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RECENT DEVELOPMENTS IN THE THEORY OF GENERAL HYPERGEOMETRIC FUNCTIONS

I.M.Gelfand, M.I.Graev, V.S.Retakh[1]

(Institute for System Analysis,
Avtozavodskaya 23, Moscow 109280, USSR)

0. This report is related to a series of papers [2-7] devoted to the theory of general hypergeometric functions. Our aim is to describe briefly the results from [8,9]. First we recall the definition of equations of hypergeometric type according to [4,5]. Let T^m be an m-dimensional complex torus acting on an N-dimensional complex space W. We fix a basis $e_1, \cdots, e_N$ in W, in which all the transformations belonging to T^m are diagonal. Let $\lambda_1, \cdots, \lambda_N$ be characters of T^m such that $te_i = \lambda_i(t)e_i$ for $t \in T^m$. We write vectors $w \in W$ in the form $w = \sum z_i e_i$. If we choose coordinates $t_1, \cdots, t_m$ in the group $T^m \simeq (\mathbf{C}^*)^m$, then each λ_i has the form $\lambda_i = \prod_1^m t_k^{\lambda_{ki}}$, where (λ_{ki}) is some $n \times N$-matrix.

Let $L = \{a = (a_1, \cdots, a_N)\}$ be the integer lattice of solutions of the system of equations $\sum_1^N a_i \lambda_{ki} = 0,\ 1 \leq k \leq m$. For any multiparameter $\beta = (\beta_1, \cdots, \beta_m) \in \mathbf{C}^m$ the system of hypergeometric type on W is defined:

$$\left\{\begin{array}{ll} \displaystyle\sum_{1\leq i\leq N} \lambda_{ki} z_i \frac{\partial \Phi}{\partial z_i} = \beta_k \Phi \qquad 1 \leq k \leq m & (1) \\ \displaystyle\left[\prod_{i:a_i>0} \left(\frac{\partial}{\partial z_i}\right)^{a_i}\right] \Phi = \left[\prod_{i:a_i<0} \left(\frac{\partial}{\partial z_i}\right)^{-a_i}\right] \Phi, \quad a \in L & (2) \end{array}\right.$$

It is not hard to check that all the equations (2) are a consequence of a finite number of them. In [4,5] the solutions of the system (1)-(2) as Γ-series and in [7] as generalized Euler integrals were described.

IMPORTANT EXAMPLE. Let $G_{k,n}$ be the Grassmanian of k-dimensional subspaces of complex space $\mathbf{C}^n$ with the coordinates $x_1, \cdots, x_n$. Suppose that such subspaces can be written in the form $x_j = v_{ij}x_1 + \cdots + v_{kj}x_k, j = k+1, \cdots, n$. Consider the coefficients v_{ij} of these equations as local coordinates on $G_{k,n}$. Let V be the space of complex matrixes (v_{ij}), $i = 1, \cdots, k, j = k+1, \cdots, n$. The action of torus T^n on V is generated by all possible dilatations of the rows and columns of matrices: $v_{ij} \mapsto t_i^{-1} v_{ij} t_j$. The corresponding equations on V can be written in the form:

$$\sum_j v_{ij} \frac{\partial \Phi}{\partial v_{ij}} = (\alpha_i + 1)\Phi \qquad i = 1, \cdots, k \tag{3}$$

[1] The report is based on a joint work with I. M. Gelfand and M. I. Graev.

$$\sum_i v_{ij}\frac{\partial\Phi}{\partial v_{ij}} = \alpha_j\Phi \qquad j = k+1,\cdots,n \tag{4}$$

$$\frac{\partial^2\Phi}{\partial v_{ij}\partial v_{i'j'}} = \frac{\partial^2\Phi}{\partial v_{i'j}\partial v_{ij'}} \tag{5}$$

where parameters α_i are connected by the formula $\sum\alpha_i = -k$, because only $(n-1)$-dimensional torus acts effectively on V.

According to the [4,5,7] one can describe the hypergeometric functions on $G_{k,n}$ as Γ-series or Euler integrals depending of local coordinates. Here we want to describe the solution of (3)-(5) as Γ-series or Euler integrals depending of Plücker coodinates which are more natural for Grassmanians. This approach gives a possibility for studying hypergeometric functions on strata in $G_{k,n}$. For example, our method gives the representation of Gaussian function F as a triple integral. We also obtain a generalization of classical reduction formulas for hypergeometric series.

