

ON A STABILITY OF HEAT KERNEL ESTIMATES UNDER FEYNMAN-KAC PERTURBATIONS FOR DIFFUSION PROCESSES

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1. PRELIMINARIES

This note is an intermediate report of [14] for diffusion cases. Let (E, d) be a locally compact complete separable metric space. We assume that E is connected and unbounded, and any closed ball of (E, d) is compact. Let $\mathbf{m}$ be a Radon measure with full support. We write $B_r(x) = \{y \in E \mid d(x, y) < r\}$ and $V(x, r) := \mathbf{m}(B_r(x))$. We consider a strongly local irreducible regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; \mathbf{m})$ and let $\mathbf{X} = (\Omega, X_t, \mathbf{P}_x)_{x \in E}$ be an $\mathbf{m}$ -symmetric diffusion process associated to $(\mathcal{E}, \mathcal{F})$ (see [6] for details). We always assume that $\mathbf{X}$ admits a jointly continuous heat kernel $p_t(x, y)$.

For $\alpha > 0$, we define the α -order resolvent kernel

$$R_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y) dt, \quad x, y \in E.$$

When the process $\mathbf{X}$ is transient, we can define 0-order resolvent kernel $R(x, y) := R_0(x, y) < \infty$ for $x, y \in E$ with $x \neq y$. $R(x, y)$ is called the Green kernel of $\mathbf{X}$. For a non-negative Borel measure ν , we write $R_\alpha \nu(x) := \int_E R_\alpha(x, y) \nu(dy)$, $R\nu(x) := R_0 \nu(x)$ and $R_\alpha f(x) = R_\alpha \nu(x)$ when $\nu(dx) = f(x)dx$ for any $f \in \mathcal{B}_+(E)$ or $f \in \mathcal{B}_b(E)$. Here $\mathcal{B}_+(E)$ (resp. $\mathcal{B}_b(E)$) denotes the space of non-negative (resp. bounded) Borel functions on E . The space of bounded continuous functions on E will be denoted as $C_b(E)$. Let E_∂ be the one point compactification of E . An increasing sequence $\{F_k\}$ of closed sets is said to be a *strict $\mathcal{E}$ -nest* if $\mathbf{P}_x(\lim_{k \rightarrow \infty} \sigma_{F_k^c} = \infty) = 1$ $\mathbf{m}$ -a.e. $x \in E$. Here $\sigma_{F_k^c} := \inf\{t > 0 \mid X_t \in E \setminus F_k\}$ is the first hitting time of X_t to $F_k^c := E \setminus F_k$. A function f defined on E (resp. E_∂) is said to be *$\mathcal{E}$ -quasi continuous* (resp. *strictly $\mathcal{E}$ -quasi continuous*) if there exists an $\mathcal{E}$ -nest (resp. a strict $\mathcal{E}$ -nest) $\{F_k\}$ of closed sets such that $f|_{F_k}$ (resp. $f|_{F_k \cup \{\partial\}}$) is continuous for each $k \in \mathbb{N}$. Since $\mathbf{X}$ is conservative, any $\mathcal{E}$ -nest $\{F_k\}$ of closed sets is automatically a strict $\mathcal{E}$ -nest. The regularity of the given Dirichlet form $(\mathcal{E}, \mathcal{F})$ for $\mathbf{X}$ tells us that there exists an $\mathcal{E}$ -nest of compact sets ([18, Chapter V, Proposition 2.12]). Denote by $QC(E_\partial)$ the family of all strictly $\mathcal{E}$ -quasi continuous functions on E_∂ . Let $(\mathcal{E}, \mathcal{F}_e)$ be the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$ and any element $f \in \mathcal{F}_e$ admits a strictly $\mathcal{E}$ -quasi continuous version $\tilde{f}$ with $\tilde{f}(\partial) = 0$ (the proof is similar to the proof of [6, Theorem 2.1.3] if $\mathbf{X}$ is transient, for general case, it can be proved by way of time change method). Throughout this paper, we always take a strictly $\mathcal{E}$ -quasi continuous version of the element of $\mathcal{F}_e$, that is, we omit *tilde* from $\tilde{f}$ for $f \in \mathcal{F}_e$.

