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An exact sequence of Grothendieck-Witt rings
(Grothendieck-Witt 環の完全系列)

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1. Introduction

Throughout this paper let G denote a finite group, Θ a finite G-set, and R a commutative ring with multiplicative unit.

C. B. Thomas [13] defined the Hermitian representation ring $\mathrm{G}_1(R, G)$ and showed that the Wall group $\mathrm{L}_n(\mathbb{Z}[G], w)$ is a module over $\mathrm{G}_1(\mathbb{Z}, G)$, providing the orientation homomorphism w is trivial. A. Dress defined the Grothendieck-Witt rings $\mathrm{GW}_0(R, G)$ and $\mathrm{GW}(G, R)$ in [7, p. 742] and [8, p. 294], respectively (cf. [12, p. 2356]) as quotient rings of $\mathrm{G}_1(R, G)$. By [8, Theorem 5], we can see that the canonical epimorphism $\mathrm{GW}(G, \mathbb{Z}) \to \mathrm{GW}_0(\mathbb{Z}, G)$ is actually an isomorphism. For the induction theory of equivariant surgery obstruction groups, the authors have defined in [2, Section 2] the (generalized) Grothendieck-Witt ring $\mathrm{GW}_0(R, G, \Theta)$. Details of the induction theory of equivariant surgery obstruction groups are described in [12] and [10]. Applications to equivariant surgery are given in [11, Section 6] and [4]. Let $\mathfrak{S}_2$ denote the group of order 2 with generator τ. Give the cartesian product $\Theta \times \Theta$ the diagonal G-action and the $\mathfrak{S}_2$-action:

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$(\tau, (x, y)) \mapsto (y, x)$ for $x, y \in \Theta$. Let $\mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R)$ denote the ring of all $G \times \mathfrak{S}_2$-maps from $\Theta \times \Theta$ to R, where R has the trivial $G \times \mathfrak{S}_2$-action. The goal of this article is to prove the following theorem.

Theorem 1. *Let R be a principal ideal domain. Then the sequence of canonical homomorphisms*

$$0 \longrightarrow \mathrm{GW}_0(R, G) \longrightarrow \mathrm{GW}_0(R, G, \Theta) \longrightarrow \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R) \longrightarrow 0$$

is split exact.

We remark that $\mathrm{GW}_0(R, G)$ and $\mathrm{GW}_0(R, G, \Theta)$ are rings with multiplicative unit and the canonical homomorphism $\mathrm{GW}_0(R, G) \to \mathrm{GW}_0(R, G, \Theta)$ preserves multiplication, but not the multiplicative unit. The R-rank of $\mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R)$ was computed by Mitsuaki Kubo in his Master thesis for $G = A_5$ and by XianMeng Ju [9] for $G = \mathrm{SL}(2, 5)$.

The definitions of the Grothendieck-Witt rings above are recalled in Section 2 for the reader's convenience. Theorem 1 is proved in Section 3.

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2. Definition of the Grothendieck-Witt rings

In this section we recall the definitions of the Grothendieck-Witt rings used in the current paper and the canonical homomorphisms

$$\mathrm{GW}_0(R, G) \longrightarrow \mathrm{GW}_0(R, G, \Theta) \longrightarrow \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta \times \Theta, R).$$

The reader can refer to Section 4 of [12] for details.

A *Hermitian $R[G]$-module* is a pair (M, B) consisting of a finitely generated R-projective $R[G]$-module M and a symmetric G-invariant R-bilinear map $B : M \times M \to R$. The map B is called *nonsingular* if the associated map $M \to \mathrm{Hom}_R(M, R)$; $x \mapsto B(x, -)$, is a bijection. A Θ-*positioned Hermitian* $R[G]$-*module* is a triple (M, B, α) consisting of a Hermitian module (M, B) and a G-map $\alpha : \Theta \to M$. If B is nonsingular then (M, B) and (M, B, α) are also called *nonsingular*. The G-map α is called *trivial* (resp. *totally isotropic*) if $\alpha(t) = 0$ for all $t \in \Theta$ (resp. $B(\alpha(t), \alpha(t')) = 0$ for all t, $t' \in \Theta$). Let $\mathcal{H}(R, G, \Theta)$ denote the category of all nonsingular Θ-positioned Hermitian $R[G]$-modules (M, B, α), where the morphisms $(M, B, \alpha) \to (M', B', \alpha')$ are isomorphisms $f : M \to M'$ such that $B'(f(x), f(y)) = B(x, y)$ for all x, $y \in M$ and the diagram

