

Symplectic volumes of double weight varieties associated with $SU(3)$, I

By

TARO SUZUKI *

Abstract

We consider double weight varieties, that is, symplectic torus quotients for a direct product of two integral coadjoint orbits of $SU(3)$, and investigate their symplectic volumes. According to a fundamental theorem for symplectic quotients, it is equivalent to studying weight multiplicities in a tensor product of two irreducible representations of $SU(3)$, and their asymptotic behavior. We assume that both of two coadjoint orbits used to define the double weight variety are flag manifolds of $SU(3)$. As a main result, we obtain an explicit formula for the symplectic volumes of double weight varieties.

§ 1. Introduction

Let M be a symplectic manifold with a Hamiltonian action of a torus T and $\Phi : M \rightarrow \mathfrak{t}^*$ its moment map. For any regular value $\mu \in \mathfrak{t}^*$ of Φ , the symplectic quotient at μ is defined by

$$M//_{\mu}T := \Phi^{-1}(\mu)/T.$$

For example, let M be a coadjoint orbit $\mathcal{O}_{\lambda}$ of a compact semi-simple Lie group G through the point $\lambda \in \mathfrak{t}^*$, and consider the action of the maximal torus $T \subset G$ on $\mathcal{O}_{\lambda}$. Then $M//_{\mu}T = \mathcal{O}_{\lambda}//_{\mu}T$ is called a *weight variety*. Many results have been obtained about weight varieties. Knutson [15] studied the relation with weight spaces of an irreducible representation (thereby $\mathcal{O}_{\lambda}//_{\mu}T$ were named weight varieties), Guillemin-Lerman-Sternberg [6] gave some formulas for the volumes of weight varieties, and Goldin [4] gave an explicit expression for the cohomology rings of weight varieties for

Received September 30, 2011. Revised January 13, 2012.

2000 Mathematics Subject Classification(s): 53D20, 22E46.

Key Words: representation; tensor product; weight space; multiplicity; coadjoint orbit; symplectic quotient.

*Department of Mathematics, Graduate School of Chuo University, 1-13-27 Kasuga Bunkyo-ku, Tokyo112-8551, Japan.

e-mail: suzuki@gug.math.chuo-u.ac.jp

© 2013 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

$G = SU(n)$. For compact semi-simple Lie groups except of type A , it is known that weight varieties are orbifolds in general (see [5]).

For $\lambda_1, \lambda_2 \in \mathfrak{t}^*$, let M be a direct product $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ of two coadjoint orbits of G , and consider the diagonal action of $T \subset G$ on $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$. Then $M//_{\mu}T = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T$ is called a *double weight variety*.

In this paper, we consider the case where $G = SU(3)$. Except for the orbit consisting only of the origin, each coadjoint orbit of $SU(3)$ is diffeomorphic to either the flag manifold $SU(3)/T$ or the complex projective space $\mathbb{P}^2$. Our aim is to express the symplectic volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ of the double weight variety in an explicit form.

First, we assume that $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, where P denotes the weight lattice of G and P_+ denotes the set of dominant integral weights of G . As we will discuss in Section 2, under certain conditions on $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, we can express the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ in terms of representations of T . Namely, $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ is equal to

$$\lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}],$$

where V_{λ} denotes the irreducible representation of G with highest weight $\lambda \in P_+$, and for a representation V of T , $[V : W_{\mu}]$ denotes the multiplicity of weight space W_{μ} ($\mu \in P$) in the weight decomposition of V . Moreover, k runs over positive integers and d is the complex dimension of $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T$. Here, the theorem of Guillemin-Sternberg (and its generalization) on the characteristic numbers of symplectic quotients (see, e.g., [8] and [17]) plays the key role, as well as the Borel-Weil theorem and the Hirzebruch-Riemann-Roch theorem. The argument above are essentially the same with those in [20] and [21].

Next, we assume that both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are flag manifolds of $SU(3)$. Namely, we assume $\lambda_1, \lambda_2 \in P_{++}$, where P_{++} denotes the set of dominant integral weights which belong to the interior of the Weyl chamber. The main result in this paper is an explicit formula for the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$. The details of the notation will be given in Sections 2 and 3.

Theorem. (See Theorem 3.4 below.) *Let $\lambda_1, \lambda_2 \in P_{++}$ and $\mu \in P$ satisfy the assumptions (A0), (A1) and (A2) in Section 3.2. Then we have*

$$\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = \sum_{w_1, w_2 \in W} \varepsilon(w_1)\varepsilon(w_2)F(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where W denotes the Weyl group of $SU(3)$ and $F(\lambda_1, \lambda_2, \mu; w_1, w_2)$ is given as follows.

(1) If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12} \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2 \rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, 2\Lambda_1 - \Lambda_2 \rangle.$$

(2) If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12} \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1 \rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, -\Lambda_1 + 2\Lambda_2 \rangle.$$

(3) Otherwise,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

§ 2. Preliminaries

§ 2.1. The representation theory of $SU(3)$ and notation

We review some standard facts about the representation theory of $SU(3)$ in order to fix our notation. We refer to [2] for the generalities on compact Lie groups and their representations.