Let $S_1(\mathbf{X})$ be the family of positive smooth measures in the strict sense ([6]). A measure $\nu \in S_1(\mathbf{X})$ is said to be of *Dynkin class* (resp. *Green-bounded*) with respect to $\mathbf{X}$ if $\sup_{x \in E} R_\beta \nu(x) < \infty$ for some $\beta > 0$ (resp. $\sup_{x \in E} R\nu(x) < \infty$). A measure $\nu \in S_1(\mathbf{X})$ is said to be of *Kato class* (resp. of *extended Kato class*) with respect to $\mathbf{X}$ if $\lim_{\beta \rightarrow \infty} \sup_{x \in E} R_\beta \nu(x) = 0$ (resp. $\lim_{\beta \rightarrow \infty} \sup_{x \in E} R_\beta \nu(x) < 1$). Denote by $S_D^1(\mathbf{X})$ (resp. $S_{D_0}^1(\mathbf{X})$) the family of measures of Dynkin class (resp. of Green-bounded) and by $S_K^1(\mathbf{X})$ (resp. $S_{EK}^1(\mathbf{X})$) the family of measures of Kato class (resp. of extended Kato class). Clearly, $S_K^1(\mathbf{X}) \subset S_{EK}^1(\mathbf{X}) \subset S_D^1(\mathbf{X})$ and $S_{D_0}^1(\mathbf{X}) \subset S_D^1(\mathbf{X})$. Note that any measure $\nu \in S_D^1(\mathbf{X})$ is a positive Radon measure in view of the Stollmann-Voigt's inequality: $\int_E u^2 d\nu \leq \|R_\beta \nu\|_\infty \mathcal{E}_\beta(u, u)$, $u \in \mathcal{F}, \beta \geq 0$ ([19, Theorem 3.1]). Conversely, any positive Radon measure ν satisfying $\sup_{x \in E} R_\alpha \nu(x) < \infty$ for some $\alpha > 0$ always belongs to $S_1(\mathbf{X})$ in view of [17, Proposition 3.1].

We say that a positive continuous additive functional (PCAF in abbreviation) in the strict sense A^ν of $\mathbf{X}$ and a positive measure $\nu \in S_1(\mathbf{X})$ are in the Revuz correspondence if they satisfy for any bounded $f \in \mathcal{B}_+(E)$,

$$\int_E f(x) \nu(dx) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \int_E \mathbf{E}_x \left[\int_0^t f(X_s) dA_s^\nu \right] m(dx).$$

It is known that the family of equivalence classes of the set of PCAFs in the strict sense and the family of positive measures belonging to $S_1(\mathbf{X})$ are in one to one correspondence under the Revuz correspondence ([6, Theorem 5.1.4]).

A function f on E is said to be *locally in $\mathcal{F}$ in the broad sense* (denoted as $f \in \dot{\mathcal{F}}_{\text{loc}}$) if there is an increasing sequence of finely open Borel sets $\{E_n\}$ with $\bigcup_{n=1}^\infty E_n = E$ q.e. and for every $n \geq 1$, there is $f_n \in \mathcal{F}$ such that $f = f_n$ m-a.e. on E_n . A function f on E is said to be *locally in $\mathcal{F}$ in the ordinary sense* (denoted as $f \in \mathcal{F}_{\text{loc}}$) if for any relatively compact open set G , there exists an element $f_G \in \mathcal{F}$ such that $f = f_G$ m-a.e. on G . Clearly $\mathcal{F}_{\text{loc}} \subset \dot{\mathcal{F}}_{\text{loc}}$. It is shown in [16, Theorem 4.1], $\mathcal{F}_e \subset \dot{\mathcal{F}}_{\text{loc}}$.