1. Euler integrals on $\wedge^k\mathbf{C}^n$. Let $X = \wedge^k\mathbf{C}^n$ and $P_I = P_{i_1\cdots i_k}$, $1 \le i_1 < \cdots < i_k \le n$, be the coordinates in X with the base $\{e_{i_1}\wedge\cdots\wedge e_{i_k}|i_1 < \cdots < i_k\}$. Here $\{e_i\}$ is a standard base in $\mathbf{C}^n$. We define also $p_{i_1\cdots i_k}$ for any unordered set $i_1,\cdots i_k$ according to the standard transposition rules.

This is a standard action of torus $T^n = (\mathbf{C}^*)^n$ on X : $\{p_I\}\mapsto\{t_Ip_I\}$, where $t_I = t_{i_1}\cdots t_{i_k}$, $I = \{i_1,\cdots,i_k\}$. The corresponding system of hypergeometric type equations for $\alpha = (\alpha_1,\cdots,\alpha_n)\in\mathbf{C}^n$ is:

$$\sum_{I\ni i} p_I\frac{\partial\Phi}{\partial p_I} = \alpha_i\Phi \qquad i = 1,\cdots,n \tag{6}$$

$$\frac{\partial^2\Phi}{\partial p_{I_1}\partial p_{I_2}} = \frac{\partial^2\Phi}{\partial p_{J_1}\partial p_{J_2}} \tag{7}$$

where $|I_1| = |I_2| = |J_1| = |J_2| = k, |I_1\cap J_1| = |I_2\cap J_2| = k-1$ (we give here only the basic equations of the system.)

The solutions of this system will be called the hypergeometric functions on $\wedge^k\mathbf{C}^n$.

Definition . Let $p = \{p_I\}$. The subset $X_\Xi = \{p\in X|p_I\ne 0 \Leftrightarrow I\in\Xi\}$ is called the Ξ-stratum in $X, \Xi = \{I_1,\cdots,I_r\}$. If Ξ consists of all $I\subset[1,n], |I| = k$, then X_Ξ is called the generic stratum.

All the strata are T^n-invariant. The stratum is called nondegenerate if every T^n-orbit on it is nondegenerate. A hypergeometric function on a stratum X_Ξ is the restriction $\varphi|X_\Xi$ of a hypergeometric function φ on X.

Consider for every point $p = \{p_I\}$ the polynomial $u(t,p)$ on $\mathbf{C}^n$:

$$u(t,p) = \sum p_It_I = \sum_{1\le i_1\le\cdots\le i_k\le n} p_{i_1\cdots i_k}\cdot t_{i_1}\cdots t_{i_k}.$$

Let θ be a differential form

$$\theta = u^{-1}(t,p)\prod_{j=1}^{n} t_j^{-\alpha_j-1}\omega(t),$$

where $\omega(t) = t_1dt_2 \wedge \cdots \wedge dt_n - t_2dt_1 \wedge dt_3 \wedge \cdots \wedge dt_n + \cdots, \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{C}^n$.

Suppose that $\sum \alpha_i = -k$, then one can consider the form θ on a projective space $\mathbf{PC}^n$. Set

$$F(\alpha,p) = \int_\gamma \theta \tag{8}$$

where $\gamma \subset \mathbf{PC}^n$ is a projectivization of $\tilde{\gamma} = \{t \in \mathbf{C}^n | t_i \in \mathbf{R}, t_i > 0,\ i = 1, \cdots, n\}$. For a stratum X_Ξ set $U_\Xi = \{p \in X_\Xi |\ \mathrm{Re}\ p_I > 0 \text{ for } I \in \Xi\}$.

Theorem 1. *For any nondegenerate stratum $X_\Xi \subset X$ there exists a domain $\mathcal{O}_\Xi \subset \mathbf{C}^n$ such that for $\alpha \in \mathcal{O}_\Xi$ the integral* (8) *absolutely converges for every $p \in U_\Xi$, the function $F(\alpha,p)$ is regular on U_Ξ and $F(\alpha,p)$ is a hypergeometric function on X_Ξ.*

We define the integral $F(\alpha,p)$ for all α by the analytic continuation.