$$\begin{tikzcd} \Theta \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & M \arrow[d, "f"] \\ & M' \end{tikzcd}$$

commutes. Let $\mathcal{H}(R, G, \Theta)^{\mathrm{triv}}$ (resp. $\mathcal{H}(R, G, \Theta)^{\mathrm{t-iso}}$) denote the full subcategory of $\mathcal{H}(R, G, \Theta)$ consisting of all $(M, B, \alpha) \in \mathcal{H}(R, G, \Theta)$ such that α is trivial (resp. totally isotropic).

The orthogonal sum

$$(M, B, \alpha) \oplus (M', B', \alpha'), \quad (= (M'', B'', \alpha'') \text{ say})$$

of (M, B, α), $(M', B', \alpha') \in \mathcal{H}(R, G, \Theta)$ is defined by $M'' = M \oplus M'$, $B''((x, x'), (y, y')) = B(x, y) + B'(x', y')$ for x, $y \in M$ and x', $y' \in M'$, and $\alpha''(t) = (\alpha(t), \alpha'(t))$ for $t \in \Theta$. The tensor product

$$(M, B, \alpha) \otimes (M', B', \alpha'), \quad (= (M'', B'', \alpha'') \text{ say})$$

is defined by $M'' = M \otimes M'$, $B''(x \otimes x', y \otimes y') = B(x, y)B'(x', y')$ for x, $y \in M$ and x', $y' \in M'$, and $\alpha''(t) = \alpha(t) \otimes \alpha'(t)$ for $t \in \Theta$. $\mathcal{H}(R, G, \Theta)^{\mathrm{triv}}$ and $\mathcal{H}(R, G, \Theta)^{\mathrm{t-iso}}$ are closed under orthogonal sum as well as tensor product. Let $\mathrm{KH}_0(R, G, \Theta)$ (resp. $\mathrm{KH}_0(R, G, \Theta)^{\mathrm{triv}}$, $\mathrm{KH}_0(R, G, \Theta)^{\mathrm{t-iso}}$) denote the Grothendieck group of the category $\mathcal{H}(R, G, \Theta)$ (resp. $\mathcal{H}(R, G, \Theta)^{\mathrm{triv}}$, $\mathcal{H}(R, G, \Theta)^{\mathrm{t-iso}}$) with respect to orthogonal sum.

Let $(M, B, \alpha) \in \mathcal{H}(R, G, \Theta)$. An $R[G]$-submodule U of M is called a *Quillen submodule* of (M, B, α) if U is an R-direct summand of M such that $B(U, U) = 0$ and $\alpha(\Theta) \subseteq U$. In this case, $((M, B, \alpha), U)$ is called a *Quillen pair*. For any $(M, B, \alpha) \in \mathcal{H}(R, G, \Theta)$,

$$\Delta M = \{(x, x) \in M \oplus M \mid x \in M\}$$

is a Quillen submodule of

$$(M,B,\alpha)\oplus(M,-B,\alpha).$$

If $((M,B,\alpha),U)$ is a Quillen pair, we obtain $(U^\perp/U,B^\perp,\alpha_0)\in\mathcal{H}(R,G,\Theta)$ where

$$U^\perp=\{y\in M\mid B(x,y)=0\ \forall x\in U\}$$

$$B^\perp(x+U,y+U)=B(x,y)\ \text{ for } x,\ y\in U^\perp$$

$$\alpha_0(t)=0+U\in U^\perp/U\ \text{ for } t\in\Theta.$$

Define the *Grothendieck-Witt group* (which will be also referred to as the *Grothendieck-Witt ring*)

$$\mathrm{GW}_0(R,G,\Theta)\ \ (\text{resp. } \mathrm{GW}_0(R,G,\Theta)^{\mathrm{triv}},\ \mathrm{GW}_0(R,G,\Theta)^{\mathrm{t-iso}})$$

by

$$\mathrm{GW}_0(R,G,\Theta)=\mathrm{KH}_0(R,G,\Theta)/\langle(M,B,\alpha)-(U^\perp/U,B^\perp,\alpha_0)\rangle$$

