there exists a class $A(p_0, p_0)$ operator T such that T is not class $A(p, p)$ for any $0 < p < p_0$.
- (iv) For each $p_0 > 0$, there exists a p_0 -paranormal operator T such that T is not p -paranormal for any $0 < p < p_0$.

Proof of Corollary 4.6.

Proofs of (i) and (ii). By putting $p_0 = k_0$, $r_0 = 1$, $\delta = 0$ and $p = k$ in Corollary 4.3, we have (i) and (ii) since class $A(k)$ equals class $F(k, 1, k + 1)$ and absolute- k -paranormality equals $(k, 1, k + 1)$ -paranormality.

Proofs of (iii) and (iv). By putting $p_0 = r_0$, $\delta = 0$ and $p = r$ in (ii) and (iii) of Theorem 4.2, we have (iii) and (iv) since class $A(p, p)$ equals class $F(p, p, 2)$ and p -paranormality equals $(p, p, 2)$ -paranormality. $\square$

Remark 2. For each $p > 0$, we can obtain an example of non-class $A(p, p)$ and p -paranormal operators by using essentially the same example as [15, (2) of Example 8] as follows: Let $p > 0$ and

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{23} \end{pmatrix}^{\frac{2}{p}} \text{ and } B = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}^{\frac{2}{p}}.$$

Then

$$(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} - B^p = \begin{pmatrix} 0.17472\dots & -3.1798\dots \\ -3.1798\dots & 11.770\dots \end{pmatrix}.$$

Eigenvalues of $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} - B^p$ are $12.585\dots$ and $-0.64001\dots$, so that $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} \not\geq B^p$. So $T_{A,B}$ is a non-class $A(p, p)$ operator by (ii) of Proposition 4.5.

On the other hand, for $\lambda > 0$, define $X(\lambda)$ as follows:

$$X(\lambda) \equiv B^{\frac{p}{2}} A^p B^{\frac{p}{2}} - 2\lambda B^p + \lambda^2 I = \begin{pmatrix} 404 - 26\lambda + \lambda^2 & -576 + 24\lambda \\ -576 + 24\lambda & 844 - 26\lambda + \lambda^2 \end{pmatrix}.$$

Put $p(\lambda) = \text{tr } X(\lambda)$ and $q(\lambda) = \det X(\lambda)$, where $\text{tr } X$ denotes the trace of a matrix X and $\det X$ denotes the determinant of a matrix X . Then

$$\begin{aligned} p(\lambda) &= 2\lambda^2 - 52\lambda + 1248 \\ &= 2(\lambda - 13)^2 + 910 > 0 \end{aligned}$$

$$\begin{aligned} q(\lambda) &= (404 - 26\lambda + \lambda^2)(844 - 26\lambda + \lambda^2) - (-576 + 24\lambda)^2 \\ &= \lambda^4 - 52\lambda^3 + 1348\lambda^2 - 4800\lambda + 9200. \end{aligned}$$

By calculation,

$$\begin{aligned} q'(\lambda) &= 4\lambda^3 - 156\lambda^2 + 2696\lambda - 4800 \\ &= 4(\lambda - 2)(\lambda^2 - 37\lambda + 600) \\ &= 4(\lambda - 2) \left\{ \left(\lambda - \frac{37}{2} \right)^2 + \frac{1031}{4} \right\}. \end{aligned}$$

So $q'(\lambda) = 0$ iff $\lambda = 2$, that is, $q(\lambda) \geq q(2) = 4592 > 0$ for all $\lambda > 0$. Hence $X(\lambda) \geq 0$ for all $\lambda > 0$ since $\text{tr } X(\lambda) = p(\lambda) > 0$ and $\det X(\lambda) = q(\lambda) > 0$ for all $\lambda > 0$. Therefore $T_{A,B}$ is a p -paranormal operator since $T_{A,B}$ is p -paranormal if and only if

$$pB^{\frac{p}{2}}A^pB^{\frac{p}{2}} - 2p\mu^pB^p + p\mu^{2p}I \geq 0 \quad \text{for all } \mu > 0$$

if and only if

$$B^{\frac{p}{2}}A^pB^{\frac{p}{2}} - 2\lambda B^p + \lambda^2I \geq 0 \quad \text{for all } \lambda > 0.$$

by (iii) of Proposition 4.5.

5 Relations between Furuta-type inequalities

In this section, we shall show a generalization of Theorem 3.A which plays an important role in the proofs of the results in Section 3. Here we recall Theorem 3.A.

