

ERGODIC PROPERTIES OF RANDOM PIECEWISE CONVEX MAPS

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ABSTRACT. In this note, we briefly sketch our current state on the investigation, for random piecewise convex maps, towards ergodic properties: the existence of σ -finite invariant measures absolutely continuous with respect to the Lebesgue measure, their ergodicity, and further limit theorems.

1. INTRODUCTION

Let $X = [0, 1]$, $\mathcal{B}$ be the Borel σ -algebra and λ be the Lebesgue probability measure on $(X, \mathcal{B})$. A piecewise convex map in the sense of Lasota–Yorke [5] is a map $\tau : X \rightarrow X$ such that there is a finite partition $0 = a_0 < a_1 < \dots < a_N = 1$ satisfying

- (1) for each $k = 1, \dots, N$ the restriction $\tau_k := \tau|_{[a_{k-1}, a_k]}$ is continuous and convex in the sense that $\tau_k(tx + (1-t)y) \leq t\tau_k(x) + (1-t)\tau_k(y)$ for any $x, y \in [a_{k-1}, a_k]$ and $0 \leq t \leq 1$;
- (2) $\tau(a_{k-1}) = 0$ for each $k = 1, \dots, N$;
- (3) $\tau'(0) > 1$ and $\tau'(a_{k-1}) > 0$ for each $k = 2, \dots, N$.

Lasota and Yorke showed in [5] the existence of a unique λ -absolutely continuous invariant probability measure μ for τ , where μ is called invariant for τ if $\mu(\tau^{-1}A) = \mu(A)$ holds for any $A \in \mathcal{B}$. Moreover, they showed that the density of μ is bounded and non-increasing, and that μ is exact: $\bigcap_{n \geq 0} \tau^{-n}\mathcal{B} = \{\emptyset, X\} \pmod{\mu}$, which implies asymptotic stability of the system.

In the piecewise C^1 setting, Inoue relaxed their condition (3) to that τ can admit an indifferent fixed point at 0 (i.e., $\tau'(0) = 1$) and critical points (i.e., $\tau'(a_{k-1}) = 0$ for some $k = 2, \dots, N$). He proved that a unique λ -absolutely continuous exact invariant probability measure is still valid under certain assumption (see [2]); otherwise any σ -finite, λ -absolutely continuous ergodic invariant measure is an infinite measure and such an invariant measure exists (see [3]), which is unique up to multiplicative constants. Here, a measure μ is called ergodic if any $E \in \mathcal{B}$ with $\tau^{-1}E = E \pmod{\mu}$ satisfies either $E = \emptyset$ or $X \pmod{\mu}$.

Motivated by their previous researches, we consider random piecewise convex maps as follows. We introduce two probability spaces $(\mathbb{A}, \nu_{\mathbb{A}})$ and $(\mathbb{B}, \nu_{\mathbb{B}})$ as parameter spaces. For each $\alpha \in \mathbb{A}$ and $\beta \in \mathbb{B}$,

a map $T_{\alpha,\beta}$ is assigned which is of the form

$$T_{\alpha,\beta}x = \begin{cases} \tau_\alpha x & (x \in [0, \frac{1}{2}]), \\ S_\beta x & ((\frac{1}{2}, 1]) \end{cases}$$

where $\tau_\alpha : [0, \frac{1}{2}] \rightarrow X$ and $S_\beta : (\frac{1}{2}, 1] \rightarrow X$ are injective and continuous maps with the following conditions:

(0) The map $\mathbb{A} \times \mathbb{B} \times X \ni (\alpha, \beta, x) \mapsto T_{\alpha,\beta}x \in X$ is measurable with respect to each variables;

for $\nu_{\mathbb{A}}$ -almost every $\alpha \in \mathbb{A}$ and $\nu_{\mathbb{B}}$ -almost every $\beta \in \mathbb{B}$,

(1) τ_α and S_β are C^1 -functions and S_β can be extended to a continuous function on $[\frac{1}{2}, 1]$ with

$$\tau_\alpha(0) = 0, \tau_\alpha(\frac{1}{2}) = 1 \text{ and } S_\beta(\frac{1}{2}) = 0;$$

(2) τ'_α and S'_β are non-decreasing with $\tau'_\alpha(0) \geq 1$, $\tau'_\alpha(x) > 1$ for $x \in (0, \frac{1}{2})$, $S'_\beta(\frac{1}{2}) \geq 0$ and $S'_\beta(x) > 0$ for $x \in (\frac{1}{2}, 1)$.