$$(\textit{resp. } \mathrm{GW}_0(R,G,\Theta)^{\mathrm{triv}}=\mathrm{KH}_0(R,G,\Theta)^{\mathrm{triv}}/\langle(M,B,\alpha)-(U^\perp/U,B^\perp,\alpha_0)\rangle,$$

$$\mathrm{GW}_0(R,G,\Theta)^{\mathrm{t-iso}}=\mathrm{KH}_0(R,G,\Theta)^{\mathrm{t-iso}}/\langle(M,B,\alpha)-(U^\perp/U,B^\perp,\alpha_0)\rangle)$$

where $((M,B,\alpha),U)$ runs over all Quillen pairs in $\mathcal{H}(R,G,\Theta)$ (resp. $\mathcal{H}(R,G,\Theta)^{\mathrm{triv}}$, $\mathcal{H}(R,G,\Theta)^{\mathrm{t-iso}}$). Note that

$$[M,-B,\alpha]=-[M,B,\alpha]$$

in $\mathrm{GW}_0(R,G,\Theta)$. $\mathrm{GW}_0(R,G,\Theta)$, $\mathrm{GW}_0(R,G,\Theta)^{\mathrm{triv}}$ and $\mathrm{GW}_0(R,G,\Theta)^{\mathrm{t-iso}}$ are commutative rings and the first two have multiplicative units. The Grothendieck-Witt ring $\mathrm{GW}_0(R,G)$ of A. Dress is obtained as $\mathrm{GW}_0(R,G,\emptyset)$. By definition, there are canonical homomorphisms

$$\mathrm{GW}_0(R,G)\longrightarrow\mathrm{GW}_0(R,G,\Theta)^{\mathrm{triv}}\longrightarrow\mathrm{GW}_0(R,G,\Theta)^{\mathrm{t-iso}}\longrightarrow\mathrm{GW}_0(R,G,\Theta)$$

and the first homomorphism is an isomorphism. In addition, we have a canonical retraction

$$\mathrm{GW}_0(R,G,\Theta)\to\mathrm{GW}_0(R,G);\ [M,B,\alpha]\mapsto[M,B].$$

We define the homomorphism

$$\kappa:\mathrm{GW}_0(R,G,\Theta)\to\mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta\times\Theta,R)$$

by

$$\kappa([M,B,\alpha])(t,t') = B(\alpha(t),\alpha(t')) \ \text{ for } t,\, t' \in \Theta.$$

3. Proof of Theorem 1

We have already proved the exactness of the sequence

$$0 \longrightarrow \mathrm{GW}_0(R,G) \longrightarrow \mathrm{GW}_0(R,G,\Theta) \xrightarrow{\kappa} \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta\times\Theta,R)$$

in Proposition 2.1 of [2]. Thus, in order to prove Theorem 1, it suffices to show the homomorphism κ splits.

Let $f:\Theta\times\Theta\to R$ be a $G\times\mathfrak{S}_2$-map. We assign a Θ-positioned Hermitian $R[G]$-module (M,B,α) to f as follows. Let Θ' be a copy of the G-set Θ. For each element $x\in\Theta$, let x' stand for the copy in Θ' of x. Let M be the free R-module with basis $\Theta \amalg \Theta'$, namely $M = R[\Theta]\oplus R[\Theta']$. Let $B: M\times M\to R$ be the R-bilinear map satisfying $B(x,y)=f(x,y)$, $B(x,y')=\delta_{x,y}$, $B(x',y)=\delta_{x,y}$ and $B(x',y')=0$ for all x, $y\in\Theta$, where

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Since f is G-equivariant and symmetric, B is G-invariant and symmetric. Clearly, B is nonsingular. Define $\alpha:\Theta\to M$ by $\alpha(x)=(x,0)\in R[\Theta]\oplus R[\Theta']$ for $x\in\Theta$. Obviously, α is a G-map. The assignment $f\mapsto[M,B,\alpha]$ defines a homomorphism

$$\sigma: \mathrm{Map}_{G\times\mathfrak{S}_2}(\Theta\times\Theta,R)\to \mathrm{GW}_0(R,G,\Theta).$$

Since

$$\kappa([M,B,\alpha])(x,y) = B(\alpha(x),\alpha(y)) = B((x,0),(y,0)) = f(x,y),$$

the homomorphism σ is a splitting of κ.

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