Take a bounded $u \in \dot{\mathcal{F}}_{\text{loc}} \cap QC(E_\partial)$. In [15, Theorem 6.2(1)], the authors proved under the condition $\mu_{\langle u \rangle} \in S_D^1(\mathbf{X})$ that the additive functional $u(X_t) - u(X_0)$ admits the following decomposition:

$$(1.1) \quad u(X_t) - u(X_0) = M_t^u + N_t^u \quad t \in [0, +\infty[\quad \mathbf{P}_x\text{-a.s.}$$

for q.e. $x \in E$, where M^u is a square integrable martingale additive functional, $\mu_{\langle u \rangle}$ is the Revuz measure associated with the quadratic variational processes (or the sharp bracket PCAF) $\langle M^u \rangle$ of M^u , and N^u is a continuous additive functional (CAF in abbreviation) locally of zero energy. We moreover note that (1.1) holds for all $x \in E$ as the strict decomposition provided u is (nearly) Borel and finely continuous on E ([15, Theorem 6.2(2)]). Note that N^u is not a process of finite variation in general. Note that $\mathcal{E}(f, f) = \frac{1}{2} \mu_{\langle f \rangle}(E)$ provided $f \in \mathcal{F}_e$.

2. GREEN-TIGHT MEASURES OF KATO CLASS

We introduce some notions of Green-tight Kato class measures in the strict sense.

Definition 2.1 (Green-tight Kato class measures). Let $\nu \in S_1(\mathbf{X})$.

- (1) ν is said to be of *Green-tight Kato class with respect to $\mathbf{X}$* if $\nu \in S_K^1(\mathbf{X})$ and for any $\varepsilon > 0$ there exists a compact subset $K = K(\varepsilon)$ of E such that

$$\sup_{x \in E} R(\mathbf{1}_{K^c} \nu)(x) < \varepsilon.$$

- (2) ν is said to be of *semi-Green-tight extended Kato class with respect to $\mathbf{X}$* if $\nu \in S_{EK}^1(\mathbf{X})$ and there exists a compact subset K of E such that

$$\sup_{x \in E} R(\mathbf{1}_{K^c} \nu)(x) < 1.$$

- (3) ν is said to be of *Green-tight Kato class in the sense of Chen with respect to $\mathbf{X}$* if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x \in E} R(\mathbf{1}_{K^c \cup B} \nu)(x) < \varepsilon.$$

- (4) ν is said to be of *semi-Green-tight extended Kato class in the sense of Chen with respect to $\mathbf{X}$* if there exists a Borel subset K of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x \in E} R(\mathbf{1}_{K^c \cup B} \nu)(x) < 1.$$

We denote by $S_{K_\infty}^1(\mathbf{X})$ (resp. $S_{CK_\infty}^1(\mathbf{X})$, $S_{K_1}^1(\mathbf{X})$, $S_{CK_1}^1(\mathbf{X})$) the family of Green-tight Kato class measures (resp. the family of Green-tight Kato class measures in the sense of Chen, the family of semi-Green-tight extended Kato class measures, the family of semi-Green-tight Kato class measures in the sense of Chen) with respect to $\mathbf{X}$.

Remark 2.2. It is known that $S_{CK_\infty}^1(\mathbf{X}) \subset S_{CK_1}^1(\mathbf{X}) \subset S_{D_0}^1(\mathbf{X}) \cap S_{EK}^1(\mathbf{X})$. Moreover, $S_{CK_\infty}^1(\mathbf{X}) \subset S_{K_\infty}^1(\mathbf{X}) \subset S_K^1(\mathbf{X})$ and $S_{CK_1}^1(\mathbf{X}) \subset S_{K_1}^1(\mathbf{X}) \subset S_{EK}^1(\mathbf{X})$ hold in general (see [3],[11]). However, we have $S_{K_\infty}^1(\mathbf{X}) = S_{CK_\infty}^1(\mathbf{X})$ and $S_{CK_1}^1(\mathbf{X}) \cap S_K^1(\mathbf{X}) = S_{K_1}^1(\mathbf{X}) \cap S_K^1(\mathbf{X})$ provided $\mathbf{X}$ has resolvent Feller property (see [11, Lemma 4.1]).