A random piecewise convex map $\{T_{\alpha,\beta}; \nu_{\mathbb{A}}, \nu_{\mathbb{B}} : \alpha \in \mathbb{A}, \beta \in \mathbb{B}\}$ is given by the transition probability

$$\mathbb{P}(x, A) = \int_{\mathbb{A} \times \mathbb{B}} 1_A(T_{\alpha,\beta}x) d\nu_{\mathbb{A}}(\alpha) d\nu_{\mathbb{B}}(\beta)$$

for each $x \in X$ and $A \in \mathcal{B}$, where $T_{\alpha,\beta}$ satisfies the above conditions.

2. MAIN RESULTS

Recall that for a transition probability $\mathbb{P}(x, A)$, a measure μ is called *invariant* if $\int_X \mathbb{P}(x, A) d\mu(x) = \mu(A)$ for any $A \in \mathcal{B}$. Moreover, μ is called *ergodic* if any set $E \in \mathcal{B}$ with $\mathbb{P}(x, E) = 1_E(x)$ μ -almost every $x \in X$ satisfies either $E = \emptyset$ or $X \pmod{\mu}$.

Theorem 2.1 ([4]). *For a random piecewise convex map $\{T_{\alpha,\beta}; \nu_{\mathbb{A}}, \nu_{\mathbb{B}} : \alpha \in \mathbb{A}, \beta \in \mathbb{B}\}$, if*

$$\int_{\mathbb{A}} \frac{1}{\frac{1}{2} - \tau_\alpha^{-1}(\frac{1}{2})} d\nu_{\mathbb{A}}(\alpha) < \infty,$$

then there exists a unique (up to multiple constants) λ -equivalent, σ -finite invariant measure which is ergodic such that

(Decreasing) the density function $\frac{d\mu}{d\lambda}$ is non-increasing;

(Upper bound) for any $\varepsilon > 0$, there is $C > 0$ such that $\frac{d\mu}{d\lambda} \leq C$ on $[\varepsilon, 1]$.

Moreover, if we assume $\text{ess sup}_{\alpha \in \mathbb{A}} \tau'_\alpha(\frac{1}{2}) < \infty$, then we have

(Lower bound) there is $c > 0$ such that $\frac{d\mu}{d\lambda} \geq c$.

Remark 2.1. In [4], we prove that the invariant measure μ in Theorem 2.1 is indeed conservative. Moreover, the asymptotic size of μ close to 0 is also established so that we can tell whether μ is finite or not.

In what follows, we will see two examples of our theorem. The first example is random intermittent maps with a critical point, which is a modified version of random LSV (Liverani–Saussol–Vaienti) maps. The second one is random piecewise linear maps which have no indifferent (on average) fixed

point but exhibit critical-type intermittency (cf. [1]). We can establish for this further statistical properties. For the precise information and motivation, see [7, 8] and references therein.

2.1. Random intermittent maps with a critical point. Let $\mathbb{A} = [\alpha_1, \alpha_2]$ for some $0 < \alpha_1 \leq \alpha_2$, $\mathbb{B} \subset (1, \infty)$ be a compact set, and $\nu_{\mathbb{A}}$ and $\nu_{\mathbb{B}}$ be probability measures. For each $\alpha \in \mathbb{A}$ and $\beta \in \mathbb{B}$, set $\tau_{\alpha} = x + 2^{\alpha}x^{1+\alpha}$ and $S_{\beta} = (2x - 1)^{\beta}$ and consider a random piecewise convex map $\{T_{\alpha, \beta}; \nu_{\mathbb{A}}, \nu_{\mathbb{B}} : \alpha \in \mathbb{A}, \beta \in \mathbb{B}\}$. Note that 0 is an indifferent fixed point for each $\alpha \in \mathbb{A}$ and $\frac{1}{2}$ is a critical point for each $\beta \in \mathbb{B}$. Then by Theorem 2.1 there always exists a λ -equivalent, ergodic, σ -finite invariant measure μ . Furthermore, $\mu(X) = \infty$ if $\nu_{\mathbb{B}}\{\beta \in \mathbb{B} : \alpha_1\beta \geq 1\} > 0$ and $\mu(X) < \infty$ if $\alpha_1 \cdot \max_{\mathbb{B}} \beta < 1$. Roughly speaking, this means that the most expanding branch τ_{α} on the left and the most contracting branch S_{β} on the right determine the statistical law of the system. We remark that when $\mathbb{B} = \{1\}$ and $S_{\beta} = 2x - 1$, that is, for random LSV maps, it holds that $\mu(X) = \infty$ if and only if $\alpha_1 \geq 1$.

