

The notion of near-complex subvariety

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(Note by S. Nakano. Nakano is responsible for possible errors.)

§1. Introduction Let X be a compact complex analytic manifold and suppose X is diffeomorphic to the complex projective space $\mathbb{P}^n(\mathbb{C})$. Is, then, X analytically isomorphic to $\mathbb{P}^n(\mathbb{C})$?

F. Hirzebruch and K. Kodaira [1] solved this problem affirmatively under the condition that X carries a Kähler metric (and an additional condition when $n = \dim_{\mathbb{C}}(X)$ is even). In this lecture we discuss this problem without Kähler condition, but in the case when X is a member of a smooth family of complex analytic manifolds, other members being isomorphic to $\mathbb{P}^n(\mathbb{C})$. The conclusion is:

Main Theorem. Let $\pi: \mathcal{X} \rightarrow D$ be a smooth proper morphism from a complex analytic manifold $\mathcal{X}$ onto the unit disc $D = \{x \in \mathbb{C} \mid |x| < 1\}$, so that $\pi^{-1}(x)$ is a compact complex analytic manifold of dimension n (independent of x) for every $x \in D$. Suppose $\pi^{-1}(x)$ is analytically isomorphic to $\mathbb{P}^n(\mathbb{C})$ except for $\pi^{-1}(0) = X_0$, then X_0 is also isomorphic to $\mathbb{P}^n(\mathbb{C})$.

In §2 we shall indicate how Kähler condition was made use of in the proof of Hirzebruch-Kodaira and give an example. In §3 we shall discuss the detour by which we arrive at our aim without Kähler condition, namely the notion of near-complex subvarieties.

§2. Let us consider our family

$$\pi: \mathcal{X} \longrightarrow D.$$

We can set up a diffeomorphism f

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & X_0 \times D \\ \pi \downarrow & \searrow \text{id} & \downarrow \text{proj} \\ D & \xrightarrow{\quad} & D \end{array}$$

so that $H^{2q}(X_0, \mathbb{Z}) \simeq H^{2q}(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} g^q$, where g is the generator of $H^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$ dual to the hyperplane. (In reality, f can be taken to be a real analytic homeomorphism. This will be used in §3.) Hence we can speak of positive or negative cohomology classes.

Now theorem 6 in Herzebruch-Kodaira [1] can be modified to

Theorem 1 Let X be an n -dimensional compact complex analytic manifold and let a complex line bundle (invertible sheaf) L on X be given. Denote by $g (\in H^2(X, \mathbb{Z}))$ the Chern class of L . Assume

- (1) $\dim_{\mathbb{C}} H^0(X, L) \geq n + 1$,
- (2) $H^{2d}(X, \mathbb{Z}) = \mathbb{Z} \cdot g^d \simeq \mathbb{Z}$ (for $0 \leq d \leq n$),
- (3) X is a Moishezon space, (i.e. there exist n algebraically independent global meromorphic functions on X ,)

(4) any complex analytic subvariety of X determines a non-negative cohomology class.

Then we can conclude that X is analytically isomorphic to $\mathbb{P}^n(\mathbb{C})$.

In case of Hirzebruch-Kodaira, Kähler condition was made use of in order to establish (4) as well as (1) and (3). In our case of a smooth family, it is not too hard to establish (1)---(3) without Kähler condition, while for (4) we need the analysis we develop in §3.

Here is an example of a manifold which satisfies (1) --- (3) but not (4):

Take a non-singular curve Γ in the product $\mathbb{P}^1 \times \mathbb{P}^1 = S$, with the property that $\Gamma \cdot (u \times \mathbb{P}^1)$ consists of three points and $\Gamma \cdot (\mathbb{P}^1 \times v)$ of 5 points for generic u and v . Embed S into $\mathbb{P}^3$ as a quadratic surface, and blow up $\mathbb{P}^3$ with Γ as the center. We obtain a projective threefold X' and X' contains the subvariety S' , the proper transform of S . S' is isomorphic to S : $S' \cong S \cong \mathbb{P}^1 \times \mathbb{P}^1$. It can be seen that S' can be blown down, i.e. there exist a compact complex manifold X and a morphism $p: X' \rightarrow X$, so that $p|_{X'-S'}$ is an isomorphism and on $S' \cong \mathbb{P}^1 \times \mathbb{P}^1$, p is nothing but the projection to the first factor.

It can be shown that X satisfies (1)---(3) but not (4). ($p(S')$ gives a negative class.) Thus we see the condition (4) is essential in the theorem.

§3 Given a complex analytic manifold Y_0 , we denote by $\mathcal{T}_{Y_0, y}$ and $T_{Y_0, y}$ the complex tangent space and real tangent space respectively, to the manifold Y_0 at a point y on it. We denote by $\mathcal{T}_{Y_0}$ and T_{Y_0} the bundles $\bigcup_{y \in Y_0} \mathcal{T}_{Y_0, y}$ and $\bigcup_{y \in Y_0} T_{Y_0, y}$ respectively. We have a canonical isomorphism

$$T_{Y_0} \cong \operatorname{Re}(\mathcal{T}_{Y_0} \oplus \overline{\mathcal{T}}_{Y_0}).$$

Now let a compact complex analytic manifold X be given. A near-complex subvariety $\mathcal{Y}$ of X is a quadruple $(Y, Y_0, \mathcal{E}_0, \rho_0)$, where

Y is a closed subset of X ,

Y_0 is a connected dense open subset of Y and has a structure of even dimensional oriented C^∞ -manifold, (a submanifold of X with induced topology)

$\mathcal{E}_0$ is a complex subbundle of $\mathcal{T}_X|_{Y_0}$,

ρ_0 is a real bundle isomorphism $\operatorname{Re}(\mathcal{E}_0 \oplus \overline{\mathcal{E}}_0) \cong T_{Y_0}$, and maps the natural orientation on $\operatorname{Re}(\mathcal{E}_0 \oplus \overline{\mathcal{E}}_0)$ to the given orientation of Y_0 .