Theorem 3.A ([19, Theorem 1]). *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$,*

- (i) *If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.*
- (ii) *If $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

For positive invertible operators A and B , it was shown in [13] that

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \quad (5.1)$$

for fixed positive numbers $p \geq 0$ and $r \geq 0$, and Theorem 3.A is a general result for a relation between two inequalities in (5.1). We remark that it was shown in [6] and [13]

(see also [3][8][25]) as an application of Theorem F that for positive invertible operators A and B ,

$$\begin{aligned} \log A \geq \log B &\iff (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ for all } p \geq 0 \text{ and } r \geq 0, \\ &\iff A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq 0. \end{aligned} \quad (5.2)$$

As an extension of (5.2) and an immediate corollary of results on operator-valued functions in [6] and [13], we have that for positive invertible operators A and B ,

$$\begin{aligned} \log A \geq \log B &\iff (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r+\gamma}{p+r}} \geq B^{\frac{r}{2}} A^{\gamma} B^{\frac{r}{2}} \text{ for all } p \geq \gamma \geq 0 \text{ and } r \geq 0, \\ &\iff A^{\frac{p}{2}} B^{\delta} A^{\frac{p}{2}} \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq \delta \geq 0. \end{aligned} \quad (5.3)$$

We remark that inequalities of type of (5.3) were initiated in [21].

Here we shall show a generalization of Theorem 3.A on inequalities in (5.3).

Theorem 5.1. *Let A and B be positive operators. Then the following assertions hold, where S^0 means the projection onto $N(S)^{\perp}$ for a positive operator S :*

(i) *For each $r \geq \delta \geq 0$ and $p \geq 0$,*

$$\begin{aligned} \text{(i-1)} \quad &(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} \geq B^{r-\delta} \text{ ensures } A^{\frac{p}{2}} B^{\delta} A^{\frac{p}{2}} \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}, \\ \text{(i-2)} \quad &A^{\frac{p}{2}} B^{\delta} A^{\frac{p}{2}} \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}} \text{ and } N(AB^{\frac{\delta}{2}}) = N(B) \text{ ensure} \\ &(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} \geq B^{r-\delta}. \end{aligned}$$

(ii) *For each $p \geq \gamma \geq 0$ and $r \geq 0$,*

$$A^{p-\gamma} \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}} \text{ is equivalent to } (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r+\gamma}{p+r}} \geq B^{\frac{r}{2}} A^{\gamma} B^{\frac{r}{2}}.$$

We remark that two inequalities in (i) and (ii) of Theorem 5.1 are mutually equivalent in case A and B are both invertible [22].

We use the following lemma in order to give a proof of Theorem 5.1. Throughout this section, $P_{\mathcal{M}}$ denotes the projection onto a closed subspace $\mathcal{M}$, and also $S^0 = P_{N(S)^{\perp}}$ for a positive operator S .

Lemma 5.2. *Let A and B be positive operators. Then the following assertions hold:*

$$\begin{aligned} \text{(i)} \quad &\lim_{\epsilon \rightarrow +0} A^{\frac{1}{2}} (A + \epsilon I)^{-1} A^{\frac{1}{2}} = \lim_{\epsilon \rightarrow +0} (A + \epsilon I)^{-1} A = P_{N(A)^{\perp}}. \\ \text{(ii)} \quad &\lim_{\epsilon \rightarrow +0} A^{\frac{1}{2}} B^{\frac{1}{2}} \{ (B^{\frac{1}{2}} A B^{\frac{1}{2}})^{\alpha} + \epsilon I \}^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{1-\alpha} \text{ for } \alpha \in (0, 1]. \\ &\text{Particularly, in case } \alpha = 1, \\ &\lim_{\epsilon \rightarrow +0} A^{\frac{1}{2}} B^{\frac{1}{2}} (B^{\frac{1}{2}} A B^{\frac{1}{2}} + \epsilon I)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} = P_{N(B^{\frac{1}{2}} A^{\frac{1}{2}})^{\perp}}. \end{aligned}$$

For positive invertible operators A and B , equivalence between two inequalities in (i) or (ii) of Theorem 5.1 can be easily proved by applying the following Lemma 5.A.

Lemma 5.A ([14]). *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ .

We remark that for non-invertible operators A and B , Lemma 5.A is valid in case $\lambda \geq 1$ but cannot be applied in case $\lambda \in [0, 1)$. For positive invertible operators A and B , Lemma 5.A can be rewritten as

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$$

for any real number α , so that we can regard (ii) of Lemma 5.2 as a non-invertible version of Lemma 5.A for $\alpha \in (0, 1]$.