Let $G = SU(3)$, $\mathfrak{g} = \mathfrak{su}(3)$, T the standard maximal torus of G consisting of diagonal matrices in G , and $\mathfrak{t}$ its Lie algebra. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the duals of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, or $\mathfrak{t}^*$ and $\mathfrak{t}$. Let $W \cong \mathfrak{S}_3$ be the Weyl group of $G = SU(3)$ with respect to T . We define an $\text{Ad}G$ -invariant positive definite inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ by

$$(X, Y) := -\frac{1}{4\pi^2} \text{Tr}(XY) \quad (X, Y \in \mathfrak{g}).$$

We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the inner product $(\cdot, \cdot)$. We regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$ by the identification

$$\mathfrak{t}^* = \{f \in \mathfrak{g}^* | t \cdot f = f \ (\forall t \in T)\}.$$

The elements

$$(2.1) \quad H_1 = 2\pi\sqrt{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = 2\pi\sqrt{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in $\mathfrak{t}$ are generators of the integral lattice $\text{Ker}(\exp : \mathfrak{t} \rightarrow T)$ and form a basis of $\mathfrak{t}$. Under the identification by the inner product, we define the simple roots α_1, α_2 in $\mathfrak{t}^*$ by the elements which correspond to $H_1, H_2 \in \mathfrak{t}$, respectively. Let $\Delta_+ := \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the positive root system. Let us set

$$Q := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2,$$

$$\gamma_1 := \mathbb{R}_{>0}\alpha_1 + \mathbb{R}_{>0}(\alpha_1 + \alpha_2), \quad \gamma_2 := \mathbb{R}_{>0}(\alpha_1 + \alpha_2) + \mathbb{R}_{>0}\alpha_2,$$

$$\bar{\gamma}_1 := \mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2), \quad \bar{\gamma}_2 := \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2) + \mathbb{R}_{\geq 0}\alpha_2.$$

We define the fundamental weights $\Lambda_1, \Lambda_2 \in \mathfrak{t}^*$ by

$$\Lambda_1 := \frac{2\alpha_1 + \alpha_2}{3}, \Lambda_2 := \frac{\alpha_1 + 2\alpha_2}{3}.$$

Under the identification by the inner product, $\Lambda_1, \Lambda_2 \in \mathfrak{t}^*$ correspond to the elements

$$I_1 = \frac{2\pi\sqrt{-1}}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_2 = \frac{2\pi\sqrt{-1}}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

in $\mathfrak{t}$, respectively. Define

$$\begin{aligned} \mathfrak{t}_+^* &:= \mathbb{R}_{\geq 0}\Lambda_1 + \mathbb{R}_{\geq 0}\Lambda_2, \quad \mathfrak{t}_{++}^* := \mathbb{R}_{> 0}\Lambda_1 + \mathbb{R}_{> 0}\Lambda_2, \\ P &:= \mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2, \quad P_+ := \mathbb{Z}_{\geq 0}\Lambda_1 + \mathbb{Z}_{\geq 0}\Lambda_2, \quad P_{++} := \mathbb{Z}_{> 0}\Lambda_1 + \mathbb{Z}_{> 0}\Lambda_2. \end{aligned}$$

The set $\mathfrak{t}_+^*$ is a fundamental domain of the action of the Weyl group W on $\mathfrak{t}^*$. Let us set

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \Lambda_1 + \Lambda_2.$$

According to the representation theory of compact Lie groups, irreducible representations of G are, by assigning their highest weights, in one-to-one correspondence with elements in P_+ . For $\lambda \in P_+$, we denote by V_λ the irreducible representation of G with the highest weight $\lambda \in P_+$, and by $\chi_\lambda : G \rightarrow \mathbb{C}$ the character of V_λ . For $\mu \in P$, we define $e^\mu : T \rightarrow \mathbb{C}$ by $e^\mu(t) := e^{2\pi\sqrt{-1}\langle \mu, X \rangle}$ for $t = \exp X \in T$ ($X \in \mathfrak{t}$). By the Weyl character formula, χ_λ is given as a function on T by

$$\chi_\lambda(t) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}(t)}{e^\rho(t) \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}(t))},$$

where $\varepsilon(w) = \pm 1$ is the signature of $w \in W$.

§ 2.2. Coadjoint orbits

Although we mainly consider the case $G = SU(3)$, most of the following still holds when G is a general compact Lie group. For further details on coadjoint orbits, see, e.g., [13] and [16]. We also refer to [12] for the Borel-Weil theorem.

The left coadjoint action of G on $\mathfrak{g}^*$ is defined by $g \cdot f := \text{Ad}^*(g)f$ for $g \in G$ and $f \in \mathfrak{g}^*$, where

$$\langle \text{Ad}^*(g)f, X \rangle = \langle f, \text{Ad}(g^{-1})X \rangle \quad (X \in \mathfrak{g}).$$

We denote by $\mathcal{O}_\lambda = G \cdot \lambda$ the coadjoint orbit through $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$. Then the intersection $\mathcal{O}_\lambda \cap \mathfrak{t}^*$ is the W -orbit through λ , and $\mathcal{O}_\lambda \cap \mathfrak{t}_+^*$ consists of a single point. In other words, coadjoint orbits are parametrized by elements in $\mathfrak{t}_+^*$.

In particular, for $G = SU(3)$, coadjoint orbits $\mathcal{O}_\lambda$ are classified as follows, where G_λ denotes the isotropy subgroup at $\lambda \in \mathfrak{t}_+^*$ for the coadjoint action of G on $\mathfrak{g}^*$.

- (1) If $\lambda \in \mathfrak{t}_{++}^*$, then $G_\lambda \cong T$ and $\mathcal{O}_\lambda \cong SU(3)/T$.
- (2) If $\lambda \in \mathfrak{t}_+^* - \mathfrak{t}_{++}^* - \{0\}$, then $G_\lambda \cong S(U(1) \times U(2))$ and $\mathcal{O}_\lambda \cong \mathbb{P}^2$.
- (3) If $\lambda = 0$, then $G_\lambda = G$ and $\mathcal{O}_\lambda = \{0\}$.