In order to introduce the new (semi-)Green-tight measures of (extended) Kato class, we explain the notion of weighted capacity of the Dirichlet form associated with the time changed process:

Let $\nu \in S_1(\mathbf{X})$ and denote by A_t^ν the PCAF in the strict sense associated to ν in Revuz correspondence. Denote by S_o^ν the support of A^ν defined by $S_o^\nu := \{x \in E \mid \mathbf{P}_x(R = 0) = 1\}$, where $R(\omega) := \inf\{t > 0 \mid A_t^\nu(\omega) > 0\}$. S_o^ν is nothing but the fine support of ν , i.e., the topological support of ν with respect to the fine topology of $\mathbf{X}$. Let $(\tilde{\mathbf{X}}, \nu)$ be the time changed process of $\mathbf{X}$ by A_t^ν and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ the associated Dirichlet form on $L^2(S^\nu; \nu)$, where S^ν is the support of ν . It is known that $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a regular Dirichlet form having $\mathcal{C}|_{S^\nu}$ as its core and $S^\nu \setminus S_o^\nu$ is $\tilde{\mathcal{E}}$ -polar, i.e., 1-capacity 0 set with respect to $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. The life time of $(\tilde{\mathbf{X}}, \nu)$ is given by A_ζ^ν . Let $C^\nu : 2^E \rightarrow [0, +\infty]$ be the weighted 1-capacity with respect to $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$.

Now we introduce a new class of (semi-)Green-tight measures of (extended) Kato class.

Definition 2.3 (Natural (semi-)Green-tight measures of (extended) Kato class). Let $\alpha \geq 0$ and $\nu \in S_1(\mathbf{X})$.

- (1) ν is said to be an α -order natural Green-tight measure of Kato class with respect to $\mathbf{X}$ if $\nu \in S_D^1(\mathbf{X})$ ($\nu \in S_{D_0}^1(\mathbf{X})$ for $\alpha = 0$) and for any $\varepsilon > 0$ there exists a closed subset $K = K(\varepsilon)$ of E and a constant $\delta > 0$ such that for all Borel set $B \subset K$ with $C^\nu(B) < \delta$,

$$\sup_{x \in E} \mathbf{E}_x \left[\int_0^{\tau_{B \cup K^c}} e^{-\alpha t} dA_t^\nu \right] < \varepsilon.$$

- (2) ν is said to be a 0-order natural semi-Green-tight measure of extended Kato class with respect to $\mathbf{X}$ if $\nu \in S_{D_0}^1(\mathbf{X})$ and there exists a closed subset K of E and a constant $\delta > 0$ such that for all Borel set $B \subset K$ with $C^\nu(B) < \delta$,

$$\sup_{x \in E} \mathbf{E}_x [A_{\tau_{B \cup K^c}}^\nu] < 1.$$

In view of the resolvent equation, for positive α , the α -order natural Green-tightness is independent of the choice of $\alpha > 0$. Let denote by $S_{NK_\infty}^1(\mathbf{X})$ the family of positive order natural Green-tight measures of Kato class with respect to $\mathbf{X}$. The class $S_{NK_\infty}^1(\mathbf{X})$ (resp. $S_{NK_1}^1(\mathbf{X})$) is then denoted as the family of 0-order natural Green-tight measures of Kato class (resp. the family of 0-order natural semi-Green-tight measures of extended Kato class) with respect to $\mathbf{X}$.

Remark 2.4.

- (1) It is proved in [11, Lemma 4.4] that $S_{CK_1}^1(\mathbf{X}) \subset S_{NK_1}^1(\mathbf{X}) \subset S_{EK}^1(\mathbf{X}) \cap S_{D_0}^1(\mathbf{X})$ and $S_{CK_\infty}^1(\mathbf{X}) \subset S_{NK_\infty}^1(\mathbf{X}) \subset S_K^1(\mathbf{X}) \cap S_{D_0}^1(\mathbf{X})$.
- (2) The advantage of the new semi-Green-tight measures of extended Kato class is that $S_{NK_1}^1(\mathbf{X})$ is stable under some Girsanov transformation (see Corollaries 5.1 and 5.2 in [11]).

