

A critical blow-up exponent in a three-dimensional chemotaxis-May–Nowak model

Yuya Tanaka

Department of Mathematical Sciences,
Kwansei Gakuin University

1. Introduction

This paper is based on a joint work with Mario Fuest (Leibniz University Hannover). The May–Nowak model

$$\begin{cases} u_t = -u - K uw + \kappa, & t > 0, \\ v_t = -v + K uw, & t > 0, \\ w_t = -w + v, & t > 0 \end{cases}$$

was introduced as one of epidemic models (e.g. HIV infection) by Nowak and May [14], where $K > 0$ and $\kappa \geq 0$ are constants, and the functions u and v model the number of uninfected and infected cells, respectively, and w denotes the concentration of virus particles. Global properties of solutions for this model have been obtained in [11].

By biological experiments it was confirmed that the cells and the virus have random spatial movements, and the uninfected cells are attracted by high concentrations of the infected cells ([8, 12, 13]). In line with this reason, the chemotaxis-May–Nowak model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - u - K uw + \kappa, & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + K uw, & x \in \Omega, t > 0, \\ w_t = \Delta w - w + v, & x \in \Omega, t > 0 \end{cases} \quad (1.1)$$

was proposed in [15], where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a smooth bounded domain, and $\chi > 0$ and $\tau \in \{0, 1\}$ are constants. The term $-\chi \nabla \cdot (u \nabla v)$ represents the chemotactic attraction of the uninfected cells. Systems with such a term, that is, so-called chemotaxis systems have been introduced by Keller and Segel [10], and also, there are a lot of works on boundedness and finite-time blow-up of solutions in chemotaxis systems (see the survey [1]). Similarly, as to the model (1.1) results not only on boundedness but also on finite-time blow-up were obtained; in the case that $n = 1$ and $\tau = 1$ solutions are always global and bounded ([20]); in the case that $n \geq 2$ and $\tau = 1$ global boundedness of solutions were established under a smallness condition for χ ([2]); on the other hand, if χ is large, then finite-time blow-up occurs in the radial setting in the case that $n \in \{2, 3\}$ and $\tau = 0$ ([19]).

Moreover, to consider more realistic situations, some modified models were proposed and investigated in [4, 9]. In particular, in [4] the model (1.1) with $\tau = 1$ and with $K uw$ replaced by $K u^\alpha w$ ($\alpha > 0$) was considered and boundedness of solutions was shown under the condition $\alpha < \frac{2}{n}$. For this result we naturally have the following question:

Does the solution remain bounded in the case $\alpha \geq \frac{2}{n}$?

In this paper we give an answer for this question in the three-dimensional case.

Problem and main result. We consider the parabolic–elliptic–parabolic version of the model in [4],

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) - u - K u^\alpha w, & x \in B_R, \ t > 0, \\ 0 = \Delta v - v + K u^\alpha w, & x \in B_R, \ t > 0, \\ w_t = \Delta w - w + v, & x \in B_R, \ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial B_R, \ t > 0, \\ u(x, 0) = u_0(x), \ w(x, 0) = w_0(x), & x \in B_R, \end{cases} \quad (1.2)$$

where $B_R \subset \mathbb{R}^3$ is a ball with some $R > 0$; $K, \alpha > 0$ are constants; ν is the outward normal vector to ∂B_R ; $u_0, w_0 \in C^0(\overline{B_R})$ are nonnegative initial data. The main result reads as follows.

Theorem 1.1 ([7]). *Let $B_R \subset \mathbb{R}^3$ be a ball with some $R > 0$, and let $K > 0$. Assume that*

$$\frac{2}{3} < \alpha \leq 1.$$

Then for all radially symmetric positive function $w_0 \in C^0(\overline{B_R})$ and all $m > 0$, $m_1 \in (0, m)$ there exist $r_1 \in (0, R)$ and $\eta > 0$ with the following property: Whenever $u_0 \in C^0(\overline{B_R})$ is radially symmetric and nonnegative and satisfies

$$\int_{B_R} u_0(x) dx = m \quad \text{and} \quad \int_{B_{r_1}} u_0(x) dx \geq m_1$$

as well as

$$u_0(x) \leq m|x|^{-3(3\alpha-1)-\eta} \quad \text{for all } x \in B_R, \quad (1.3)$$

there exist $T_{\max} \in (0, \infty)$ and a unique nonnegative classical solution (u, v, w) of (1.2) blowing up in finite time in the sense that

$$\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(B_R)} = \infty.$$

Remark 1.1. This theorem gives us the answer such that in the three-dimensional case the value $\alpha = \frac{2}{3}$ is the critical exponent distinguishing between boundedness and finite-time blow-up of solutions.