On each coadjoint orbit $\mathcal{O}_\lambda$, there exists a natural G -invariant symplectic structure ω_λ , called the Kirillov-Kostant-Souriau symplectic form, defined by

$$(\omega_\lambda)_x(\tilde{X}_x, \tilde{Y}_x) = \langle x, [X, Y] \rangle \quad (x \in \mathcal{O}_\lambda, X, Y \in \mathfrak{g}),$$

where $\tilde{X}$ is the vector field on $\mathcal{O}_\lambda$ given by

$$\tilde{X}_x := \left. \frac{d}{dt} \right|_{t=0} (\exp tX) \cdot x.$$

The action of G on $\mathcal{O}_\lambda$ is Hamiltonian and the associated moment map is given by the inclusion $\iota_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$, that is, we have $d\langle \iota_\lambda, X \rangle(\cdot) = \omega_\lambda(\tilde{X}, \cdot)$.

In addition, there exists a G -invariant complex structure J_λ on $\mathcal{O}_\lambda$, which is compatible with the symplectic structure ω_λ , that is, $\omega_\lambda(\cdot, J_\lambda \cdot)$ becomes a Riemannian metric on $\mathcal{O}_\lambda$, and makes $\mathcal{O}_\lambda$ into a Kähler manifold.

Moreover, in the case $\lambda \in P_+$, there exists a G -equivariant holomorphic line bundle L_λ over $\mathcal{O}_\lambda$ such that $c_1(L_\lambda) = [\omega_\lambda]$. The Borel-Weil theorem shows that

$$H^0(\mathcal{O}_\lambda, \mathcal{O}(L_\lambda)) \cong V_\lambda, \quad H^k(\mathcal{O}_\lambda, \mathcal{O}(L_\lambda)) = 0 \quad (k > 0)$$

as representations of G , where $\mathcal{O}(L_\lambda)$ denotes the sheaf of germs of holomorphic sections of L_λ .

Remark 1. (1) For $k \in \mathbb{R}_{>0}$ and $\lambda \in \mathfrak{t}_+^*$, $\mathcal{O}_{k\lambda} = \mathcal{O}_\lambda$ as complex manifolds. If we compare the symplectic forms and the moment maps under this identification, we have $\omega_{k\lambda} = k\omega_\lambda$ and $\iota_{k\lambda} = k\iota_\lambda$. In the case $k \in \mathbb{Z}_{>0}$ and $\lambda \in P_+$, we have $L_{k\lambda} \cong L_\lambda^{\otimes k}$.

(2) For $\lambda \in P_+$, the action on $\mathcal{O}_\lambda$ of the center $Z(G) \cong \mathbb{Z}/3\mathbb{Z}$ of $G = SU(3)$ is trivial, while that on L_λ is not trivial in general. However, by the construction of L_λ (see, e.g., [12]), if we suppose $\lambda \in P_+ \cap Q$, this action becomes trivial, too.

§ 2.3. Double weight varieties

For general properties of symplectic and Kähler quotients, see, e.g., [10], [14] and [18]. The following still holds for a general compact Lie group G .

Let $\lambda_1, \lambda_2 \in \mathfrak{t}_+^*$. The diagonal action of the maximal torus T of G on the direct product $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$ is also Hamiltonian and the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \rightarrow \mathfrak{t}^*$ is

given by the composition of the map

$$\begin{aligned}\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} &\rightarrow \mathfrak{g}^* \\ (x_1, x_2) &\rightarrow x_1 + x_2\end{aligned}$$

and the projection $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$. For $\mu \in \mathfrak{t}^*$, we define the symplectic (or Kähler) quotient at μ by

$$(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T := \Phi^{-1}(\mu) / T.$$

Here we assume that for $\mu \in \mathfrak{t}^*$,

$$(a0) \quad \Phi^{-1}(\mu) \neq \emptyset,$$

$$(a1) \quad \mu \text{ is a regular value of the moment map } \Phi, \text{ and } (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T \text{ is a smooth manifold.}$$

Then there exist a natural symplectic structure $\omega = \omega(\lambda_1, \lambda_2, \mu)$ and a compatible complex structure J on $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T$, induced from those on $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$, which make $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T$ a Kähler manifold. The complex dimension d of $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T$ is

$$d = \dim_{\mathbb{R}} G - \frac{1}{2}(\dim_{\mathbb{R}} G_{\lambda_1} + \dim_{\mathbb{R}} G_{\lambda_2}) - \dim_{\mathbb{R}} T.$$

We call $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T$ a *double weight variety*.

Now, suppose $\lambda_1, \lambda_2 \in P_+$. Let L_{λ_i} be the T -equivariant holomorphic line bundle over $\mathcal{O}_{\lambda_i}$ as in Section 2.2, and let us set

$$\begin{aligned}\mathcal{L} &= (L_{\lambda_1} \boxtimes L_{\lambda_2}) //_{\mu} T := ((\mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2})|_{\Phi^{-1}(\mu)}) / T, \\ \mathcal{L}_i &= L_{\lambda_i} //_{\mu} T := (\mathrm{pr}_i^* L_{\lambda_i}|_{\Phi^{-1}(\mu)}) / T \quad (i = 1, 2),\end{aligned}$$

where

$$\mathrm{pr}_i : \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \rightarrow \mathcal{O}_{\lambda_i} \quad (i = 1, 2)$$

is the i -th projection.