Definition 2.5. Let $R^z(x, y)$ be the Green kernel of Doob's $R(\cdot, z)$ -transformed process $\mathbf{X}^z$ of $\mathbf{X}$ defined by

$$R^z(x, y) := \frac{R(x, y)R(y, z)}{R(x, z)}, \quad x, y \in E \setminus \{z\} \text{ with } x \neq y$$

and $R^z\nu(x) := \int_E R^z(x, y)\nu(dy)$.

- (1) A measure $\nu \in S_1(\mathbf{X})$ is said to be *conditionally Green-bounded in the sense of Chen with respect to $\mathbf{X}$* if

$$\sup_{(x, z) \in E \times E, x \neq z} R^z\nu(x) < \infty.$$

- (2) A measure $\nu \in S_1(\mathbf{X})$ is said to be *of conditionally Green-tight Kato class in the sense of Chen with respect to $\mathbf{X}$* if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{(x, z) \in E \times E, x \neq z} R^z(\mathbf{1}_{K^c \cup B}\nu)(x) < \varepsilon.$$

- (3) A measure $\nu \in S_1(\mathbf{X})$ is said to be *of conditionally semi-Green-tight extended Kato class in the sense of Chen with respect to $\mathbf{X}$* if there exists a Borel subset K of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{(x,z) \in E \times E, x \neq z} R^z(\mathbf{1}_{K^c \cup B} \nu)(x) < 1.$$

Let denote by $S_{CS_\infty}^1(\mathbf{X})$ (resp. $S_{CS_1}^1(\mathbf{X})$, $S_{DS_0}^1(\mathbf{X})$) the family of conditionally Green-tight Kato class measures (resp. the family of conditionally semi-Green-tight extended Kato class measures, the family of conditionally Green-bounded measures) in the sense of Chen. It is known in general that $S_{CS_\infty}^1(\mathbf{X}) \subset S_{CS_1}^1(\mathbf{X}) \subset S_{DS_0}^1(\mathbf{X})$, $S_{CS_\infty}^1(\mathbf{X}) \subset S_{CK_\infty}^1(\mathbf{X})$, $S_{CS_1}^1(\mathbf{X}) \subset S_{CK_1}^1(\mathbf{X})$ and similarly $S_{DS_0}^1(\mathbf{X}) \subset S_{D_0}^1(\mathbf{X})$ (cf. [3, 4]).

3. RESULTS

Throughout this note, we assume that Ψ is a fixed continuous increasing bijection on $]0, +\infty[$ satisfying that for all $0 < r < R$,

$$(3.1) \quad C_\Psi^{-1} \left(\frac{R}{r} \right)^\beta \leq \frac{\Psi(R)}{\Psi(r)} \leq C_\Psi \left(\frac{R}{r} \right)^{\beta'}$$

for some $1 < \beta \leq \beta'$ and $C_\Psi \geq 1$. We consider the following condition: there exists large $L > 0$ such that

$$(3.2) \quad \text{ess-sup}_{s \geq L} \frac{s\Psi'(s)}{\Psi(s)} < \infty.$$

Example 3.1. Take $\beta_1, \beta_2 \in]1, +\infty[$ and set

$$\Psi(s) := \begin{cases} s^{\beta_1}, & s \in [0, 1], \\ s^{\beta_2}, & s \in [1, +\infty[. \end{cases}$$

Then (3.1) and (3.2) for some large $L > 0$ are satisfied.

We further assume the *volume doubling condition* (VD) (see [8, Definition 1.1]): there exists a constant $C_D > 0$ such that

$$V(x, 2r) \leq C_D V(x, r) \quad \text{for all } x \in E, \quad r \in]0, +\infty[.$$

We set

$$\Phi(s, t) := \sup_{r > 0} \left\{ \frac{s}{r} - \frac{t}{\Psi(r)} \right\}.$$

Definition 3.2. For $c \in]0, 1]$, we say that $(E, d, \mathbf{m}, \mathbf{X})$ satisfies $(UE)_\Psi^{c,*}$ if the heat kernel $p_t(x, y)$ of $\mathbf{X}$ exists and satisfies the following upper estimate

$$(3.3) \quad p_t(x, y) \leq \frac{C e^{kt}}{V(x, \Psi^{-1}(t))} \exp \left(-\frac{1}{2} \Phi(cd(x, y), t) \right)$$

for all $t > 0$ and $\mathbf{m}$ -a.e. $x, y \in E$, where $C > 0$ and $k \geq 0$ are constants independent of x, y, t .

