

On a certain difference-differential equation
related to nonlinear lattices

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In this note I should like to point out that the solutions
of equations of the form

$$\frac{\psi(u_1+v_1, u_2+v_2)\psi(u_1-v_1, u_2-v_2) - \psi(u_1, u_2)^2}{\psi(u_1, u_2)^2} \\ = \left(F_1^2(v_1) \frac{\partial^2}{\partial u_1^2} + 2X(v_1, v_2) F_1(v_1) F_2(v_2) \frac{\partial^2}{\partial u_1 \partial u_2} + F_2^2(v_2) \frac{\partial^2}{\partial u_2^2} \right) \log \psi(u_1, u_2) \quad (1)$$

are related to multi-soliton solutions of a discrete-time
lattice equation.

Example 1. The function

$$\psi(u) = 1 + e^u$$

satisfies

$$\frac{\psi(u+v)\psi(u-v) - \psi(u)^2}{\psi(u)^2} = F(v) \frac{d^2}{du^2} \log \psi(u)$$

where

$$F(v) = 4 \sinh^2 \frac{v}{2}$$

Example 2. The function

$$\psi(u_1, u_2) = 1 + e^{u_1} + e^{u_2} + B e^{u_1+u_2}$$

satisfies (1) with

$$F_1 = 2 \sinh \frac{v_1}{2}, \quad F_2 = 2 \sinh \frac{v_2}{2} \quad (v_1, v_2 \geq 0)$$

and

$$X = \frac{1}{2(B-1) \sinh \frac{v_1}{2} \sinh \frac{v_2}{2}} \left[B \sinh^2 \frac{v_1+v_2}{2} + \sinh^2 \frac{v_1-v_2}{2} - (B+1) \left\{ \sinh^2 \frac{v_1}{2} + \sinh^2 \frac{v_2}{2} \right\} \right]$$

(1)

Application. Let n be lattice position and t the discrete-time,

and

$$u_1 = 2\eta_1 = 2(P_1 n + \beta_1 t + \eta_1^0)$$

$$u_2 = 2\eta_2 = 2(P_2 n + \beta_2 t + \eta_2^0)$$

and let

$$\psi_n(t) = 1 + e^{2\eta_1} + e^{2\eta_2} + B e^{2\eta_1 + 2\eta_2}$$

We assume a discrete-time lattice equation

$$\delta^{-2} [\psi_n(t+\delta)\psi_n(t-\delta) - \psi_n(t)^2] = \psi_{n+1}(t)\psi_{n-1}(t) - \psi_n(t)^2$$

Then we see that β_i ($i=1,2$) and B are determined as

$$\delta^{-1} \sinh \beta_i \delta = \sinh P_i \quad (i=1,2)$$

$$B = \frac{\sinh^2(P_1 - P_2) - \delta^{-2} \sinh(\beta_1 - \beta_2) \delta}{\delta^{-2} \sinh(\beta_1 + \beta_2) \delta - \sinh^2(P_1 + P_2)}$$

and $\psi_n(t)$ gives a two-soliton solution. $\psi_n(t)$ reduces to a solution for the ordinary exponential lattice in the limit as $t \rightarrow 0$.

Extension 1. As an extension of (1) we have

$$\frac{\psi(u_1 + v_1, u_2 + v_2, \dots) \psi(u_1 - v_1, u_2 - v_2, \dots)}{\psi(u_1, u_2, \dots)^2} = 1 + \sum_{i,j=1}^N E_{ij}(v_1, v_2, \dots) \frac{\partial^2}{\partial u_i \partial u_j} \log \psi(u_1, u_2, \dots)$$

Solutions to this equation will lead us to a N -soliton solution.

Extension 2. As is well known, the elliptic $\mathcal{J}$ -function $\mathcal{J}_3(u)$ satisfies an equation of the form

$$(2)$$

$$\frac{\mathcal{L}_3(u+v)\mathcal{L}_3(u-v)}{\mathcal{L}_3(u)^2} = C(v) + A(v) \frac{d^2}{du^2} \log \mathcal{L}_3(u)$$

and $\mathcal{L}_3$ is related to the simplest periodic solution of the exponential lattice. For the future problem to obtain general periodic solutions of the discrete-time lattice equation, we will have to deal with a certain equation of the form

$$\frac{\psi(u_1+v_1, u_2+v_2, \dots) \psi(u_1-v_1, u_2-v_2, \dots)}{\psi(u_1, u_2, \dots)^2} = C(v_1, v_2, \dots) + \sum_{i,j} E_{ij}(v_1, v_2, \dots) \frac{\partial^2}{\partial u_i \partial u_j} \log \psi(u_1, u_2, \dots)$$

Reference

R. Hirota, J. Phys. Soc. Japan 43 (1977) 2074.