2. Euler integrals on $G_{k,n}$. Let $Z_{k,n}$ be a space of $k \times n$-matrices. Consider a map $\pi : Z_{k,n} \to X = \wedge^k\mathbf{C}^n$, $\pi(\| z_{ij} \|) = \{p_{i_1\cdots i_k} = \det \| z_{r,i_s} \|_{r,s=1,\cdots,k}\}$. The image π in X is denoted by P, we call P the Plücker manifold. One can consider the hypergeometric functions on Grassmanian $G_{k,n}$ as functions on P. So we will use the terminology "hypergeometric functions on P" instead of "hypergeometric functions on $G_{k,n}$".

For any $\lambda, p \in X$ denote by $\lambda \circ p$ the vector in X with the coordinates $\{\lambda_I p_I\}$.

Theorem 2. *If φ is a hypergeometric function on X then for every $\lambda \in P$ the function*

$$\psi(p) = \varphi(\lambda \circ p) \tag{9}$$

is a hypergeometric function on P. If ψ is a hypergeometric function on P and ψ is regular in a domain $\mathcal{O} \subset P$ then there exists a hypergeometric function φ on X and a vector $\lambda \in P$ such that equality (9) *is valid.*

A (nondegenerate) stratum P_Ξ in P is by definition the intersection $X_\Xi \cap P$ for a (nondegenerate) stratum X_Ξ in X. A hypergeometric function on P_Ξ is by definition the restriction of hypergeometric function on P. We use the theorem 2 for a description of hypergeometric function on strata in P.

Theorem 3. a) *Let P_Ξ be a nondegenerate stratum and $\mathcal{O}_\Xi$ the domain of multiparameter α defined by the theorem* 1. *For any $P_0 \in P_\Xi$ there exists its neighbourhood $V \subset P$ such that for any $\lambda \in V \cap P_\Xi, \alpha \in \mathcal{O}_\Xi$, the integral*

$$\Phi_{\bar{\lambda}}(\alpha,p) = \int_\gamma u^{-1}(t, \bar{\lambda} \circ p)\prod_{j=1}^{n} t_j^{-\alpha_j-1}\omega(t) \tag{10}$$

absolutely converges on V. The function $\Phi_{\bar{\lambda}}$ is a hypergeometric function on V.

b) *The restrictions of integrals* (10) *on $V \cap P_\Xi$ for all $\lambda \in V \cap P_\Xi$ linearly generate the space of hypergeometric functions on P_Ξ regular on the neighbourhood $V \cap P_\Xi$.*

If $\{\varphi_i(\alpha,p)\}$ is a base in a space of hypergeometric functions on P_Ξ regular in a neighbourhood $\tilde{V}$ of a point $p_0 \in P_\Xi$ then according to the theorem 3

$$\Phi_{\bar{\lambda}}(\alpha,p) = \sum_{i,j} c_{ij}\varphi_i(\alpha,\bar{\lambda})\varphi_j(\alpha,p), \quad \alpha, p \in \tilde{V}$$

Here the matrix $\| c_{ij} \|$ is nondegenerate. It depends only of α. One can choose the base $\{\varphi_i\}$ such that $\| c_{ij} \|$ will be a diagonal matrix. I want to mention that solutions of a hypergeometric system are described here by varying the parameter λ and integrating over the fixed cycle γ. On the contrary, usually the solutions are described by integrating over different cycles [7].

Example . $k=2, n=4$. If $p, \lambda \in P$ have the coordinates $p_{12} = p_{13} = -p_{23} = -p_{24} = 1$, $p_{14} = x$; $\lambda_{12} = \lambda_{13} = -\lambda_{23} = -\lambda_{24} = 1$, $\lambda_{14} = \rho$ then the solution of Gaussian equation $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$ according to the theorem 3 is given by the formula

$$\begin{aligned} y_\rho(x) &= \int\int\int (t_1t_2 + t_1t_3 + t_2t_3 + t_2t_4 + \rho x t_1 t_4 \\ &\quad + (1-\rho)(1-x)t_3t_4)^{-1} t_1^{c-b-1} t_2^{-a} t_3^{a-c} t_4^{b-1} \omega(t) \\ &= \int_0^\infty \int_0^\infty \int_0^\infty (t_2 + t_3 + t_2t_3 + t_2t_4 + \rho x t_4 \\ &\quad + (1-\rho)(1-x)t_3t_4)^{-1} t_2^{-a} t_3^{a-c} t_4^{b-1} dt_2 dt_3 dt_4 \end{aligned}$$

From this formula one can represent the Gaussian function F as triple Euler integral.

