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# The statement AN is equivalent to the statement $n(\beta\omega\setminus\omega) > c$

Shizuo Kamo (Univ. of Osaka Prefecture)

1. Introduction and results. A filter $\mathfrak{F}$ on $\omega$ is said to be ample, if there is an infinite subset $a$ of $\omega$ such that, whenever $x \in \mathfrak{F}$, $a \setminus x$ is finite. A filter $\mathfrak{F}$ on $\omega$ is said to be weakly ample, if for each free ultrafilter (uf) $\mathcal{U}$ on $\omega$, there is a function f on $\omega$ such that $f(\mathcal{U}) \supset \mathfrak{F}$. Let us denote by AN the statement: "every free weakly ample filter on $\omega$ is ample." In [2], we showed

PROPOSITION 1.

(i) AN implies the existence of $c^+$ Ramsey ufs on $\omega$, where c denotes the cardinality of $2^\omega$.

(ii) The existence of $c^+$ Ramsey ufs on $\omega$ does not imply AN.

(iii) Ⓟ implies AN, where Ⓟ denotes the statement: "every free filter on $\omega$ generated by a set cardinality less than c is ample."

It seems to be interesting to consider how strong the statement AN is. As to this, we first show

THEOREM 1. AN is equivalent to the statement that $\beta\omega\setminus\omega$ can not be covered by a family of c nowhere dense sets, where $\beta\omega$ denotes the Čech-Stone compactification of $\omega$.

Let us denote by $n(\beta\omega\setminus\omega)$ the Baire number of $\beta\omega\setminus\omega$ (i.e. the minimal cardinal of a family of nowhere dense sets covering $\beta\omega\setminus\omega$). As to the Baire number of $\beta\omega\setminus\omega$, the systematical estimation was given and several consistencies were shown in [1]. In [1], it is shown

PROPOSITION 2 (5.2.V in [1]) The statement $n(\beta\omega\setminus\omega) > c$ does not imply that $\forall\kappa < c\ (\ |2^{\kappa}| \leq c\ )$.

Since Ⓟ implies that $\forall\kappa < c\ (\ |2^{\kappa}| \leq c\ )$ ([3]), by Theorem 1 and Proposition 2, AN does not imply Ⓟ.

Define the pseudo-ordering $<^*$ on $\omega^{\omega}$ by $f <^* g$ iff $\lim_{n\to\infty}(\ g(n) - f(n)\ ) = \infty$. A family F of subsets of $\omega^{\omega}$ is said to be unbounded, if there does not exist $g \in \omega^{\omega}$ such that, whenever $f \in F$, $f <^* g$. Then, it holds

PROPOSITION 3 (4.6 and 4.7 in [1])

(i) The statement $n(\beta\omega\setminus\omega) > c$ implies the statement Ⓓ : "every unbounded family of subsets of $\omega^{\omega}$ has the cardinality c."

(ii) Ⓓ does not imply that $n(\beta\omega\setminus\omega) > c$.

By Propositions 1~3 and Theorem 1, the following diagram holds.

$$\begin{tikzcd}
 & Ⓟ \arrow[d, shift right] & \\
 & \mathrm{AN} \arrow[u, shift right, "/" marking] \arrow[r, shift left, "/" marking] \arrow[dl, shift right] \arrow[dr, shift left, "/" marking] & \forall\kappa<c\ (\ |2^{\kappa}| \leq c\ ) \arrow[l, shift left, "/" marking] \\
Ⓓ \arrow[ur, shift right, "/" marking] & & \exists\, c^{+} \text{ Ramsey ufs on} \arrow[ul, shift left]
\end{tikzcd}$$

The only interesting in the above diagram which is not mentioned is whether AN + $\forall\kappa < c$ ( $|2^{\kappa}| \leq c$ ) implies (P) or not. As to this, we show

THEOREM 2. AN + $\forall\kappa < c$ ( $|2^{\kappa}| \leq c$ ) does not imply (P).

We shall prove Theorem 1 in the following section and Theorem 2 in section 3.

2. Proof of Theorem 1. We first show that $n(\beta\omega\setminus\omega) > c$ implies AN. So, assume that $n(\beta\omega\setminus\omega) > c$. Let $\mathfrak{F}$ be any weakly ample filter on $\omega$. For each $f \in \omega^{\omega}$, set

$$D_f = \{ \mathcal{U} \in \beta\omega\setminus\omega \ ; \ f(\mathcal{U}) \supset \mathfrak{F} \}.$$

Since $\mathfrak{F}$ is weakly ample, it holds that

$$\bigcup\{ D_f \ ; \ f \in \omega^{\omega} \} = \beta\omega\setminus\omega.$$

So, there is some $f \in \omega^{\omega}$ such that $D_f$ is not nowhere dense in $\beta\omega\setminus\omega$. Take an infinite subset $a_0$ of $\omega$ such that

(*) $\quad \forall x \subset a_0 \ ( \ |x| = \omega \Rightarrow \exists \mathcal{U} \in D_f \ ( \ x \in \mathcal{U} \ ) \ ).$

Set $a_1 = f''a_0$. Then, by (*), it holds that $a_1$ is infinite and $\forall x \in \mathfrak{F}$ ( $a_1 \setminus x$ is finite ). Hence, $\mathfrak{F}$ is ample.

