

MOURRE THEORY FOR TIME-PERIODIC SYSTEMS

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We consider the following Schrödinger equation with time-dependent Hamiltonian on $\mathbb{R}^\nu$,

$$(1) \quad i \frac{\partial}{\partial t} u(t, x) = H(t)u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^\nu,$$

$$(2) \quad H(t) = -\Delta_x + V(t),$$

where $V(t)$ is a multiplicative operator by a function $V(t, x)$ which is periodic in t with period 2π :

$$(3) \quad V(t + 2\pi, x) = V(t, x).$$

As is well-known, with some suitable conditions on $V(t, x)$, $H(t)$ generates a unique unitary propagator $\{U_1(t, s)\}_{-\infty < t, s < \infty}$. For $H_0 = -\Delta_x$, the associated unitary propagator is denoted by $U_0(t, s) = e^{-i(t-s)H_0}$. A traditional way to study the temporal asymptotics as $t \rightarrow \pm\infty$ of $U_1(t, s)$ is to introduce an operator $K = -i \frac{d}{dt} + H(t)$ on $\mathbb{T} \times \mathbb{R}^\nu$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and to investigate the asymptotic behavior of $e^{-i\sigma K}$. They are mutually related through the following formula

$$(4) \quad (e^{-i\sigma K} f)(t, x) = (U_1(t, t - \sigma)f(t - \sigma, \cdot))(x),$$

for $f \in \mathbb{H} = L^2(\mathbb{T} \times \mathbb{R}^\nu)$. Let

$$(5) \quad K_0 = -i \frac{d}{dt} + H_0.$$

Definition 1 (conjugate operator).

$$(6) \quad A = \frac{1}{2}(L_D \cdot x + x \cdot L_D)$$

where $D_x = \frac{1}{i}\nabla_x$ and $L_D = (L_j)_{1 \leq j \leq \nu}$ with $L_j = D_{x_j} < D_x >^{-2}$.

The following assumption is imposed on $V(t)$.

Assumption 1. Let V be the operator of multiplication by the function $V(t, x)$ on $\mathbb{H}$. We assume that

(i) $V, [V, A]$ are extended to K_0 -compact operators.

(ii) $[[V, A], A]$ is extended to a K_0 -bounded operator.

We denote the extension of the form $[K, A]$ as $[K, A]^0$.

Theorem 1. Suppose Assumption 1 is satisfied. For $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, let $d(\lambda, \mathbb{Z})$ denote the distance from λ to $\mathbb{Z}$. Then, Eigenvalues of K (the set of which are denoted by $\sigma_{pp}(K)$) are discrete with possible accumulation points in $\mathbb{Z}$. If $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, for each $\epsilon > 0$ there exists $0 < \delta < d(\lambda, \mathbb{Z})$ such that

$$(7) \quad f(K) i[K, A]^0 f(K) \geq \left(\frac{2d(I, \mathbb{Z})}{d(I, \mathbb{Z}) + 1} - \epsilon \right) f(K)^2$$

for all $f \in C_0^\infty([\lambda - \delta, \lambda + \delta])$.

Let $\mathfrak{B}(\mathbb{H})$ be the set of bounded operators on $\mathbb{H}$.

Theorem 2. Suppose $\alpha > 1/2$.

(i) For each closed interval $I \subset \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$ the following inequalities hold:

$$(8) \quad \sup_{\operatorname{Im} z \neq 0, \operatorname{Re} z \in I} \| \langle x \rangle^{-\alpha} (K - z)^{-1} \langle x \rangle^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty.$$

(ii) There exist the norm limits in $\mathfrak{B}(\mathbb{H})$.

$$\lim_{\operatorname{Im} z \rightarrow 0, \operatorname{Re} z \in I} \langle x \rangle^{-\alpha} (K - z)^{-1} \langle x \rangle^{-\alpha}.$$

$$\langle x \rangle^{-\alpha} (K - \lambda \mp i0)^{-1} \langle x \rangle^{-\alpha}$$

is Hölder continuous with respect to $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$.

Next we proceed to the propagation estimates. We need the following stronger assumption on the potential.

Assumption 2. There exists $\delta_0 > 0$ such that

$$(9) \quad V(t, \cdot) \in C(\mathbb{T}; C^\infty(\mathbb{R}^\nu)), \quad |\partial_x^\alpha V(t, x)| \leq C_\alpha \langle x \rangle^{-\delta_0 - |\alpha|}, \quad \forall \alpha.$$

Theorem 3. Suppose Assumption 2 is satisfied. Let $E \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, and $\epsilon > 0$ be given. Then there exists a small open interval I containing E such that for any $f \in C_0^\infty(I)$ and $s' > s > 0$,

$$(10) \quad \|\chi \left(\frac{|x|^2}{4\sigma^2} - \frac{d(I, \mathbb{Z})}{d(I, \mathbb{Z}) + 1} < -\epsilon \right) e^{-i\sigma K} f(K) \langle x \rangle^{-s'} \|_{\mathfrak{B}(\mathbb{H})} = O(\sigma^{-s}) \quad \text{as } \sigma \rightarrow \infty$$

where $\chi(x < a)$ denotes the characteristic function of the interval $(-\infty, a)$.