For $c \in [1, +\infty[$, we say that $(E, d, m, \mathbf{X})$ satisfies $(LE)_{\Psi}^{c,*}$ if the heat kernel $p_t(x, y)$ of $\mathbf{X}$ exists and satisfies the following lower estimate

$$(3.4) \quad p_t(x, y) \geq \frac{Ce^{-kt}}{V(x, \Psi^{-1}(t))} \exp(-c\Phi(cd(x, y), t))$$

for all $t > 0$ and m -a.e. $x, y \in E$, where $C > 0$ and $k \geq 0$ are constants independent of x, y, t .

We say that $(E, d, m, \mathbf{X})$ satisfies $(UE)_{\Psi}^*$ (resp. $(LE)_{\Psi}^*$) if it satisfies $(UE)_{\Psi}^{c,*}$ (resp. $(LE)_{\Psi}^{c,*}$) for some $c \in]0, 1]$ (resp. $c \in [1, +\infty[$). In particular, we say that $(E, d, m, \mathbf{X})$ satisfies $(UE)_{\Psi}$ (resp. $(LE)_{\Psi}$) if it satisfies $(UE)_{\Psi}^*$ (resp. $(LE)_{\Psi}^*$) with $k = 0$.

Remark 3.3.

- (1) Clearly, $\Phi(s, t) = t\Phi(s/t, 1)$. If $\Psi(r) = Cr^{\beta}$ with some $C > 0$ and $\beta > 1$, then $\Phi(s, 1) = cs^{\beta/(\beta-1)}$. Consequently, under (3.1), we always have $\Phi(s, 1) \geq cs^{\beta/(\beta-1)}$ for some $c > 0$.
- (2) It is known (cf. [2],[7]) that $(UE)_{\Psi} + (LE)_{\Psi}$ implies that the heat kernel $p_t(x, y)$ admits a locally Hölder continuous in x, y version, so that (3.3) is a posteriori true for all $x, y \in E$.
- (3) By [8, Theorem 1.17], $(UE)_{\Psi} + (LE)_{\Psi}$ is stable under bounded perturbations, that is, if two strongly local regular Dirichlet forms $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ having common domain $\mathcal{F}^{(1)} = \mathcal{F}^{(2)}$, satisfy $C^{-1}\mathcal{E}^{(1)} \leq \mathcal{E}^{(2)} \leq C\mathcal{E}^{(1)}$ on $\mathcal{F}^{(1)} \times \mathcal{F}^{(1)}$ and $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ admits $(UE)_{\Psi} + (LE)_{\Psi}$ for some positive constants, then $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ also does for some constants.
- (4) By [8, Theorem 1.17], $(UE)_{\Psi} + (LE)_{\Psi}$ is equivalent to $(G)_{\Psi}$ (see [8] for $(G)_{\Psi}$). In this case, under the transience of $\mathbf{X}$, we know the existence of global Green kernel $R(x, y)$ and it satisfies that there exist $\kappa \in]0, 1[$ and $C \in]0, +\infty[$ such that for $x, y \in E$ with $x \neq y$

$$C^{-1} \int_{\kappa d(x, y)}^{\infty} \frac{\Psi(s)ds}{sV(x, s)} \leq R(x, y) \leq C \int_{\kappa d(x, y)}^{\infty} \frac{\Psi(s)ds}{sV(x, s)}.$$

