

SURJECTIVE ISOMETRIES ON $C^1[0, 1]$ WITH RESPECT TO SEVERAL NORMS

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ABSTRACT. Let $C^1[0, 1]$ be a complex linear space of all continuously differentiable complex valued functions on the unit interval $[0, 1]$. We give a characterization of surjective, not necessarily linear, isometries on $C^1[0, 1]$ with respect to the following norms: $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$, $\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0, 1]\}$ and $\|f\|_\sigma = |f(0)| + \|f'\|_\infty$ for $f \in C^1[0, 1]$, respectively.

1. INTRODUCTION

Let M and N be real or complex normed linear spaces with norms $\|\cdot\|_M$ and $\|\cdot\|_N$, respectively. We say that a mapping $T: M \rightarrow N$ is an *isometry* if and only if

$$\|T(a) - T(b)\|_N = \|a - b\|_M \quad (a, b \in M).$$

It should be emphasized that we never assume linearity of isometries unless otherwise stated. Let X be a compact Hausdorff space and $C(X)$ the Banach space of all continuous complex valued functions on X with the supremum norm $\|\cdot\|_\infty$. Denote by $C_{\mathbb{R}}(X)$ the real Banach space of all continuous real valued functions on X . Banach [1, Theorem 3 in Chapter XI] proved that if $T: C_{\mathbb{R}}(X) \rightarrow C_{\mathbb{R}}(Y)$ is a surjective isometry and if X and Y are compact metric spaces, then there exist a continuous function $u: Y \rightarrow \{\pm 1\}$ and a homeomorphism $\varphi: Y \rightarrow X$ such that $T(f)(y) = T(0)(y) + u(y)f(\varphi(y))$ for all $f \in C_{\mathbb{R}}(X)$ and $y \in Y$. Stone [18, Theorem 83] generalized the result by Banach for compact Hausdorff spaces X and Y . On the other hand, the so-called Banach-Stone theorem states that if $T: C(X) \rightarrow C(Y)$ is a surjective *complex linear* isometry, then there exist a continuous function $u: Y \rightarrow \mathbb{C}$ with $|u(y)| = 1$ for $y \in Y$ and a homeomorphism $\varphi: Y \rightarrow X$ such that $T(f)(y) = u(y)f(\varphi(y))$ for all $f \in C(X)$ and $y \in Y$.

Let $C^1[0, 1]$ be the Banach space of all continuously differentiable complex valued functions on the unit interval $[0, 1]$ with the norm $\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0, 1]\}$ for $f \in C^1[0, 1]$. Cambern [4, Theorem 1.5] gave a characterization for surjective complex linear isometries from $C^1[0, 1]$ onto itself; to be more explicit, if $T: C^1[0, 1] \rightarrow C^1[0, 1]$ is a surjective *complex linear* isometry, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $T(f)(t) = cf(t)$ for all $f \in C^1[0, 1]$ and $t \in [0, 1]$, or $T(f)(t) = cf(1-t)$ for all $f \in C^1[0, 1]$ and $t \in [0, 1]$. The result by Cambern has been extended in various directions; Pathak [16, Theorem 2.5] described surjective complex linear isometries between the Banach space of

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all n times continuously differentiable functions. Rao and Roy [17, Theorem 4.1] considered surjective complex linear isometries on $C^1[0, 1]$ with the norm $\|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0, 1]$. Jarosz and Pathak [9, Theorem 3] gave a scheme to verify that surjective complex linear isometries are given by homeomorphisms. Botelho and Jamison [2, Theorem 3.5] investigated surjective complex linear isometries between $C^1([0, 1], E)$, where E denotes a finite dimensional Hilbert space. We refer the reader to [6, 7] for a survey of the study of isometries on various function spaces.

The purpose of this paper is to describe surjective isometries on $C^1[0, 1]$ without assuming linearity of maps. In fact, the following is the main theorem of this paper, which extends the result by Rao and Roy [17, Theorem 4.1]:

2. MAIN RESULTS

Theorem 2.1. *Let $T: C^1[0, 1] \rightarrow C^1[0, 1]$ be a surjective isometry, which need not be linear, with respect to the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$. Then there exists a constant $c \in \mathbb{C}$ with $|c| = 1$ such that*

$$\begin{aligned} T(f)(t) &= T(0)(t) + cf(t) & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T(f)(t) &= T(0)(t) + cf(1-t) & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T(f)(t) &= T(0)(t) + \overline{cf(t)} & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T(f)(t) &= T(0)(t) + \overline{cf(1-t)} & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), \end{aligned}$$

where $\bar{\cdot}$ denotes the complex conjugate.