By the assumption (a1), the isotropy subgroup at each point in $\Phi^{-1}(\mu)$ is a finite group. Since its action on $\mathrm{pr}_i^* L_{\lambda_i}$ ($i = 1, 2$) and $L_{\lambda_1} \boxtimes L_{\lambda_2} = \mathrm{pr}_1^* L_{\lambda_1} \otimes \mathrm{pr}_2^* L_{\lambda_2}$ may not be trivial, $\mathcal{L}$ and $\mathcal{L}_i$ ($i = 1, 2$) are, in general, orbifold holomorphic line bundles over $(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T$. We assume that

$$(a2) \quad \mathcal{L}_i \text{ is a genuine holomorphic line bundle over } (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T \text{ for } i = 1, 2.$$

Then we have $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $c_1(\mathcal{L}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) = [\omega(\lambda_1, \lambda_2, \mu)]$.

In general, let (M, ω) be a compact Kähler manifold and suppose that there exists a holomorphic line bundle L over M such that $c_1(L) = [\omega]$. In this case, we define

$$\chi(M, L) := \sum_i (-1)^i H^i(M, \mathcal{O}(L))$$

as a virtual vector space. The dimension $\dim_{\mathbb{C}} \chi(M, L)$ is called the Riemann-Roch number of (M, L) . We define the symplectic volume $\text{vol}(M)$ of (M, ω) by

$$\text{vol}(M) := \int_M \frac{\omega^d}{d!},$$

where d is the complex dimension of M .

Lemma 2.1. *Suppose that (M, ω) and L satisfy the assumptions above. Then we have*

$$\text{vol}(M) = \lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi(M, L^{\otimes k}).$$

Proof. By the Hirzebruch-Riemann-Roch theorem, we have

$$\dim_{\mathbb{C}} \chi(M, L) = \int_M \text{Ch}(L) \text{Td}(M) = \int_M e^{c_1(L)} \text{Td}(M),$$

where $\text{Ch}(L)$ is the Chern character of L and $\text{Td}(M)$ is the Todd class of M . Hence we have

$$\lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi(M, L^{\otimes k}) = \lim_{k \rightarrow \infty} \int_M \frac{e^{kc_1(L)}}{k^d} \text{Td}(M) = \int_M \frac{c_1(L)^d}{d!} = \text{vol}(M).$$

□

In addition, we assume that a torus T acts holomorphically on M , and this action lifts to $L \rightarrow M$. Then the action of T on M is Hamiltonian, and the moment map $\Phi : M \rightarrow \mathfrak{t}^*$ is determined. For suitably chosen $\mu \in P$, we obtain the Kähler manifold $M//_{\mu}T = \Phi^{-1}(\mu)/T$, the line bundle $L//_{\mu}T = (L|_{\Phi^{-1}(\mu)})/T$ over $M//_{\mu}T$ and $\chi(M//_{\mu}T, L//_{\mu}T)$ (see, e.g., [8] and [17]).

Remark 2. (1) It follows from the multiplicative property of χ (see the appendix in [11]) that $\chi(\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}, L_{\lambda_1} \boxtimes L_{\lambda_2}) = \chi(\mathcal{O}_{\lambda_1}, L_{\lambda_1}) \otimes \chi(\mathcal{O}_{\lambda_2}, L_{\lambda_2})$.

(2) As in Remark 1 (1), $(\mathcal{O}_{k\lambda_1} \times \mathcal{O}_{k\lambda_2})//_{k\mu}T = (\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T$ as complex manifolds, and $\omega(k\lambda_1, k\lambda_2, k\mu) = k\omega(\lambda_1, \lambda_2, \mu)$ for $k \in \mathbb{R}_{>0}$, $\lambda_1, \lambda_2 \in \mathfrak{t}_+^*$ and $\mu \in \mathfrak{t}^*$. In particular, it follows that $\text{vol}((\mathcal{O}_{k\lambda_1} \times \mathcal{O}_{k\lambda_2})//_{k\mu}T) = k^d \cdot \text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$. In the case $k \in \mathbb{Z}_{>0}$, $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, we have $(L_{k\lambda_1} \boxtimes L_{k\lambda_2})//_{k\mu}T = ((L_{\lambda_1} \boxtimes L_{\lambda_2})//_{\mu}T)^{\otimes k}$.

(3) Even if $\lambda_1, \lambda_2 \in P_+$ does not satisfy (a2), $n\lambda_1$ and $n\lambda_2$ does satisfy (a2) for some positive integer n . Hence, as far as the symplectic volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ is concerned, we can assume (a2) without loss of generality.

On the other hand, we can regard $\chi(M, L)$ as a representation of T . Then we obtain the weight decomposition

$$\chi(M, L) = \sum_{\mu \in P} m_{\mu} W_{\mu},$$

where W_μ denotes the weight space associated with $\mu \in P$ and

$$m_\mu = [\chi(M, L) : W_\mu] := \dim_{\mathbb{C}}(\chi(M, L) \otimes W_\mu^*)^T$$

is the weight multiplicity of W_μ . Besides, for a representation V of T ,

$$V^T := \{v \in V \mid t \cdot v = v \ (\forall t \in T)\}$$

is the set of invariants in V .

The Guillemin-Sternberg theorem and its generalization (see, e.g., [8] and [17]) tell us the following.