Our approach is based on an analysis of the moment-type functional ϕ defined by $\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) \left[\int_0^{s^{1/3}} \rho^2 u(\rho, t) d\rho \right] ds$ for $t \in (0, T_{\max})$ with some $s_0 \in (0, R^3)$ and $\gamma \in (0, 1)$, which was used in studies of finite-time blow-up for chemotaxis systems with logistic source (see e.g. [16, 17, 18, 21]). The goal is to establish the differential inequality

$$\phi'(t) \geq C\phi^2(t) - C'\phi(t) - C' \quad (1.4)$$

with some $C, C' > 0$, which leads to $T_{\max} < \infty$. One of the key to the proof is to show upper and lower estimates for w . This together with (1.3) and [6, Theorem 1.4] leads to the estimate $u(x, t) \leq C_3|x|^{-3(3\alpha-1)-\eta}$ with some $C_3 > 0$, which plays an important role in controlling the term $Ku^\alpha w$.

2. Preliminaries

We first state a result on local existence, which can be proved as in [19, Lemma 2.1].

Lemma 2.1. *Let $m > 0$. Suppose that*

$$u_0 \in C^0(\overline{B_R}) \text{ is nonnegative and radially symmetric with } \int_{B_R} u_0(x) dx = m \quad (2.1)$$

and

$$w_0 \in C^0(\overline{B_R}) \text{ is positive and radially symmetric.} \quad (2.2)$$

Then there exist $T_{\max} \in (0, \infty]$ and a classical solution (u, v, w) of (1.2), uniquely determined by the inclusions

$$\begin{aligned} u &\in C^0(\overline{B_R} \times [0, T_{\max})) \cap C^{2,1}(\overline{B_R} \times (0, T_{\max})), \\ v &\in \bigcap_{q>n} C^0([0, T_{\max}); W^{1,q}(B_R)) \cap C^{2,0}(\overline{B_R} \times (0, T_{\max})), \\ w &\in C^0(\overline{B_R} \times [0, T_{\max})) \cap C^{2,1}(\overline{B_R} \times (0, T_{\max})), \end{aligned}$$

such that

$$\text{if } T_{\max} < \infty, \text{ then } \lim_{t \nearrow T_{\max}} \left\{ \|u(\cdot, t)\|_{L^\infty(B_R)} + \|w(\cdot, t)\|_{L^\infty(B_R)} \right\} = \infty. \quad (2.3)$$

Moreover, u, v , and w are nonnegative and radially symmetric.

In the model (1.2) an upper bound for the total mass $\int_{B_R} u$ can be immediately obtained. As a preparation we introduce the following statement.

Lemma 2.2. *Suppose (2.1) and (2.2). Then the following holds.*

$$\int_{B_R} u(x, t) dx \leq \int_{B_R} u_0(x) dx \quad \text{for all } t \in [0, T_{\max}). \quad (2.4)$$

Proof. Integrating the first equation in (1.2), we have $\frac{d}{dt} \int_{B_R} u = - \int_{B_R} u - K \int_{B_R} u^\alpha w \leq 0$ for all $t \in (0, T_{\max})$, which proves this lemma. $\square$

In the case $\alpha = 1$, Theorem 1.1 has been established without the condition (1.3) in [19]. Therefore, throughout the sequel, we consider only the case

$$\frac{2}{3} < \alpha < 1. \quad (2.5)$$

Moreover, without further explicit mentioning, we always assume $K > 0$, (2.1) and (2.2). Also, introducing $r := |x|$, we denote by $(u, v, w) = (u(r, t), v(r, t))$ the radially symmetric classical solution of (1.2) in $\overline{B_R} \times (0, T_{\max})$ given by Lemma 2.1.