For a bounded finely continuous function $u \in \dot{\mathcal{F}}_{\text{loc}} \cap QC(E_{\partial})$ satisfying $\mu_{\langle u \rangle} \in S_D^1(\mathbf{X})$, we consider the transforms by the additive functionals $A_t := N_t^u + A_t^{\mu}$ of the form

$$(3.5) \quad e_A(t) := \exp(A_t), \quad t \geq 0,$$

where N_t^u is the continuous additive functional of zero quadratic variation appeared in a Fukushima decomposition of $u(X_t) - u(X_0)$, (see (1.1)), A_t^{μ} is the continuous additive functional of $\mathbf{X}$ with a signed smooth measure $\mu := \mu_1 - \mu_2$ in the strict sense as its Revuz measure. Note that N^u is not necessarily of bounded variation. The transform (3.5) defines a semigroup, namely, the generalized Feynman-Kac semigroup

$$(3.6) \quad P_t^A f(x) := \mathbf{E}_x[e_A(t)f(X_t)], \quad f \in \mathcal{B}_b(E), \quad t \geq 0.$$

The purpose of this note is to give the analytic condition on u and μ under which $p_t^A(x, y)$ on $]0, +\infty[\times E \times E$ also satisfies $(UE)_{\Psi} + (LE)_{\Psi}$.

Define the quadratic form $(Q, \mathcal{F})$ by

$$(3.7) \quad Q(f, g) := \mathcal{E}(f, g) + \frac{1}{2} \int_E f(x) \mu_{\langle u, g \rangle}(dx) + \frac{1}{2} \int_E g(x) \mu_{\langle u, f \rangle}(dx) - \int_E f g d\mu.$$

Then it is well-defined for $f, g \in \mathcal{F}$ provided $\mu_{\langle u \rangle} \in S_D^1(\mathbf{X})$, $\mu_1 + \mu_2 \in S_D^1(\mathbf{X})$. Moreover, if $\mathbf{X}$ is transient, Q is extended to $\mathcal{F}_e \times \mathcal{F}_e$ with the same expression (3.7) provided $\mu_{\langle u \rangle} \in S_{D_0}^1(\mathbf{X})$, $\mu_1 + \mu_2 \in S_{D_0}^1(\mathbf{X})$. The L^2 -generator $\mathcal{L}^Q$ associated to $(Q, D(Q))$ can be expressed in the formal form $\mathcal{L}^Q := \mathcal{L}^\varepsilon + \mathcal{L}^\varepsilon u + \mu$, where $\mathcal{L}^\varepsilon$ is the infinitesimal generator for the semigroup of $\mathbf{X}$.

Under $\mu_{\langle u \rangle} + \mu_1 + \mu_2 \in S_D^1(\mathbf{X})$ (resp. $\mu_{\langle u \rangle} + \mu_1 + \mu_2 \in S_{D_0}^1(\mathbf{X})$), we set for $\alpha > 0$ (resp. $\alpha = 0$)

$$(3.8) \quad \lambda^{Q_\alpha}(\bar{\mu}_1) := \inf \left\{ Q_\alpha(f, f) \mid f \in \mathcal{C}, \int_E f^2 d\bar{\mu}_1 = 1 \right\},$$

where $\bar{\mu}_1 := \frac{1}{2} \mu_{\langle u \rangle} + \mu_1$. Let $\lambda^{Q_0}(\bar{\mu}_1) := \lambda^{Q_\alpha}(\bar{\mu}_1)$.

Our main theorem is the following:

Theorem 3.4. *Assume that $\mathbf{X}$ is transient. Suppose that the heat kernel $p_t(x, y)$ satisfies $(UE)_\Psi + (LE)_\Psi$ for all $x, y \in E$. Let $u \in \mathcal{F}_{\text{loc}} \cap QC(E_\partial)$ be a bounded finely continuous (nearly) Borel function on E . Assume $\mu_1 \in S_{NK_1}^1(\mathbf{X})$, $\mu_{\langle u \rangle} \in S_{NK_\infty}^1(\mathbf{X})$ and $\mu_2 \in S_{D_0}^1(\mathbf{X})$. Then we have the following:*

- (1) $\lambda^{Q_\alpha}(\bar{\mu}_1) > 0$ implies that the integral kernel $p_t^A(x, y)$ satisfies $(UE)_\Psi + (LE)_\Psi$ for all $x, y \in E$.
- (2) Suppose that $\mu_1 \in S_{CS_1}^1(\mathbf{X})$, $\mu_{\langle u \rangle} \in S_{CS_\infty}^1(\mathbf{X})$ and $\mu_2 \in S_{DS_0}^1(\mathbf{X})$ hold. If $p_t^A(x, y)$ satisfies $(UE)_\Psi$ for all $x, y \in E$ and (3.2) holds for some $L > 0$, then $\lambda^{Q_\alpha}(\bar{\mu}_1) > 0$.