Theorem 2.2. *Assume that the action of T on $\Phi^{-1}(\mu)$ is free. Then we have*

$$\dim_{\mathbb{C}} \chi(M//_{\mu}T, L//_{\mu}T) = \dim_{\mathbb{C}}(\chi(M, L) \otimes W_\mu^*)^T = [\chi(M, L) : W_\mu].$$

In our situation, we obtain the following.

Proposition 2.3. *Suppose that $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$ satisfy the assumptions (a0), (a1) and (a2). Then we have*

- (1) $\dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T, (L_{\lambda_1} \boxtimes L_{\lambda_2})//_{\mu}T) = [V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu],$
- (2) $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) = \lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}].$

Proof. (1) It follows from Theorem 2.2, Remark 2 (1), and the Borel-Weil theorem.

(2) By Lemma 2.1, Remark 2 (2), and the assertion (1), we obtain

$$\begin{aligned} & \text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T, ((L_{\lambda_1} \boxtimes L_{\lambda_2})//_{\mu}T)^{\otimes k}) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot \dim_{\mathbb{C}} \chi((\mathcal{O}_{k\lambda_1} \times \mathcal{O}_{k\lambda_2})//_{k\mu}T, (L_{k\lambda_1} \boxtimes L_{k\lambda_2})//_{k\mu}T) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k^d} \cdot [V_{k\lambda_1} \otimes V_{k\lambda_2} : W_{k\mu}]. \end{aligned}$$

□

In the next section, we will compute the Rimann-Roch number $\dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T, (L_{\lambda_1} \boxtimes L_{\lambda_2})//_{\mu}T)$ and the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu}T)$ for $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$.

§ 3. Main results

§ 3.1. Combinatorial expression

For $\lambda \in P_+$, let us set

$$\chi_\lambda := \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)}}{e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})} \in \mathbb{C}[e^{\Lambda_1}, e^{\Lambda_2}][[e^{-\Lambda_1}, e^{-\Lambda_2}]]$$

and define

$$F_{\lambda_1, \lambda_2}^\mu := \chi_{\lambda_1} \cdot \chi_{\lambda_2} \cdot e^{-\mu}$$

for $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$. As in the Weyl character formula, we also regard them as functions on T .

Lemma 3.1. *For $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$, $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu]$ is equal to the coefficient of e^0 (i.e., the constant term) in $F_{\lambda_1, \lambda_2}^\mu$.*

Proof. Let $d\mu_T$ be the normalized invariant measure on T . Then we have

$$(3.1) \quad [V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = \int_T \chi_{\lambda_1}(t) \chi_{\lambda_2}(t) e^{-\mu}(t) d\mu_T.$$

Let H_1, H_2 be the basis of the integral lattice in $\mathfrak{t}$ as (2.1). we write an element $t \in T$ as $t = \exp(x_1 H_1 + x_2 H_2)$ with $x_1, x_2 \in [0, 1]$. Let us set $u_i = e^{2\pi\sqrt{-1}x_i}$ ($i = 1, 2$) and define an isomorphism $T \cong U(1)^2$ by $t \mapsto (u_1, u_2)$. Then we have

$$d\mu_T = dx_1 dx_2 = \frac{du_1}{2\pi\sqrt{-1}u_1} \frac{du_2}{2\pi\sqrt{-1}u_2}.$$

Hence (3.1) is equal to the coefficient of $u_1^0 u_2^0$ in $F_{\lambda_1, \lambda_2}^\mu(t)$, which is regarded as a Laurent series of (u_1, u_2) . If we write $F_{\lambda_1, \lambda_2}^\mu$ as

$$F_{\lambda_1, \lambda_2}^\mu = \sum C_{m_1, m_2} e^{m_1 \Lambda_1 + m_2 \Lambda_2},$$

then we have

$$F_{\lambda_1, \lambda_2}^\mu(u_1, u_2) = \sum C_{m_1, m_2} u_1^{m_1} u_2^{m_2}.$$

Therefore, (3.1) is equal to the coefficient of e^0 in $F_{\lambda_1, \lambda_2}^\mu$. □

Proposition 3.2. *Let $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$. Then we have*

$$[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = \sum_{w_1, w_2 \in W} \varepsilon(w_1) \varepsilon(w_2) E(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where for $w_1, w_2 \in W$, we define

$$E(\lambda_1, \lambda_2, \mu; w_1, w_2) = \sum_{(j_1, j_2, j_3)} (j_1 + 1)(j_2 + 1)(j_3 + 1)$$

and the sum is taken over all $(j_1, j_2, j_3) \in (\mathbb{Z}_{\geq 0})^3$ which satisfy the condition

$$(3.2) \quad w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho - j_1\alpha_1 - j_2\alpha_2 - j_3(\alpha_1 + \alpha_2) = 0.$$

Proof. Applying the generalized binomial theorem to $(e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}))^{-2}$ in $F_{\lambda_1, \lambda_2}^\mu$, we have a power series expansion

$$\begin{aligned}
F_{\lambda_1, \lambda_2}^\mu &= \sum_{w_1, w_2 \in W} \sum_{(j_1, j_2, j_3)} \varepsilon(w_1) \varepsilon(w_2) (-1)^{j_1 + j_2 + j_3} \binom{-2}{j_1} \binom{-2}{j_2} \binom{-2}{j_3} \\
&\quad \cdot e^{w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - j_1 \alpha_1 - j_2 \alpha_2 - j_3(\alpha_1 + \alpha_2) - 2\rho} \\
&= \sum_{w_1, w_2 \in W} \sum_{(j_1, j_2, j_3)} \varepsilon(w_1) \varepsilon(w_2) \binom{j_1 + 1}{1} \binom{j_2 + 1}{1} \binom{j_3 + 1}{1} \\
&\quad \cdot e^{w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - j_1 \alpha_1 - j_2 \alpha_2 - j_3(\alpha_1 + \alpha_2) - 2\rho} \\
&= \sum_{w_1, w_2 \in W} \sum_{(j_1, j_2, j_3)} \varepsilon(w_1) \varepsilon(w_2) (j_1 + 1)(j_2 + 1)(j_3 + 1) \\
&\quad \cdot e^{w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - j_1 \alpha_1 - j_2 \alpha_2 - j_3(\alpha_1 + \alpha_2) - 2\rho}.
\end{aligned}$$

