

Examples of convolution equations in tube domains

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In the present paper, we study convolution equations of type $\mu * u = f$, where the kernel μ is a hyperfunction with compact support, the given data f and the unknown function u are holomorphic functions in the tube domain $U \in \mathbb{C}^n$ with the form $U = \mathbb{R}^n \times \sqrt{-1}\Omega$ by an open convex subset $\Omega \subset \mathbb{R}^n$. First we recall the notions of the condition (S) due to T. Kawai [3] and the characteristics $\text{Char}(\mu^*)$ (see [2]), which are deeply related to the existence and the continuation of holomorphic solutions. After that we give some examples in non-local operator case.

1 The condition (S) and the characteristics

Let μ be a hyperfunction with compact support on $\mathbb{R}^n$, and Ω be a convex open set in $\mathbb{R}^n$. We consider a convolution equation:

$$\mu * u = f \quad f \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega).$$

We denote by $\hat{\mu}(\zeta)$ the Fourier-Borel transform of μ defined by

$$\hat{\mu}(\zeta) := \langle \mu, e^{x\zeta} \rangle_x = \int_{\mathbb{R}^n} \mu(x) e^{x\zeta} dx.$$

$\hat{\mu}$ is an entire function of exponential type, precisely $\hat{\mu}$ satisfies

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \quad |\hat{\mu}(\zeta)| \leq C_\varepsilon \exp(H_K(\zeta) + \varepsilon |\zeta|),$$

where $K = \text{supp } \mu$ and

$$H_K(\zeta) := \sup_{x \in K} \text{Re } x\zeta$$

is the supporting function of K .

For the convolution operator $\mu*$, we introduce the condition (S) due to Prof. Kawai and the notion of characteristics of $\hat{\mu}$.

Definition 1.1. $\hat{\mu}$ satisfies the condition (S) if and only if for any $\varepsilon > 0$, there exists $N = N_\varepsilon > 0$, such that for any $\eta \in \mathbb{R}^n$ with $|\eta| > N$, we can find $\zeta \in \mathbb{C}^n$ satisfying:

- $|\zeta - \sqrt{-1}\eta| < \varepsilon |\eta|$,
- $|\hat{\mu}(\zeta)| > -\varepsilon |\eta|$.

Definition 1.2. We define the characteristics $\text{Char}(\mu*) \subset \sqrt{-1}S^{n-1}$ by: the vector $\sqrt{-1}\rho \in \sqrt{-1}S^{n-1}$ with $(|\rho| = 1)$, belongs to $\text{Char}(\mu*)$ if and only if there exists a sequence $\{\zeta_\nu\}_\nu \subset \mathbb{C}^n$ satisfying:

- $\hat{\mu}(\zeta_\nu) = 0$ for any ν ,
- $|\zeta_\nu| \rightarrow \infty$ as $\nu \rightarrow \infty$,
- $\zeta_\nu / |\zeta_\nu| \rightarrow \sqrt{-1}\rho$ as $\nu \rightarrow \infty$.

Moreover we define the polar enveloping of Ω and $U = \mathbb{R}^n \times \sqrt{-1}\Omega$. We put

$$\langle \Omega \rangle_\mu := \text{the interior of } \bigcap_{\eta \in \text{Char}(\mu*)} \{y \in \mathbb{R}^n; y\eta < \sup_{y' \in \Omega} y'\eta\}$$

and

$$\langle U \rangle_\mu := \mathbb{R}^n \times \sqrt{-1}\langle \Omega \rangle_\mu.$$

Under these notations, we recall our result about existence and continuation problem (see [2]), which tell us the importance of the condition (S) and the notion of characteristics.