Now, we shall prove that the inverse implication holds. The following fact which we shall use in the proof is well-known and easy.

FACT 1. There is a family W of subsets of $\omega$ such that

(1) $|W| = c$,

(2) $\forall x \in W$ ( $|x| = \omega$ ),

(3) $\forall x, y \in W$ ( $x \neq y \Rightarrow x \cap y$ is finite ).

Assume that AN holds. Let $\mathcal{D} = \{ D_\alpha ; \alpha < c \}$ be any family of nowhere dense subsets of $\beta\omega\setminus\omega$. Let $\langle a_\alpha \mid \alpha < c \rangle$ be a monotone enumeration of a family W of subsets of $\omega$ which satisfies (1) $\sim$ (3) in Fact 1. For each $\alpha < c$, take $f_\alpha \in \omega^\omega$ such that $f_\alpha : \omega \longrightarrow a_\alpha$ one-to-one and onto. Define the filter $\mathfrak{F}$ on $\omega$ by

$$x \in \mathfrak{F} \quad \text{iff} \quad \forall \alpha < c \, \forall \mathcal{U} \in D_\alpha \, ( x \in f_\alpha(\mathcal{U}) ).$$

Then, it is easy to see that $\mathfrak{F}$ is free and not ample. So, by AN, there is $\mathcal{U} \in \beta\omega\setminus\omega$ such that

$$\forall g \in \omega^\omega ( g(\mathcal{U}) \not\supset \mathfrak{F} ).$$

Then, it holds that, for any $\alpha < c$, $\mathcal{U} \notin D_\alpha$, since $f_\alpha(\mathcal{U}) \not\supset \mathfrak{F}$. Hence, $\mathcal{U} \notin \bigcup \mathcal{D}$. ∎

3. Proof of Theorem 2. Let $\mathcal{M}$ be a countable transitive model of ZFC + GCH. We shall show that a generic extension of $\mathcal{M}$ on the poset $P \times Q$ which will be defined below satisfies that AN + $\forall \kappa < c \, ( |2^\kappa| < c )$ + $\neg$ (P). The poset $P \times Q$ is alike the poset used in 5.V of [1]. Let P be the Solovey-Tennenbaum's poset used for the consistency of MA + $|2^\omega| = \omega_2$. Define the poset $Q \in \mathcal{M}$ by, in $\mathcal{M}$,

$$Q = \{ q ; \exists \alpha < \omega_1 ( q : \alpha \longrightarrow 2 ) \}.$$

Let $G \times H$ be $\mathcal{M}$-generic on $P \times Q$ and $\widetilde{\mathcal{M}} = \mathcal{M}[G \times H]$. Then, similar arguments in [1] show that

$$\widetilde{\mathcal{M}} \models \text{" } |2^\omega| = |2^{\omega_1}| = \omega_2 + \text{AN "}.$$

We shall show that $\widetilde{\mathcal{M}} \models \neg$ (P). Since CH holds in $\mathcal{M}$, it holds that, in $\mathcal{M}$, there is a dence embedding from Q to $P(\omega)$/finite.

So, we may assume that $H$ is $\mathfrak{M}$-generic on $(P(\omega)/\text{finite})^{\mathfrak{M}}$.

Define $\mathfrak{F} \in \tilde{\mathfrak{M}}$ by

$$\tilde{\mathfrak{M}} \models \text{" } \mathfrak{F} = \{ x \subset \omega \ ; \ \exists\, a/\text{finite} \in H \ ( a \setminus x \text{ is finite} ) \}. \text{"}$$

Since $\tilde{\mathfrak{M}} \models$ " $|H| = \omega_1$ ", it holds that

$$\tilde{\mathfrak{M}} \models \text{" } \mathfrak{F} \text{ is an } \omega_1\text{-generated free filter on } \omega. \text{"}$$

Moreover, since $H$ is not in $\mathfrak{M}[G]$, we have that

$$\tilde{\mathfrak{M}} \models \text{" } \mathfrak{F} \text{ is not ample. "}$$

Hence, $\tilde{\mathfrak{M}} \models \neg \text{Ⓟ}$. ▨

References

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[3] Rothberger, F., On some problems of Hausdorff and Sierpiński, Fund. Math.. 35 (1948), pp.109-126.