Hence our claim follows from Proposition 3.1. $\square$

§ 3.2. Formulas for the Rimann-Roch number and the volume

In the following, we assume that $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$ satisfy the following three conditions.

(A0) $\mu \in \text{conv.}\{w_1 \lambda_1 + w_2 \lambda_2 | w_1, w_2 \in W\}$,

(A1) For any $w_1, w_2 \in W$,

$$\begin{aligned}
&\langle w_1 \lambda_1 + w_2 \lambda_2 - \mu, \Lambda_1 \rangle \neq 0, \\
&\langle w_1 \lambda_1 + w_2 \lambda_2 - \mu, \Lambda_2 \rangle \neq 0, \\
&\langle w_1 \lambda_1 + w_2 \lambda_2 - \mu, \Lambda_1 \rangle \neq \langle w_1 \lambda_1 + w_2 \lambda_2 - \mu, \Lambda_2 \rangle,
\end{aligned}$$

(A2) $\lambda_1, \lambda_2, \mu \in Q$,

where $\text{conv.}\{w_1 \lambda_1 + w_2 \lambda_2 | w_1, w_2 \in W\}$ denotes the convex full of $\{w_1 \lambda_1 + w_2 \lambda_2 | w_1, w_2 \in W\}$.

Remark 3. We mention the relation between the conditions (a0)–(a2) and the conditions (A0)–(A2). By the convexity theorem of Hamiltonian torus actions on symplectic manifolds (see, e.g., [1], [6] and [7]), (A0) implies (a0), and (A1) implies the former part of (a1). Moreover, for $G = SU(3)$, the former part of (a1) implies the latter part of (a1). Besides, by Remark 1 (2), (A2) implies (a2). As we noted in Remark 2 (3), as far as the symplectic volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T)$ is concerned, we may assume (A2) without loss of generality.

Under the conditions above, let us concretely compute $E(\lambda_1, \lambda_2, \mu; w_1, w_2)$ in Proposition 3.2. The condition (3.2) in Proposition 3.2 means that $(j_1, j_2, j_3) \in \mathbb{Z}^3$ satisfies

$$\begin{cases} j_1 = \langle w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho, \Lambda_1 \rangle - j_3 \geq 0, \\ j_2 = \langle w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho, \Lambda_2 \rangle - j_3 \geq 0, \\ j_3 \geq 0. \end{cases}$$

We write

$$\begin{aligned} A &= \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1 \rangle, \quad B = \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2 \rangle, \\ C &= \langle w_1\rho + w_2\rho, \Lambda_1 \rangle, \quad D = \langle w_1\rho + w_2\rho, \Lambda_2 \rangle \end{aligned}$$

for brevity. Note that $\langle \rho, \Lambda_1 \rangle = \langle \rho, \Lambda_2 \rangle = 1$. Then the condition (3.2) means that $(j_1, j_2, j_3) \in \mathbb{Z}^3$ satisfies

$$\begin{cases} j_1 = A + C - 2 - j_3 \geq 0, \\ j_2 = B + D - 2 - j_3 \geq 0, \\ j_3 \geq 0. \end{cases}$$

Therefore we consider the following cases. Recall that

$$\bar{\gamma}_1 = \mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2), \quad \bar{\gamma}_2 = \mathbb{R}_{\geq 0}(\alpha_1 + \alpha_2) + \mathbb{R}_{\geq 0}\alpha_2.$$

Case 1. Suppose that $A + C - 2 \geq 0$ and $B + D - 2 \geq 0$.

(1-1) If $A + C \geq B + D$, that is, $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma}_1$, then we have

$$\begin{aligned} &E(\lambda_1, \lambda_2, \mu; w_1, w_2) \\ &= \sum_{j_3=0}^{B+D-2} (A + C - 1 - j_3)(B + D - 1 - j_3)(j_3 + 1) \\ &= \frac{1}{12}(B + D - 1)(B + D)(B + D + 1)(2(A + C) - (B + D)). \end{aligned}$$

(1-2) If $A + C \leq B + D$, that is, $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma}_2$, then we have

$$\begin{aligned} &E(\lambda_1, \lambda_2, \mu; w_1, w_2) \\ &= \sum_{j_3=0}^{A+C-2} (A + C - 1 - j_3)(B + D - 1 - j_3)(j_3 + 1) \\ &= \frac{1}{12}(A + C - 1)(A + C)(A + C + 1)(-(A + C) + 2(B + D)). \end{aligned}$$

Case 2. Suppose that $A + C - 2 < 0$ or $B + D - 2 < 0$.

We have $E(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0$.

Combining all the results above, we obtain the following explicit formula for the Rimann-Roch number $\dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu} T, (L_{\lambda_1} \boxtimes L_{\lambda_2})//_{\mu} T)$.