4. Formulas of reduction and Γ-series. Consider the space $Z = Z_{k,n}$ of complexes $(k \times n)$-matrices. The action of torus T^{k+n} on Z is generated by all possible dilatations of the rows and columns of matrices $z = (z_{ij})$. By the general theory we have the system of hypergeometric equations on Z:

$$\sum_i z_{ij} \frac{\partial \varphi}{\partial z_{ij}} = \alpha_j \varphi \quad j = 1, \cdots, n \tag{11}$$

$$\sum_j z_{ij} \frac{\partial \varphi}{\partial z_{ij}} = \beta_i \varphi \quad i = 1, \cdots, k \tag{12}$$

$$\frac{\partial^2 \varphi}{\partial z_{ij} \partial z_{i'j'}} = \frac{\partial^2 \varphi}{\partial z_{i'j} \partial z_{ij'}} \tag{13}$$

where parameters α_j, β_i are connected by the formula $\sum \alpha_j = \sum \beta_i$.

There exists a map χ from Z to V - the space of local coordinates over Grassmanian $G_{k,n}$. For $z = (u,v)$, where u is $k \times k$-matrix, $\chi z = u^{-1}v$. Then Ψ is a solution of (3)-(5) if

$$\Phi(z) = (\det u)^{-1} \Psi(\chi z) \tag{14}$$

and $\beta_i = -1$, $i = 1, \cdots, k$.

According to [8] this means that the system (3-5) is subordinated to the system (11)-(13).

We give now a combinatorical description of Γ-series Φ satisfying (14).

Definition . A set $I \subset [1,k] \times [1,n]$ is a base if $|I| = k+n-1$ and the manifold $\{z \in Z | z_{ij} \neq 0 \Leftrightarrow (i,j) \in I\}$ is T^{k+n}-orbit.

Consider for every base the series

$$\Phi_I(z) = \sum_m \prod_{(i,j)\in I} \frac{z_{ij}^{m_{ij}+\gamma_{ij}}}{\Gamma(m_{ij}+\gamma_{ij}+1)} \cdot \prod_{(i,j)\in I'} \frac{z_{ij}^{m_{ij}}}{m_{ij}!}. \tag{15}$$

Here $I' = [1,k] \times [1,n] \setminus I$; the sum is taken over all $m_{ij} \geq 0$, $(i,j) \in I'$; the integers $m_{ij}, (i,j) \in I$ are linear combinations of $m_{ij}, (i,j) \in I'$ such that $\sum_i m_{ij} = \sum_j m_{ij} = 0$. The complex numbers γ_{ij}, $(i,j) \in I$ are defined from the formulas $\sum_i' \gamma_{ij} = \alpha_{j'}$, $j = [1,n]$, $\sum_j' \gamma_{ij} = \beta_i$, $i \in [1,k]$. Here $\sum'$ means the summation over $(i,j) \in I$.

The series $\Phi_I(z)$ converge and give the complete system of solutions of the equations (11)-(13).

Proposition 4. *The function Φ_I for $\beta_i = -1$, $i \in [1,k]$ satisfies (14) if and only if the base I is admissible: i.e. for every $i \in [1,k]$ the base I contains at least two elements (i,j) and (i,j').*

At least we pass to the formulas of reduction. Let $Z_{\mathfrak{A}} = \{z \in Z | z_{ij} = 0 \text{ for } (i,j) \in \mathfrak{A}\}$, where $\mathfrak{A} \subset [1,k] \times [1,n]$. We call $Z_{\mathfrak{A}}$ the general subspace of Z if $\chi Z_{\mathfrak{A}} = V$. If $I \cap \mathfrak{A} = \emptyset$; then the serie (14) for $I' \setminus \mathfrak{A}$ instead of I' gives us a hypergeometric function on $Z_{\mathfrak{A}}$ and for this function the proposition 4 is valid.

Suppose a pair $(I, \mathfrak{A})$ is given such that $I \cap \mathfrak{A} = \emptyset$; the base I is admissible and $Z_{\mathfrak{A}}$ is a general subspace of minimal dimension. In this case $\mathfrak{A} = \{(i,j) | j \in J_i, i \in [1,k]\}$ where $|J_i| = k-1$.