Theorem 1.3. *If $\hat{\mu}$ satisfies (S), then for any open convex subset $\Omega \subset \mathbb{R}^n$, $\mu* : \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega) \rightarrow \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega)$ is surjective. Conversely, assume that $\mu* : \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega) \rightarrow \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega)$ is surjective for a bounded open convex subset $\Omega \subset \mathbb{R}^n$ with C^2 -boundary. Then $\hat{\mu}$ satisfies (S).*

Theorem 1.4. *Assume that $\hat{\mu}$ satisfies (S). We put*

$$\text{Sol}^\mu(U) := \{u \in \mathcal{O}(U); \mu * u = 0\}$$

for $U = \mathbb{R}^n \times \sqrt{-1}\Omega$. Then the restriction map $\text{Sol}^\mu(\langle U \rangle_\mu) \rightarrow \text{Sol}^\mu(U)$ is surjective.

Remark that a kind of the converse statement of this theorem is also true (see [6]).

2 The case of differential-difference equations

Let us consider the case that $\text{supp } \mu$ consists of finite points. We set

$$\text{supp } \mu = \{\lambda_1, \dots, \lambda_\ell\}$$

with $\lambda_j \in \mathbb{R}^n$, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

By the standard structure theorem of hyperfunctions, we can find a family $\{P_j(\zeta)\}_{j=1, \dots, \ell}$ of entire functions of infra-exponential type such that

$$\mu(x) = \sum_{j=1}^{\ell} P_j(D) \delta(x - \lambda_j),$$

where $P_j(D)$'s are the differential operators of infinite order with constant coefficients defined by P_j 's. Thus we have:

$$\mu * u = \sum_{j=1}^{\ell} P_j(D) u(z - \lambda_j)$$

and the convolution equation is differential-difference equation of infinite order.

In this case, we give

Theorem 2.1. *Let μ be a hyperfunction whose support consists of finite points. Then $\hat{\mu}$ satisfies (S). Moreover if $\#\text{supp } \mu > 2$, then $\text{Char}(\mu*) = \sqrt{-1}S^{n-1}$.*

Corollary 2.2. *Differential-difference equations in tube domains are always solvable. Moreover all pure imaginary vectors are characteristic except the case the equations coincides with a differential equation under a suitable translation.*

This theorem can be proved by the theory of entire functions of completely regular growth and the asymptotic estimate of zeros of entire functions of this form due to Ronkin [7].

3 An example of elliptic operator

In this section, we give an example of non-local elliptic operator in the case $n = 1$.

For positive constant a and b with $(a < b)$, we will construct a hyperfunction μ with the following properties:

1. the convex hull of $\text{supp } \mu$ coincides with the segment $[a, b]$,
2. $\hat{\mu}$ satisfies (S),
3. $\text{Char}(\mu*) = \emptyset$.

Moreover we give a remark that $\hat{\mu}$ is not of completely regular growth for any direction $\zeta_0 \in \mathbb{C} \setminus \sqrt{-1}\mathbb{R}$.

Take a sequence $\{a_n\}_{n=1,2,\dots}$ in $\mathbb{R}$ with

- $\{|a_n|\}_n$ is strictly increasing,
- $n/|a_n| \rightarrow 0$ as $n \rightarrow \infty$,
- $\sum_n 1/|a_n|$ diverges,
- $\limsup_{N \rightarrow \infty} \sum_{n < N} 1/a_n = b$, and $\liminf_{N \rightarrow \infty} \sum_{n < N} 1/a_n = a$.

For example, $a_n = \epsilon_n \cdot n \log n$ with $\epsilon_n = \pm 1$ satisfies the conditions for any a and b , if we choose the signs ϵ_n suitably according to a and b .

Put

$$\delta(r) := \sum_{|a_n| < r} \frac{1}{a_n},$$

$$f(\zeta) := \prod_n \left(1 - \frac{\zeta}{a_n}\right) e^{\zeta/a_n},$$

then we can show that f is a Fourier-Borel transform of a hyperfunction μ satisfying the condition 1 and 3. We can also show the condition 2, by the estimate

$$|f(\sqrt{-1}\eta)| \geq 1 \quad \text{for any } \sqrt{-1}\eta \in \sqrt{-1}\mathbb{R}.$$

For this μ , we have

- the convolution equation $\mu * u = f$ is always solvable in any tube domain,
- any solution u of $\mu * u = 0$ can be continued analytically to $\mathbb{C}$.

References

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