Proposition 3.3. *Let $\lambda_1, \lambda_2 \in P_+$ and $\mu \in P$ satisfy the assumptions (A0), (A1) and (A2). Then we have*

$$\dim_{\mathbb{C}} \chi((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu} T, (L_{\lambda_1} \boxtimes L_{\lambda_2})//_{\mu} T) = \sum_{w_1, w_2 \in W} \varepsilon(w_1) \varepsilon(w_2) E(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where $E(\lambda_1, \lambda_2, \mu; w_1, w_2)$ is given as follows. We write

$$\begin{aligned} A &= \langle w_1 \lambda_1 + w_2 \lambda_2 - \mu, \Lambda_1 \rangle, \quad B = \langle w_1 \lambda_1 + w_2 \lambda_2 - \mu, \Lambda_2 \rangle, \\ C &= \langle w_1 \rho + w_2 \rho, \Lambda_1 \rangle, \quad D = \langle w_1 \rho + w_2 \rho, \Lambda_2 \rangle \end{aligned}$$

for brevity.

(1) If $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma}_1$,

$$\begin{aligned} E(\lambda_1, \lambda_2, \mu; w_1, w_2) \\ = \frac{1}{12} (B + D - 1)(B + D)(B + D + 1)(2(A + C) - (B + D)). \end{aligned}$$

(2) If $w_1(\lambda_1 + \rho) + w_2(\lambda_2 + \rho) - \mu - 2\rho \in \bar{\gamma}_2$,

$$\begin{aligned} E(\lambda_1, \lambda_2, \mu; w_1, w_2) \\ = \frac{1}{12} (A + C - 1)(A + C)(A + C + 1)(-(A + C) + 2(B + D)). \end{aligned}$$

(3) Otherwise,

$$E(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

In the following, we assume that both of two coadjoint orbits $\mathcal{O}_{\lambda_1}$ and $\mathcal{O}_{\lambda_2}$ are flag manifolds of $SU(3)$. Namely, we assume $\lambda_1, \lambda_2 \in P_{++}$. Combining Propositions 2.3 and 3.3, we obtain the following formula for the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu} T)$.

Theorem 3.4. *Let $\lambda_1, \lambda_2 \in P_{++}$ and $\mu \in P$ satisfy the assumptions (A0), (A1) and (A2). Then we have*

$$(3.3) \quad \text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu} T) = \sum_{w_1, w_2 \in W} \varepsilon(w_1) \varepsilon(w_2) F(\lambda_1, \lambda_2, \mu; w_1, w_2),$$

where $F(\lambda_1, \lambda_2, \mu; w_1, w_2)$ is given as follows.

(1) If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12} \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_2 \rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, 2\Lambda_1 - \Lambda_2 \rangle.$$

(2) If $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = \frac{1}{12} \langle w_1\lambda_1 + w_2\lambda_2 - \mu, \Lambda_1 \rangle^3 \langle w_1\lambda_1 + w_2\lambda_2 - \mu, -\Lambda_1 + 2\Lambda_2 \rangle.$$

(3) Otherwise,

$$F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0.$$

Proof. We first note that for $\lambda_1, \lambda_2 \in P_{++}$ and $\mu \in P$ satisfying the assumptions (A0), (A1) and (A2), we have

$$d = \dim_{\mathbb{C}}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}) //_{\mu} T) = 4.$$

According to Propositions 2.3 and 3.3, we must compute $E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2)$. In this case, the condition in (1) of Proposition 3.3 means that $kB + D - 2 \geq 0$ and

$$\begin{aligned} 0 &\leq \langle w_1(k\lambda_1 + \rho) + w_2(k\lambda_2 + \rho) - 2\rho - k\mu, \Lambda_1 - \Lambda_2 \rangle \\ &= k(A - B) + (C - D). \end{aligned}$$

Let us take k large enough. By the assumption (A1), these inequalities above means that $A > B > 0$, that is, $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_1$. Then we have

$$\begin{aligned} E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2) &= \frac{1}{12} (kB + D - 1)(kB + D)(kB + D + 1)(2(kA + C) - (kB + D)) \\ &= \frac{k^4}{12} B^3(2A - B) + (\text{lower terms of } k). \end{aligned}$$

Similarly, when $k \gg 0$, the condition in (2) of Proposition 3.3 means that $B > A > 0$, that is, $w_1\lambda_1 + w_2\lambda_2 - \mu \in \gamma_2$. Then we have

$$\begin{aligned} E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2) &= \frac{1}{12} (kA + C - 1)(kA + C)(kA + C + 1)(-(kA + C) + 2(kB + D)) \\ &= \frac{k^4}{12} A^3(-A + 2B) + (\text{lower terms of } k). \end{aligned}$$

Finally, when $k \gg 0$, the condition in (3) of Proposition 3.3 means that $A < 0$ or $B < 0$. Then we have

$$E(k\lambda_1, k\lambda_2, k\mu; w_1, w_2) = 0.$$

Combining all the results above and Proposition 2.3, we obtain our claim. $\square$

Remark 4. (1) In the case where λ_1 or λ_2 is in $P_+ - P_{++}$, although $F(\lambda_1, \lambda_2, \mu; w_1, w_2) \neq 0$ in general, the right hand side of the formula (3.3) is always equal to 0. We need another consideration for this. We will discuss it in the forthcoming paper [19].