Conversely, each of the above maps is a surjective isometry on $C^1[0, 1]$ with respect to $\|\cdot\|_\Sigma$, where $T(0)$ is an arbitrary element of $C^1[0, 1]$.

The following result is a special case of [2, Theorem 3.5] by Botelho and Jamison; in fact, they consider surjective linear isometries on $C^1([0, 1], H)$ with respect to the norm $\sup\{\|f(t)\|_H + \|f'(t)\|_H : t \in [0, 1]\}$, where H denotes a finite dimensional Hilbert space. We can identify $C^1[0, 1]$ with $C^1([0, 1], \mathbb{R}^2)$. If T_0 is a surjective real linear isometry on $C^1[0, 1]$, then we may regard T_0 as a surjective linear isometry on $C^1([0, 1], \mathbb{R}^2)$. Thus, T_0 is characterized by [2, Theorem 3.5]. On the other hand, we can prove the following result as a corollary to Theorem 2.1.

Corollary 2.2. *Let $T: C^1[0, 1] \rightarrow C^1[0, 1]$ be a surjective isometry, which need not be linear, with respect to the norm $\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0, 1]\}$. Then there exists a constant $c \in \mathbb{C}$ with $|c| = 1$ such that*

$$\begin{aligned} T(f)(t) &= T(0)(t) + cf(t) & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T(f)(t) &= T(0)(t) + cf(1-t) & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T(f)(t) &= T(0)(t) + \overline{cf(t)} & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T(f)(t) &= T(0)(t) + \overline{cf(1-t)} & (\forall f \in C^1[0, 1], \forall t \in [0, 1]). \end{aligned}$$

Conversely, each of the above maps is a surjective isometry on $C^1[0, 1]$ with respect to $\|\cdot\|_C$, where $T(0)$ is an arbitrary element of $C^1[0, 1]$.

Theorem 2.3. Let $T: C^1[0, 1] \rightarrow C^1[0, 1]$ be a surjective isometry, which need not be linear, with respect to the norm $\|f\|_\sigma = |f(0)| + \|f'\|_\infty$. Then there exist a constant $c \in \mathbb{C}$ with $|c| = 1$, a continuous unimodular function $\beta: [0, 1] \rightarrow \mathbb{C}$ and a homeomorphism $\rho: [0, 1] \rightarrow [0, 1]$ such that

$$\begin{aligned} T_0(f)(t) &= cf(0) + \int_0^t \beta(s) f'(\rho(s)) ds & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T_0(f)(t) &= c\overline{f(0)} + \int_0^t \beta(s) f'(\rho(s)) ds & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T_0(f)(t) &= cf(0) + \int_0^t \beta(s) \overline{f'(\rho(s))} ds & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), & \text{ or} \\ T_0(f)(t) &= c\overline{f(0)} + \int_0^t \beta(s) \overline{f'(\rho(s))} ds & (\forall f \in C^1[0, 1], \forall t \in [0, 1]), \end{aligned}$$

where $T_0(f)(t) = T(f)(t) - T(0)(t)$.

Conversely, each of the above maps is a surjective isometry on $C^1[0, 1]$ with respect to $\|\cdot\|_\sigma$, where $T(0)$ is an arbitrary element of $C^1[0, 1]$.

A key of proofs of the main results is a significant result related to isometries proven by Mazur and Ulam. The Mazur-Ulam theorem [13] states that if T is a surjective isometry between normed linear spaces, then $T - T(0)$ is real linear; consequently $T - T(0)$ is a surjective, *real linear* isometry. Väisälä [19] gave a simple proof of the Mazur-Ulam theorem. Theorem 2.1 states that surjective real linear isometry $T - T(0)$ on $C^1[0, 1]$ is the same as complex linear one up to the complex conjugate; similar results were proven for function algebras [5, 8, 14] and for function spaces under additional assumptions [12]. On the other hand, real linear isometries are quite different from complex linear ones in general; such an elementary example is given in [12, Example 6.2]. A characterization is obtained in [15] in order that surjective real linear isometries on function spaces with respect to the supremum norm be of the canonical form, that is, a combination of weighted composition operators and the complex conjugate. Surjective, non-canonical isometries are investigated in [10].