Next corollary gives a stability of the short time estimates for integral kernel $p_t^A(x, y)$ without assuming the transience of $\mathbf{X}$. Denote by $\mathbf{X}^{(\alpha)}$ the α -subprocess killed at rate αm .

Corollary 3.5. *Suppose that $p_t(x, y)$ satisfies $(UE)_\Psi + (LE)_\Psi$ for all $x, y \in E$. Let $u \in \mathcal{F}_{\text{loc}} \cap QC(E_\partial)$ be a bounded finely continuous (nearly) Borel function on E . Assume $\mu_1 \in S_{NK_1}^1(\mathbf{X}^{(\alpha)})$, $\mu_{\langle u \rangle} \in S_{NK_\infty}^1(\mathbf{X})$ and $\mu_2 \in S_D^1(\mathbf{X})$.*

Then we have the following:

- (1) $\lambda^{Q_\alpha}(\bar{\mu}_1) > 0$ implies that the integral kernel $p_t^A(x, y)$ satisfies $(UE)_\Psi^* + (LE)_\Psi^*$ for all $x, y \in E$.
- (2) Suppose that $\mu_1 \in S_{CS_1}^1(\mathbf{X}^{(\alpha)})$, $\mu_{\langle u \rangle} \in S_{CS_\infty}^1(\mathbf{X}^{(\alpha)})$ and $\mu_2 \in S_{DS_0}^1(\mathbf{X}^{(\alpha)})$ hold. If $p_t^A(x, y)$ satisfies $(UE)_\Psi^*$ for all $x, y \in E$ and $\alpha > k$, then $\lambda^{Q_\alpha}(\bar{\mu}_1) > 0$. Here k is the constant appeared in $(UE)_\Psi^*$.

Our result, in particular Theorem 3.4(1),(2), extend the result on the stability of Li-Yau estimates for the heat kernel of Riemannian manifold proved by Takeda [20]. One of the main progress in our result is to add the perturbation by continuous additive functional of locally of zero energy. The global integral kernel estimate under such perturbations was firstly done by Glover-Rao-Song [9, Theorem 2.9] (see

also Glover-Rao-Šikić-Song [10, Proposition 1.4]) in the framework of Brownian motion. But the stability of global full integral kernel estimates have not been treated in this direction. Indeed, the global estimates shown in [9, Theorem 2.9] is weaker than the heat kernel of Brownian motion, because of the lack of Green-tightness of Kato class measures.

As a corollary of Corollary 3.5(1), we have the following theorem:

Theorem 3.6. *Let $u \in \mathcal{F}_{\text{loc}} \cap QC(E_\partial)$ be a bounded finely continuous (nearly) Borel function on E . Assume $\mu_1 \in S_{EK}^1(\mathbf{X})$, $\mu_{\langle u \rangle} \in S_K^1(\mathbf{X})$ and $\mu_2 \in S_D^1(\mathbf{X})$. Then, $p_t^A(x, y)$ satisfies $(UE)_\Psi^* + (LE)_\Psi^*$ (resp. $(UE)_\Psi^* + (LE)_\Psi^*$) for all $x, y \in E$ provided $p_t(x, y)$ satisfies $(UE)_\Psi + (LE)_\Psi$ (resp. $(UE)_\Psi^* + (LE)_\Psi^*$) for all $x, y \in E$.*

Theorem 3.6 also extends the previous known results on the integral kernel estimates under the perturbation by measures of Kato classes and provides a stability of the short time estimates for integral kernel. The conditions for measures in Theorem 3.6 are very mild comparing the known results.

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