3. Upper and lower estimates for w

In this section we first show an upper estimate for w . The following lemma can be proved as in the proof of [19, Lemma 3.4]. However, due to the importance of this lemma, we give a full proof here.

Lemma 3.1. *There exists $t_* \leq 1$ such that*

$$\|w(\cdot, t)\|_{L^\infty(B_R)} \leq 2\|w_0\|_{L^\infty(B_R)} =: K^* \quad \text{for all } t \in (0, \min\{t_*, T_{\max}\}).$$

Proof. We define $\widehat{T} > 0$ as

$$\widehat{T} := \sup \left\{ \tau \in (0, \min\{t_*, T_{\max}\}) \mid \|w\|_{L^\infty(B_R)} < 2\|w_0\|_{L^\infty(B_R)} \text{ on } [0, \tau) \right\}, \quad (3.1)$$

where $t_* \leq 1$ will be given later. We shall make sure that $\widehat{T} = \min\{t_*, T_{\max}\}$ by a contradiction argument. To this end, let us assume that $\widehat{T} < \min\{t_*, T_{\max}\}$. Then we have $\|w(\cdot, \widehat{T})\|_{L^\infty(B_R)} = 2\|w_0\|_{L^\infty(B_R)}$. Let $p \in [\frac{3}{2}, 3)$. By virtue of [3, Lemma 23] with the Sobolev embedding theorem and the second equation in (1.2) there is $c_1 > 0$ such that

$$\|v\|_{L^p(B_R)} \leq c_1 \| -\Delta v + v \|_{L^1(B_R)} = c_1 K \|u^\alpha w\|_{L^1(B_R)} \leq c_1 K \|u^\alpha\|_{L^1(B_R)} \|w\|_{L^\infty(B_R)},$$

which together with Hölder's inequality, (2.4) and (3.1) yields

$$\|v\|_{L^p(B_R)} \leq 2c_1 K |B_R|^{1-\alpha} \|u_0\|_{L^1(B_R)}^\alpha \|w_0\|_{L^\infty(B_R)} =: c_2 \quad \text{on } (0, \widehat{T}).$$

Noting that $t_* \leq 1$, from the third equation in (1.2) and semigroup estimates ([19, Lemma 3.3]) we obtain $c_3 > 0$ such that

$$\begin{aligned} \|w(\cdot, t)\|_{L^\infty(B_R)} &\leq \|w_0\|_{L^\infty(B_R)} + c_3 \int_0^t (t-s)^{-\frac{3}{2} \cdot \frac{1}{p}} \|v(\cdot, s)\|_{L^p(B_R)} ds \\ &\leq \|w_0\|_{L^\infty(B_R)} + c_2 c_3 \cdot \frac{t^{1-\frac{3}{2} \cdot \frac{1}{p}}}{1 - \frac{3}{2} \cdot \frac{1}{p}} \quad \text{for all } t \in [0, \widehat{T}]. \end{aligned}$$

Now we choose

$$t_* := \min \left\{ 1, \left(\frac{\left[1 - \frac{3}{2} \cdot \frac{1}{p}\right] \|w_0\|_{L^\infty(B_R)}}{2c_2 c_3} \right)^{\frac{1}{1-\frac{3}{2} \cdot \frac{1}{p}}} \right\}.$$

Then $\|w(\cdot, t)\|_{L^\infty(B_R)} \leq \|w_0\|_{L^\infty(B_R)} + \frac{1}{2}\|w_0\|_{L^\infty(B_R)}$ for all $t \in [0, \widehat{T}]$, which contradicts $\|w(\cdot, \widehat{T})\|_{L^\infty(B_R)} = 2\|w_0\|_{L^\infty(B_R)}$. Thus we can see that $\widehat{T} = \min\{t_*, T_{\max}\}$, and moreover, (3.1) leads to the conclusion. $\square$

Next we prove a lower bound for w .