Theorem 5. *There exists a formula of reduction for every such pair. It connects Γ-series on Z and $Z_{\mathfrak{A}}$:*

$$\Phi_I(z) = p_{j_1\cdots j_k}^{-1} \cdot \begin{vmatrix} p_{j_1 J_1} & \cdots & p_{j_k J_k} \\ \cdots & \cdots & \cdots \\ p_{j_1 J_k} & \cdots & p_{j_k J_k} \end{vmatrix} \times$$

$$\sum_n \left(\prod_{(i,j)\in I} \frac{p_{jJ_i}^{n_{ij}+\gamma_{ij}}}{\Gamma(n_{ij}+\gamma_{ij}+1)} \prod_{(i,j)\in I'\setminus\mathfrak{A}} \frac{p_{jJ_i}^{n_{ij}}}{n_{ij}!} \right) \tag{16}$$

Here Φ_I is Γ-serie on Z given by the formula (15), $p_{i_1,\cdots,i_k}$ - the Plücker coordinate of z. The sum is taken over $n_{ij} \geq 0$, $(i,j) \in I' \setminus \mathfrak{A}$; the integers n_{ij}, $(i,j) \in I$ are the linear combinations of n_{ij}, $(i,j) \in I' \setminus \mathfrak{A}$, given by the formulas $\sum_i n_{ij} = \sum_j n_{ij} = 0$ where $(i,j) \notin \mathfrak{A}$. The formula (16) does not depend of the choice $j_1, \cdots j_k \in [1,n]$ such that $p_{j_1\cdots j_k} \neq 0$.

The multiciplicities of the series from (16) are $N = kn - (k+n-1)$ and $N - k(k-1)$ respectively. The restrictions of Φ_I on different coordinate subspaces in Z gives us many reduction formulas for the series of other multiciplicities.

Example . Let $k = 2$, $n = 4$, $I = \{(2,1),(2,2),(2,3),(1,3),(1,4)\}, \mathfrak{a} = \{(1,1),(2,4)\}$. Then (16) turns to

$$z_{13}^{-\alpha_4-1} z_{14}^{\alpha_4} z_{21}^{\alpha_1} z_{22}^{\alpha_2} z_{23}^{-\alpha_1-\alpha_2-1}$$
$$\times \sum c(n_1,n_2,n_3)\left(\frac{z_{11}z_{23}}{z_{21}z_{13}}\right)^{n_1}\left(\frac{z_{12}z_{23}}{z_{22}z_{13}}\right)^{n_2}\left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{n_3}$$
$$= p_{31}^{-\alpha_4-1} p_{41}^{\alpha_1+\alpha_4+1} p_{42}^{\alpha_2} p_{43}^{-\alpha_1-\alpha_2-1} \sum c(n)\left(\frac{p_{21}p_{43}}{p_{31}p_{42}}\right)^{n},$$

where $c^{-1}(n_1,n_2,n_3) = n_1!n_2!n_3!\Gamma(-n_1+\alpha_1+1)\Gamma(-n_2+\alpha_2+1)\Gamma(-n_3+\alpha_4+1)\cdot\Gamma(n_3-n_1-n_2-\alpha_4)\Gamma(n_1+n_2-n_3-\alpha_1-\alpha_2)$; $c^{-1}(n) = \Gamma(\alpha_1+1)\Gamma(\alpha_4+1)\cdot\Gamma(-n+\alpha_2+1)\Gamma(-n-\alpha_4)\Gamma(n-\alpha_1-\alpha_2)n!$.

Setting $x_1 = \frac{z_{11}z_{23}}{z_{21}z_{13}}, x_2 = \frac{z_{12}z_{23}}{z_{22}z_{13}}, x_3 = \frac{z_{13}z_{24}}{z_{14}z_{23}}$ we obtain a reduction formula for Pondy function G_B [10]:

$$\sum c(n_1,n_2,n_3)x_1^{n_1}x_2^{n_2}x_3^{n_3} = (1-x_1)^{-\alpha_4-1}\cdot(1-x_3)^{-\alpha_1-\alpha_2-1}\cdot$$
$$(1-x_1x_3)^{\alpha_1+\alpha_4+1}(1-x_2x_3)^{\alpha_2}\cdot\sum c(n)\left(\frac{(1-x_3)(x_2-x_1)}{(1-x_1)(1-x_2x_3)}\right)^{n} \tag{17}$$

Setting $x_3 = 0$ we receive a classical formula of reduction for the Appel function F_1. For $x_1 = 0$ or $x_2 = 0$ we obtain reduction formulas for the Horn function G_2.

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