

Operator means and comparison of their norms: general theory and examples

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The classical Heinz inequality ([4]) states

$$(1) \quad \|H^\theta X K^{1-\theta} + H^{1-\theta} X K^\theta\| \leq \|HX + XK\| \quad (0 \leq \theta \leq 1)$$

for Hilbert space operators H, K, X with $H, K \geq 0$. This inequality remains valid for an arbitrary unitarily invariant norm $|||\cdot|||$, and the special case $\theta = 1/2$ of this generalized version is the “arithmetic-geometric mean” inequality obtained in [1]:

$$|||H^{1/2} X K^{1/2}||| \leq \frac{1}{2} |||HX + XK|||.$$

In recent years such operator (and/or matrix) means and comparison of their norms are under active investigation (see [2, 5, 6, 8, 10] for instance). We will briefly explain the general apparatus (obtained in [7]) to deal with such problems. More details as well as a more complete list of references can be found in my survey article in “Sugaku” to be published shortly (or in [7]).

1. OPERATOR (MATRIX) MEANS

In this article a scalar (symmetric homogeneous) mean will mean a continuous function $M(s, t)$ on $[0, \infty) \times [0, \infty)$ satisfying

- (a) $M(s, t) = M(t, s)$,
- (b) $M(\alpha s, \alpha t) = \alpha M(s, t)$ for $\alpha \geq 0$,
- (c) $M(s, t)$ increasing in each variable,
- (d) $\min\{s, t\} \leq M(s, t) \leq \max\{s, t\}$.

The set of all such means will be denoted by $\mathfrak{M}$. Typical examples are

$$(st)^{1/2}, \quad (s-t)/(\log s - \log t) \quad \left(= \int_0^1 s^x t^{1-x} dx\right), \\ \frac{s+t}{2}, \quad \frac{s^\theta t^{1-\theta} + s^{1-\theta} t^\theta}{2} \quad (\text{with } 0 \leq \theta \leq 1).$$

To each $M(s, t) \in \mathfrak{M}$ a corresponding operator mean (denoted by $M(H, K)X$) will be associated.

To get more intuition on the subject matter, we begin with the matrix case ($H, K, X \in M_n(\mathbb{C})$ and $H, K \geq 0$). At first we diagonalize H, K :

$$H = U \text{diag}(t_1, t_2, \dots, t_n) U^*, \quad K = V \text{diag}(s_1, s_2, \dots, s_n) V^*$$

with unitary matrices U, V . For each $M \in \mathfrak{M}$ we define

$$M(H, K)X = U \left(\left[M(s_i, t_j) \right]_{i,j=1,2,\dots,n} \circ (U^* X V) \right) V^*$$

with the Schur product $\circ$. If $M(s, t)$ is of the form $\sum_{k=1}^{\ell} f_k(s)g_k(t)$, we simply have

$$M(H, K)X = \sum_{k=1}^{\ell} f_k(H)Xg_k(K).$$

Let us consider the projections $P_i = UE_{ii}U^*$, $Q_j = VE_{jj}V^*$. Then, $H = \sum_{i=1}^n s_i P_i$, $K = \sum_{j=1}^n t_j Q_j$ are the spectral decompositions of H, K , and we observe that the above matrix mean $M(H, K)X$ can be also expressed as

$$M(H, K)X = \sum_{i,j=1}^n M(s_i, t_j)P_i X Q_j.$$

We now move to the general (operator) case. Let H, K be positive operators with the spectral decompositions

$$H = \int_0^{\|H\|} s dE_s, \quad K = \int_0^{\|K\|} t dF_t.$$

The above expression involving $\sum_{i,j}$ suggests that an operator mean $M(H, K)X$ should be something like

$$M(H, K)(X) = \int_0^{\|H\|} \int_0^{\|K\|} M(s, t) dE_s X dF_t$$

(at least formally). Of course the meaning of this double integral has to be justified, however fortunately the well-developed theory of Stieltjes double integral transformations (see the recent survey article [3]) is at our disposal. Also the problem on multipliers has to be taken care of.

2. STIELTJES DOUBLE INTEGRAL TRANSFORMATIONS

Given a function $\phi(s, t) \in L^\infty([0, \|H\|] \times [0, \|K\|]; \lambda \times \mu)$ (with $\lambda \sim dE_s$ and $\mu \sim dF_t$ in the absolute continuity sense), we would like to make a sense out of

$$(2) \quad \Phi(X) = \int_0^{\|H\|} \int_0^{\|K\|} \phi(s, t) dE_s X dF_t \quad (\in B(\mathcal{H})) \text{ for } X \in B(\mathcal{H}).$$

We begin with the case $X \in \mathcal{C}_2(\mathcal{H})$, the Hilbert-Schmidt operators. For Borel subsets

$$\Lambda \subseteq [0, \|H\|], \quad \Xi \subseteq [0, \|K\|]$$

$\pi_\ell(E_\Lambda), \pi_r(F_\Xi)$ are commuting projections acting on $\mathcal{C}_2(\mathcal{H})$, where $\pi_\ell(\cdot), \pi_r(\cdot)$ mean the left and right multiplications acting on $\mathcal{C}_2(\mathcal{H})$. Let us consider the correspondence

$$\Lambda \times \Xi \subseteq [0, \|H\|] \times [0, \|K\|] \longrightarrow \pi_\ell(E_\Lambda)\pi_r(F_\Xi) \in B(\mathcal{C}_2(\mathcal{H}))_{proj},$$

which can be easily extended to a spectral family. Therefore, we can perform functional calculus relative to this spectral family. Thus, we have

$$\Phi(\cdot) = \int_0^{\|H\|} \int_0^{\|K\|} \phi(s, t) d(\pi_\ell(E) \pi_r(F)) \in B(\mathcal{C}_2(\mathcal{H})),$$

and $\Phi(X) (\in \mathcal{C}_2(\mathcal{H}))$ makes a perfect sense. The right side of (2) should be understood in this way (for $X \in \mathcal{C}_2(\mathcal{H})$).