(2) In general, for an element μ in the weight lattice P , there exist $w \in W$ and $\mu' \in P_+$ such that $\mu = w\mu'$. Since $[V_{\lambda_1} \otimes V_{\lambda_2} : W_\mu] = [V_{\lambda_1} \otimes V_{\lambda_2} : W_{\mu'}]$, we have $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) = \text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_{\mu'} T)$. But the computation of the right hand side of the formula (3.3) for $\mu' \in P_+$ becomes more simpler than that for $\mu \in P$.

Now, let $\lambda_1, \lambda_2 \in (P \otimes \mathbb{Q}) \cap \mathfrak{t}_{++}^*$ and $\mu \in P \otimes \mathbb{Q}$ satisfy the conditions (A0) and (A1). Then there exists $n \in \mathbb{Z}_{>0}$ such that $n\lambda_1, n\lambda_2 \in P_{++}$ and $n\mu \in P$. Therefore it follows from Remark 2 (2) and Proposition 2.3 that the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ is given by the right hand side of the formula (3.3). Furthermore, by continuity of the symplectic volume of symplectic quotients (see, e.g., [3] and [9]), we obtain the following.

Corollary 3.5. *Suppose that $\lambda_1, \lambda_2 \in \mathfrak{t}_{++}^*$ and $\mu \in \mathfrak{t}^*$ satisfy the conditions (A0) and (A1) in Section 3.2. Then the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ is given by the right hand side of the formula (3.3).*

§ 4. An example

As a simple example of Theorem 3.4 or Corollary 3.5, let us consider the case where μ is sufficiently close to $\lambda_1 + \lambda_2$. In this case, we have $F(\lambda_1, \lambda_2, \mu; w_1, w_2) = 0$ unless the case $w_1 = w_2 = e$. Hence we have

$$\begin{aligned} \text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) &= F(\lambda_1, \lambda_2, \mu; w_1 = e, w_2 = e) \\ &= \begin{cases} \frac{1}{12} \langle \lambda_1 + \lambda_2 - \mu, \Lambda_2 \rangle^3 \langle \lambda_1 + \lambda_2 - \mu, 2\Lambda_1 - \Lambda_2 \rangle & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12} \langle \lambda_1 + \lambda_2 - \mu, \Lambda_1 \rangle^3 \langle \lambda_1 + \lambda_2 - \mu, -\Lambda_1 + 2\Lambda_2 \rangle & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases} \end{aligned}$$

Now if we write $\lambda_1 = p\alpha_1 + q\alpha_2, \lambda_2 = r\alpha_1 + s\alpha_2 \in P_{++}$ and $\mu = x\alpha_1 + y\alpha_2 \in P$, we can express the volume $\text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T)$ as a polynomial of p, q, r, s, x, y as follows.

$$\begin{aligned} \text{vol}((\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2})//_\mu T) &= \begin{cases} \frac{1}{12} (q + s - y)^3 (2p - q + 2r - s - 2x + y) & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_1), \\ \frac{1}{12} (p + r - x)^3 (-p + 2q - r + 2s + x - 2y) & (\text{if } \lambda_1 + \lambda_2 - \mu \in \gamma_2). \end{cases} \end{aligned}$$

ACKNOWLEDGEMENTS. The author is grateful to Professor Tatsuru Takakura for many discussion and powerful encouragement.

References

- [1] M. Atiyah, Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1–15.
- [2] T. Bröcker and T. Tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Math. **98**, Springer, 1995.
- [3] J. J. Duistermaat and G. J. Heckman, On the variation in the cohomology of the symplectic form of the reduced phase space, *Invent. Math.* **69** (1982), 259–268.
- [4] R. F. Goldin, The cohomology ring of weight varieties and polygon spaces, *Adv. Math.* **160** (2001), 175–204.
- [5] R. Goldin, T. S. Holm, and A. Knutson, Orbifold cohomology of torus quotients, *Duke Math. J.* **139** (2007), 89–139.
- [6] V. Guillemin, E. Lerman, and S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge Univ. Press, 1996.
- [7] V. Guillemin and S. Sternberg, Convexity properties of the moment mapping, *Invent. Math.* **67** (1982), 491–513.
- [8] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.
- [9] V. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Springer, 1999.
- [10] V. Guillemin and S. Sternberg, *Symplectic techniques in Physics*, Cambridge Univ. Press, 1990.
- [11] F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Springer, 1966.
- [12] A. Huckleberry, *Introduction to group actions in symplectic and complex geometry*, Infinite Dimensional Kähler Manifolds, D.M.V.Seminar **31**, Birkhäuser, 2001.
- [13] A. A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Math. **64**, Amer. Math. Soc., 2004.
- [14] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes **31**, Princeton Univ. Press, 1984.
- [15] A. Knutson, Weight varieties, MIT Ph.D. thesis (1996).
- [16] B. Kostant, *Quantization and unitary representation I: Prequantization*, Lecture Notes in Math. **170**, 87–207, Springer, 1970.
- [17] E. Meinrenken, On Riemann-Roch formulas for multiplicities, *J. Amer. Math. Soc.* **9** (1996), no. 2, 373–389.
- [18] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Springer, 1994.
- [19] T. Suzuki, Symplectic volumes of double weight varieties associated with $SU(3)$, II, preprint.
- [20] T. Suzuki and T. Takakura, Symplectic volumes of certain symplectic quotients associated with the special unitary group of degree three, *Tokyo J. Math.*, **31** (2008), 1–26.
- [21] T. Suzuki and T. Takakura, Asymptotic dimension of invariant subspace in tensor product representation of compact Lie group, *J. Math. Soc. Japan*, **61** (2009), 921–969.