Proof of Lemma 5.2. (i) is well known and a proof was given in [19], for example.

Proof of (ii). Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition. For $\alpha \in (0, 1]$, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow +0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^\alpha + \varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \\ &= \lim_{\varepsilon \rightarrow +0} U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^\alpha(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2\alpha} + \varepsilon I)^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^\alpha|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^* \\ &= U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}P_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|)^\perp}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^* \quad \text{by (i)} \\ &= U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2(1-\alpha)}U^* = |B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2(1-\alpha)} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}. \end{aligned}$$

We remark that in case $\alpha = 1$ particularly,

$$U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^0U^* = UP_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|)^\perp}U^* = UU^*UU^* = UU^* = P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})^\perp} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^0.$$

Hence the proof is complete. $\square$

Proof of Theorem 5.1.

Proof of (i). Let $r > \delta \geq 0$ since the case $r = \delta$ is obvious. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} \geq B^{r-\delta}$, then

$$A^{\frac{p}{2}}B^{\frac{\delta}{2}}B^{\frac{r-\delta}{2}}(B^{r-\delta} + \varepsilon I)^{-1}B^{\frac{r-\delta}{2}}B^{\frac{\delta}{2}}A^{\frac{p}{2}} \geq A^{\frac{p}{2}}B^{\frac{r}{2}}\{(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} + \varepsilon I\}^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}$$

for $\varepsilon > 0$, so that

$$A^{\frac{p}{2}}B^\delta A^{\frac{p}{2}} = A^{\frac{p}{2}}B^{\frac{\delta}{2}}P_{N(B)^\perp}B^{\frac{\delta}{2}}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}$$

by tending $\varepsilon \rightarrow +0$ and Lemma 5.2, hence we obtain (i-1). On the other hand, if $A^{\frac{p}{2}}B^\delta A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}}$, then

$$B^{\frac{r}{2}}A^{\frac{p}{2}}\{(A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{\delta+p}{p+r}} + \varepsilon I\}^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}} \geq B^{\frac{r-\delta}{2}}B^{\frac{\delta}{2}}A^{\frac{p}{2}}(A^{\frac{p}{2}}B^\delta A^{\frac{p}{2}} + \varepsilon I)^{-1}A^{\frac{p}{2}}B^{\frac{\delta}{2}}B^{\frac{r-\delta}{2}}$$

for $\varepsilon > 0$, so that

$$\begin{aligned} (B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{r-\delta}{p+r}} &\geq B^{\frac{r-\delta}{2}}P_{N(A^{\frac{p}{2}}B^{\frac{\delta}{2}})^{\perp}}B^{\frac{r-\delta}{2}} \quad \text{by tending } \varepsilon \rightarrow +0 \text{ and (ii) of Lemma 5.2} \\ &= B^{\frac{r-\delta}{2}}P_{N(B)^{\perp}}B^{\frac{r-\delta}{2}} \quad \text{by } N(AB^{\frac{\delta}{2}}) = N(B) \\ &= B^{r-\delta}, \end{aligned}$$

hence we obtain (i-2).

Proof of (ii). Let $p > \gamma \geq 0$ since the case $p = \gamma$ is obvious. If $A^{p-\gamma} \geq (A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}}$, then

$$B^{\frac{r}{2}}A^{\frac{p}{2}}\{(A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}} + \varepsilon I\}^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}} \geq B^{\frac{r}{2}}A^{\frac{\gamma}{2}}A^{\frac{p-\gamma}{2}}(A^{p-\gamma} + \varepsilon I)^{-1}A^{\frac{p-\gamma}{2}}A^{\frac{\gamma}{2}}B^{\frac{r}{2}}$$

for $\varepsilon > 0$, so that

$$(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{\frac{r}{2}}A^{\frac{\gamma}{2}}P_{N(A)^{\perp}}A^{\frac{\gamma}{2}}B^{\frac{r}{2}} = B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}}$$

by tending $\varepsilon \rightarrow +0$ and Lemma 5.2, hence we obtain ($\Rightarrow$). On the other hand, if $(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}}$, then

$$A^{\frac{p-\gamma}{2}}A^{\frac{\gamma}{2}}B^{\frac{r}{2}}(B^{\frac{r}{2}}A^{\gamma}B^{\frac{r}{2}} + \varepsilon I)^{-1}B^{\frac{r}{2}}A^{\frac{\gamma}{2}}A^{\frac{p-\gamma}{2}} \geq A^{\frac{p}{2}}B^{\frac{r}{2}}\{(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} + \varepsilon I\}^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}$$

for $\varepsilon > 0$, so that

$$A^{p-\gamma} \geq A^{\frac{p-\gamma}{2}}P_{N(B^{\frac{r}{2}}A^{\frac{\gamma}{2}})^{\perp}}A^{\frac{p-\gamma}{2}} \geq (A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{p-\gamma}{p+r}}$$

by tending $\varepsilon \rightarrow +0$ and (ii) of Lemma 5.2, hence we obtain ($\Leftarrow$). $\square$

Theorem 3.A can be obtained as a corollary of Theorem 5.1 as follows.