2.2. Random piecewise linear maps with low slopes. Let $\mathbb{B} = \mathbb{N}$ and $p_{\beta} = \nu_{\mathbb{B}}(\{\beta\})$ be a point measure on $\mathbb{B}$. We set $\tau_{\alpha} = 2x$ (independent of the choice of $\alpha \in \mathbb{A}$) and $S_{\beta}x = \frac{2x-1}{2^{\beta}}$ and consider a random piecewise convex map $\{T_{\beta}; p_{\beta} : \beta \in \mathbb{N}\}$. Note that this system has no indifferent (even on average) fixed point nor critical point, but it does have strong contracting property (as $\beta \rightarrow \infty$) on average depending on p_{β} . As in Theorem 2.1, the invariant measure μ always exists for any p_{β} . If we choose $p_{\beta} = 2^{-\beta}$ for $\beta \in \mathbb{N}$ then $\mu(X) < \infty$, since the probability of the choice of low slopes (i.e., large β) is exponentially small. However, when we set $p_{\beta} = C\beta^{-t}$ for $\beta \in \mathbb{N}$ where $t > 1$ and C is a normalizing constant, one can see that $\mu(X) = \infty$ if and only if $t \in (1, 2]$. Since we are interested in statistical properties of infinite measure preserving (random) systems, hereafter we assume $t \in (1, 2]$ and set $s := t - 1 \in (0, 1]$. Since each map T_{β} is piecewise linear, we can obtain exact formula of the density function of μ and consequently, we get

$$\mu\left(\left(2^{-(n+1)}, 2^{-n}\right]\right) \propto n^{-s}. \quad (2.1)$$

Then, by the equation (2.1), one can calculate the wandering rate and the asymptotic entrance density for the reference set $[\frac{1}{2}, 1]$ (see [6] for the definitions) corresponding to the skew-product transformation associated with the random map. That is, one can apply Thaler–Zweimüller’s distributional limit theorem in [6] to the system. Let $\{T^{(n)}\}_n$ be a random dynamics given by $T^{(n)} = f_n \circ f_{n-1} \circ \cdots \circ f_1$, where $\{f_n\}_n$ denotes an i.i.d. sequence of random maps such that $f_n = T_{\beta}$ with probability p_{β} for any $n \geq 1$. Then one gets the Darling–Kac law as follows. Below $\xrightarrow[N \rightarrow \infty]{d}$ means the convergence in distribution and $\mathcal{M}_s$ denotes the normalized Mittag-Leffler distribution of order s , which is characterized by $\mathbb{E}[e^{z\mathcal{M}_s}] = \sum_{k=0}^{\infty} \frac{\Gamma(1+s)^k z^k}{\Gamma(1+ks)}$.

Theorem 2.2 ([8]). Fix $s \in (0, 1)$. For any random variable Θ with values in $[\frac{1}{2}, 1]$, which is independent of the random map $\{f_n\}_n$ and whose distribution is absolutely continuous with respect to

λ and for any $E \subset [\frac{1}{2}, 1]$, it holds that

$$\frac{1}{N^s} \sum_{n=0}^{N-1} 1_{\{T^{(n)}(\Theta) \in E\}} \xrightarrow[N \rightarrow \infty]{d} C_s \mu(E) \mathcal{M}_s,$$

where C_s is a positive constant depending only on s .

- Remark 2.2.** (1) Since the wandering rate and the asymptotic entrance density are estimated in the proof of the above theorem, one can also establish the Dynkin–Lamperti arcsine law for waiting times in [6] for this system.
- (2) Even when $s = 1$, the Darling–Kac law and the Dynkin–Lamperti arcsine law above are still valid under a small modification.
- (3) For the case when $\mu(X) < \infty$ (i.e., $t > 2$), it seems not hard to see polynomial decay of (annealed) correlation for appropriate observables. We will investigate them (and more general random piecewise convex maps as in §2.1).

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