The discussion so far is indeed the starting point of the theory of Stieltjes double transformations by Birman-Solomjak. For each practical purpose the definition domain of $\Phi(\cdot)$ should be enlarged as much as possible (to $\mathcal{C}_p(\mathcal{H}), \mathcal{C}_1(\mathcal{H}), B(\mathcal{H})$, etc. depending upon available regularity assumption). Various important applications to many subjects (such as perturbation theory, Volterra operators, Hankel operators and so on) are known.

Definition. $\phi(s, t)$ is called a Schur multiplier (more precisely, $\mathcal{C}_1$ -Schur multiplier relative to (H, K)) when $\Phi(\mathcal{C}_1(\mathcal{H})) \subseteq \mathcal{C}_1(\mathcal{H})$.

Theorem (V.V. Peller's characterization, [9])

For $\phi \in L^\infty([0, \|H\|] \times [0, \|K\|]; \lambda \times \mu)$ the following conditions are all equivalent:

- (i) ϕ is a Schur multiplier (relative to (H, K));
- (ii) whenever a measurable function $k : [0, \|H\|] \times [0, \|K\|] \rightarrow \mathbf{C}$ is the kernel of a trace class operator $L^2([0, \|H\|]; \lambda) \rightarrow L^2([0, \|K\|]; \mu)$, so is the product $\phi(s, t)k(s, t)$;
- (iii) one can find a finite measure space (Ω, σ) and functions $\alpha \in L^\infty([0, \|H\|] \times \Omega; \lambda \times \sigma)$, $\beta \in L^\infty([0, \|K\|] \times \Omega; \mu \times \sigma)$ satisfying

$$(3) \quad \phi(s, t) = \int_{\Omega} \alpha(s, x) \beta(t, x) d\sigma(x);$$

- (iv) one can find a measure space (Ω, σ) and measurable functions α, β on $[0, \|H\|] \times \Omega$, $[0, \|K\|] \times \Omega$ respectively satisfying (3) and

$$\left\| \int_{\Omega} |\alpha(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\lambda)} \left\| \int_{\Omega} |\beta(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\mu)} < \infty.$$

When $\phi(s, t)$ is a Schur multiplier, $\Phi : \mathcal{C}_1(\mathcal{H}) \rightarrow \mathcal{C}_1(\mathcal{H})$ is a bounded linear operator (by the closed graph theorem) so that we have the transpose ${}^t\Phi : B(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^* \rightarrow B(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$. Starting from the decomposition (3), one can prove

$$\int_0^{\|H\|} \int_0^{\|K\|} \phi(s, t) dE_s X dF_t = \int_{\Omega} \alpha(H, x) X \beta(K, x) d\sigma(x).$$

3. NORM INEQUALITIES FOR OPERATOR MEANS

When a scalar mean $M(s, t)$ ($\in \mathfrak{M}$) is a Schur multiplier, we define

$$M(H, K)X = \int_0^{\|H\|} \int_0^{\|K\|} M(s, t) dE_s X dF_t \in B(\mathcal{H}) \quad (\text{for each } X \in B(\mathcal{H})).$$

Theorem (F. Hiai and H. Kosaki, [6, 7])

For $M, N \in \mathfrak{M}$ the following conditions are all equivalent:

- (i) There exists a symmetric probability measure ν on $\mathbf{R}$ with the following property: if N is a Schur multiplier relative to (H, K) of non-singular positive operators, then so is M and

$$M(H, K)X = \int_{-\infty}^{\infty} H^{ix} (N(H, K)X) K^{-ix} d\nu(x) \quad \text{for } X \in B(\mathcal{H});$$

- (ii) If N is a Schur multiplier relative to a pair (H, K) of positive operators, then so is M and

$$|||M(H, K)X||| \leq |||N(H, K)X|||$$

for all unitarily invariant norms and all $X \in B(\mathcal{H})$;

- (iii) $\|M(H, H)X\| \leq \|N(H, H)X\|$ for all X of finite rank and for all $H \geq 0$;
 (iv) For each n and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$

$$\left[\frac{M(\lambda_i, \lambda_j)}{N(\lambda_i, \lambda_j)} \right]_{i,j=1,2,\dots,n} \geq 0;$$

- (v) $M \prec N$, i.e., $\frac{M(e^x, 1)}{N(e^x, 1)}$ is a positive definite function.

This theorem explains the Heinz inequality (1) as follows: We set

$$M(s, t) = \frac{s^{\frac{1+\alpha}{2}} t^{\frac{1-\alpha}{2}} + s^{\frac{1-\alpha}{2}} t^{\frac{1+\alpha}{2}}}{2}, \quad N(s, t) = \frac{s+t}{2} \quad (\alpha \in [0, 1]).$$

The ratio is

$$\frac{M(e^x, 1)}{N(e^x, 1)} = \frac{e^{(\frac{1+\alpha}{2})x} + e^{(\frac{1-\alpha}{2})x}}{e^x + 1} = \frac{\cosh(\alpha x/2)}{\cosh(x/2)}$$

whose Fourier transform is given by

$$\int_{-\infty}^{\infty} \frac{\cosh(\alpha x/2)}{\cosh(x/2)} e^{ixy} dx = \frac{4\pi \cosh(\pi y) \cos(\alpha\pi/2)}{\cosh(2\pi y) + \cos(\alpha\pi)} > 0.$$

Bochner's theorem thus yields $M \prec N$.

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