In reality, we add further conditions of technical character, e.g. Y can be triangulated so that $Y - Y_0$ be a subcomplex of dimension $\leq \dim_{\mathbb{R}} Y_0 - 2$. In this way, Y determines a homology class in $H_*(X, \mathbb{Z})$ and, if we go over to rational coefficients, determines a cohomology class in $H^*(X, \mathbb{Q})$.

An example of a near-complex subvariety is given by an analytic subvariety Y of X . We take Y_0 to be the set of the simple points of Y and $\mathcal{E}_0 = \mathcal{T}_{Y_0}$, $\rho_0: \operatorname{Re}(\mathcal{E}_0 \oplus \overline{\mathcal{E}}_0) \rightarrow T_{Y_0}$ is the canonical one.

Another example will appear in connection with a smooth family of compact complex analytic manifolds: Let $\pi: \mathcal{X} \rightarrow D$ be a smooth family over the disc $D = \{x \in \mathbb{C} \mid |x| < 1\}$. We take a differentiable trivialization of $\mathcal{X}$:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & X_0 \times D \\
 \pi \downarrow & \searrow \text{id} & \downarrow \text{proj} \\
 D & \xrightarrow{\quad} & D
 \end{array}
 \quad , \quad X_0 = \pi^{-1}(0).$$

If Y is an analytic subvariety of X_0 , then it determines a near-complex subvariety in X_0 as in the preceding example. We put this in $X_0 \times x$ on the right hand side of the above diagram and pull back everything onto $X_x = \pi^{-1}(x)$ by f . Then a near-complex subvariety on X_x is obtained.

If we restrict ourselves to the family in the main theorem, all X_x are isomorphic to $\mathbb{P}^n(\mathbb{C})$ except $x = 0$, and we shall have a family of near-complex subvarieties

$$(*) \quad \{Y_x \mid x \in D - \{0\}\}$$

of $\mathbb{P}^n(\mathbb{C})$. These can be and will be taken to be real analytic near-complex subvarieties, by choosing a real analytic trivialization f .

For a near-complex subvariety $y = (Y, Y_0, \mathcal{E}_0, \rho_0)$ in $X = \mathbb{P}^n(\mathbb{C})$ with $\dim_{\mathbb{R}} Y_0 = 2(n-d)$, and for a point $y \in Y_0$, we consider the set

$$P(y) = \{L \in \text{Grass}_{\mathbb{C}}(\mathcal{T}_{X,y}, d) \mid L \text{ and } T_{Y_0,y} \text{ intersect properly with multiplicity } +1\},$$

where $\text{Grass}_{\mathbb{C}}(V, d)$ denotes the complex Grassmann variety of d -dimensional vector subspaces of the given vector space V . Since L and $T_{Y_0,y}$ are oriented, we can speak of positivity of the intersection. $P(y)$ is an open set of

$\text{Grass}_{\mathbb{C}}(\mathcal{T}_{X,Y}, d)$ and $B = \bigcup_{y \in Y_0} P(y)$ is an open set of

$$A = \bigcup_{y \in Y_0} \text{Grass}_{\mathbb{C}}(\mathcal{T}_{X,Y}, d).$$

Making use of a suitable metric, we estimate volumes of A and B , and define the positivity rate $r(y)$ of y by

$$r(y) = \text{vol}(B)/\text{vol}(A).$$

On the other hand, we define the absolute degree $\delta(y)$ of y . This is the maximal number of intersection points of Y and the variable linear subvariety L^d of $\mathbb{P}^n(\mathbb{C})$, each intersection point being counted once irrespective of orientation, and maximum being taken for $L \in \text{Grass}(\mathbb{P}^n(\mathbb{C}), d)$ - (a set of measure 0). This number is well defined because we have a real analytic near-complex subvariety.

Going back to our family $\pi: \mathcal{X} \rightarrow D$, $\pi^{-1}(x) \simeq \mathbb{P}^n(\mathbb{C})$ for $x \neq 0$, we can derive the non-negativity of the class of an analytic subvariety of $X_0 = \pi^{-1}(0)$ from the following facts:

Theorem 2 If y is a real analytic near-complex subvariety of $\mathbb{P}^n(\mathbb{C})$ with the property

$$\delta(y)(1 - r(y)) < 1,$$

then y determines a non-negative homology class.

Proposition For the near-complex subvarieties y_x of $\mathbb{P}^n(\mathbb{C})$ described in (*), $\delta(y_x)$ remains bounded and $r(y_x) \rightarrow 1$ for $x \rightarrow 0$. (Hermitean metrics on the fibers $\mathbb{P}^n(\mathbb{C})$ are induced by a fixed one on the total space $\mathcal{X}$.)

Reference

- [1] F.Hirzebruch and K.Kodaira: On the complex projective spaces, Jour. Math. pures appl. (9) vol.36 (1957) pp.201-216