Alternative proof of Theorem 3.A. Put $\delta = 0$ in (i-1) of Theorem 5.1, then $(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ ensures

$$A^p \geq A^{\frac{p}{2}}P_{N(B)^{\perp}}A^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{p}{p+r}},$$

hence we obtain (i). On the other hand, put $\gamma = 0$ in (ii) of Theorem 5.1, then $A^p \geq (A^{\frac{p}{2}}B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ ensures

$$(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{\frac{r}{2}}P_{N(A)^{\perp}}B^{\frac{r}{2}} \geq B^{\frac{r}{2}}P_{N(B)^{\perp}}B^{\frac{r}{2}} = B^r$$

since $N(A) \subseteq N(B)$ is equivalent to $P_{N(A)^{\perp}} \geq P_{N(B)^{\perp}}$, hence we obtain (ii). $\square$

References

- [1] A.Aluthge and D.Wang, *Powers of p -hyponormal operators*, J. Inequal. Appl., **3** (1999), 279–284.
- [2] T.Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged), **33** (1972), 169–178.
- [3] T.Ando, *On some operator inequalities*, Math. Ann., **279** (1987), 157–159.
- [4] R.G.Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.
- [5] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.
- [6] M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl., **179** (1993), 161–169.
- [7] M.Fujii, S.Izumino and R.Nakamoto, *Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality*, Nihonkai Math. J., **5** (1994), 61–67.
- [8] M.Fujii, J.F.Jiang and E.Kamei, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., **125** (1997), 3655–3658.
- [9] M.Fujii, D.Jung, S.H.Lee, M.Y.Lee and R.Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japon., **51** (2000), 395–402.
- [10] M.Fujii and R.Nakamoto, *Some classes of operators derived from Furuta inequality*, Sci. Math., **3** (2000), 87–94.
- [11] T.Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$* , Proc. Amer. Math. Soc., **101** (1987), 85–88.
- [12] T.Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci., **65** (1989), 126.
- [13] T.Furuta, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl., **59** (1992), 180–190.
- [14] T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219** (1995), 139–155.
- [15] T.Furuta, M.Ito and T.Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math., **1** (1998), 389–403.
- [16] F.Hansen, *An operator inequality*, Math. Ann. **246** (1979/80), 249–250.
- [17] M.Ito, *Several properties on class A including p -hyponormal and log-hyponormal operators*, Math. Inequal. Appl., **2** (1999), 569–578.

- [18] M.Ito, *On some classes of operators by Fujii and Nakamoto related to p -hyponormal and paranormal operators*, Sci. Math., **3** (2000), 319–334.
- [19] M.Ito and T.Yamazaki, *Relations between two inequalities $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications*, Integral Equations Operator Theory, **44** (2002), 442–450.
- [20] D.Jung, M.Y.Lee and S.H.Lee, *On classes of operators related to paranormal operators*, Sci. Math. Jpn., **53** (2001), 33–43.
- [21] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886.
- [22] E.Kamei, *Chaotic order and Furuta inequality*, Sci. Math. Jpn., **53** (2001), 289–293.
- [23] K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146.
- [24] A.Uchiyama, *An example of a p -quasihyponormal operator*, Yokohama Math. J., **46** (1999), 179–180.
- [25] M.Uchiyama, *Some exponential operator inequalities*, Math. Inequal. Appl., **2** (1999), 469–471.
- [26] M.Uchiyama, *Inequalities for semi bounded operators and their applications to log-hyponormal operators*, Oper. Theory Adv. Appl., **127** (2001), 599–611.
- [27] T.Yamazaki, *On powers of class $A(k)$ operators including p -hyponormal and log-hyponormal operators*, Math. Inequal. Appl., **3** (2000), 97–104.
- [28] T.Yamazaki and M.Yanagida, *A further generalization of paranormal operators*, Sci. Math., **3** (2000), 23–32.
- [29] M.Yanagida, *Powers of class $wA(s, t)$ operators associated with generalized Aluthge transformation*, J. Inequal. Appl., **7** (2002), 143–168.