Let $C^1[0, 1]$ be the Banach space of all continuously differentiable complex valued functions on the unit interval $[0, 1]$ with respect to the following norms:

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty, \quad \|f\|_\sigma = |f(0)| + \|f'\|_\infty \quad \text{and}$$

$$\|f\|_C = \sup\{|f(t)| + |f'(t)| : t \in [0, 1]\}$$

for $f \in C^1[0, 1]$, where $\|\cdot\|_\infty$ denotes the supremum norm on $[0, 1]$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$, and set $X_\Sigma = [0, 1] \times [0, 1] \times \mathbb{T}$,

$$X_\sigma = \{(r, s, z) \in X_\Sigma : r = 0\} \quad \text{and} \quad X_c = \{(r, s, z) \in X_\Sigma : r = s\}$$

with the product topology. Define

$$(1) \quad \tilde{f}(r, s, z) = f(r) + zf'(s)$$

for $f \in C^1[0, 1]$ and $(r, s, z) \in X_\Sigma$; thus $\tilde{f}(r, s, z) = f(0) + zf'(s)$ if $(r, s, z) \in X_\sigma$, and $\tilde{f}(r, s, z) = f(s) + zf'(s)$ if $(r, s, z) \in X_c$. The function $\tilde{f}$ is continuous on X_Σ . Let $C(K)$ be the Banach space of all continuous complex valued functions on a compact Hausdorff space K with respect to the supremum norm $\|\cdot\|_\infty$. We define $A_\Sigma = \{\tilde{f} \in C(X_\Sigma) : f \in C^1[0, 1]\}$, $A_\sigma = A_\Sigma|_{X_\sigma}$ and $A_c = A_\Sigma|_{X_c}$. Let $(A, X) \in \{(A_\Sigma, X_\Sigma), (A_\sigma, X_\sigma), (A_c, X_c)\}$. Then A is a normed linear subspace of $C(X)$. Let $\mathbf{1} \in C^1[0, 1]$ be the constant function such that $\mathbf{1}(t) = 1$ for all $t \in [0, 1]$. By (1), we see that A has constant function $\tilde{\mathbf{1}}$. Notice that A separates points of X in the sense that for each pair of distinct points $x_1, x_2 \in X$ there exists $\tilde{f} \in A$ such that $\tilde{f}(x_1) \neq \tilde{f}(x_2)$. The correspondence $f \mapsto \tilde{f}$ is a complex linear isometry from $(C^1[0, 1], \|\cdot\|)$ onto $(A, \|\cdot\|_\infty)$; where, $\|f\| = \|f\|_\Sigma$ if $A = A_\Sigma$, $\|f\| = \|f\|_\sigma$ if $A = A_\sigma$ and $\|f\| = \|f\|_c$ if $A = A_c$. Note that $\tilde{if} = i\tilde{f}$ for $f \in C^1[0, 1]$. We denote by A^* the complex dual space of $(A, \|\cdot\|_\infty)$. Let $\delta_x: A \rightarrow \mathbb{C}$ be the point evaluation defined as $\delta_x(\tilde{f}) = \tilde{f}(x)$ for $\tilde{f} \in A$ and $x \in X$. We see that the set of all extreme points of the unit ball of A^* is $\{\lambda\delta_x : x \in X, \lambda \in \mathbb{T}\}$.

Let $T: C^1[0, 1] \rightarrow C^1[0, 1]$ be a surjective isometry. Define a mapping $T_0: C^1[0, 1] \rightarrow C^1[0, 1]$ as $T_0 = T - T(0)$. By the Mazur-Ulam theorem, T_0 is a surjective, *real linear* isometry from $C^1[0, 1]$ onto itself. We define $S: A \rightarrow A$ as

$$(2) \quad \begin{array}{ccc} S(\tilde{f}) = \widetilde{T_0(f)} & (\tilde{f} \in A). \\ C^1[0, 1] & \xrightarrow{T_0} & C^1[0, 1] \\ \downarrow \tilde{\cdot} & & \downarrow \tilde{\cdot} \\ A & \xrightarrow{S} & A \end{array}$$

Since $f \mapsto \tilde{f}$ is a surjective isometry from $C^1[0, 1]$ onto A , it is a bijection, and thus S is well defined. As $f \mapsto \tilde{f}$ is a surjective complex linear isometry, S is a surjective real linear isometry on A . We define a mapping $S_*: A^* \rightarrow A^*$ as

$$(3) \quad S_*(\eta)(\tilde{f}) = \operatorname{Re} \eta(S(\tilde{f})) - i \operatorname{Re} \eta(S(i\tilde{f}))$$

for $\eta \in A^*$ and $\tilde{f} \in A$. It is routine to check that the mapping S_* is a surjective real linear isometry with respect to the operator norm on A^* (cf. [15, Proposition 1]).

Proof of Theorem 2.1, Corollary 2.2 and Theorem 2.3 are given in [11]. In fact, Kawamura, Koshimizu and the author of this paper generalize these results.

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