Lemma 3.2. *There exists $t_{**} > 0$ such that*

$$w(x, t) \geq \frac{1}{2} \inf_{x \in B_R} w_0(x) =: K_* \quad \text{for all } (x, t) \in B_R \times (0, \min\{t_{**}, T_{\max}\}).$$

Proof. We put

$$t_{**} := \min \left\{ t_*, \frac{1}{2K_*} \inf_{x \in B_R} w_0(x) \right\},$$

where $t_* > 0$ and $K_* > 0$ are given by Lemma 3.1. This together with the third equation in (1.2) and a simple comparison argument yields this lemma (see [19, Lemma 3.5]). $\square$

4. Differential inequality for a moment-type functional

We put $T := \min\{t_{**}, T_{\max}\}$, where t_{**} is given by Lemma 3.2. Following [19], we set

$$U(s, t) = \int_0^{s^{\frac{1}{3}}} \rho^2 u(\rho, t) d\rho \quad \text{and} \quad V(s, t) = \int_0^{s^{\frac{1}{3}}} \rho^2 v(\rho, t) d\rho$$

for $(s, t) \in [0, R^3] \times [0, T)$, which transform (1.2) to a scalar equation by a straightforward calculations; more precisely, from the second and first equations in (1.2) we have

$$s^{\frac{2}{3}} v_r(s^{\frac{1}{3}}, t) = V(s, t) - 3^{\alpha-1} \int_0^s U_s^\alpha(\sigma, t) w(\sigma^{\frac{1}{3}}, t) d\sigma$$

and

$$\begin{aligned} U_t(s, t) &= 9s^{\frac{4}{3}} U_{ss}(s, t) + 3^\alpha K U_s(s, t) \int_0^s U_s^\alpha(\sigma, t) w(\sigma^{\frac{1}{3}}, t) d\sigma - 3V(s, t) U_s(s, t) \\ &\quad - U(s, t) - 3^{\alpha-1} K \int_0^s U_s^\alpha(\sigma, t) w(\sigma^{\frac{1}{3}}, t) d\sigma \end{aligned} \quad (4.1)$$

for all $(s, t) \in (0, R^3) \times (0, T)$. Moreover, as in [21] (also, e.g. [16, 17, 18]), introducing the moment-type functional

$$\phi(s_0, t) := \int_0^{s_0} s^{-\gamma} (s_0 - s) U(s, t) ds \quad \text{for } (s_0, t) \in [0, R^3] \times [0, T), \quad (4.2)$$

where $\gamma \in (0, 1)$ and ϕ belongs to $C^0([0, T)) \cap C^1((0, T))$ for each $s_0 \in (0, R^3)$, we will derive the desired superlinear differential inequality (1.4). To achieve this, the following pointwise estimate for u plays an important role.

Lemma 4.1. *Let $\eta > 0$. Then there is $L > 0$ such that for each u_0 with (2.1) and (1.3),*

$$u(x, t) \leq L|x|^{-3(3\alpha-1)-\eta} \quad \text{for all } (x, t) \in B_R \times (0, T). \quad (4.3)$$

Proof. By applying Lemma 3.1 to the last term in the second equation in (1.2), we have $Ku^\alpha w \leq K^*Ku^\alpha$ on $(0, T)$. Also, it follows from the first equation in (1.2) that $u_t \leq \Delta u - \nabla \cdot (u \nabla v)$. Noting (2.4) holds, we employ [6, Theorem 1.4] to obtain (4.3). $\square$

This lemma implies that $u^{\alpha-1}(r, t) \geq L^{\alpha-1} r^{3(1-\alpha)(3\alpha-1+\frac{\eta}{3})}$ for all $(r, t) \in (0, R) \times (0, T)$, that is, $U_s^{\alpha-1}(s, t) \leq (\frac{L}{3})^{\alpha-1} s^{(1-\alpha)(3\alpha-1+\frac{\eta}{3})}$ for all $(s, t) \in (0, R^3) \times (0, T)$. By utilizing this estimate and Lemma 3.2 to the second term in the right of (4.1) and integrating by parts, we can derive the following estimate.

Lemma 4.2. *Let $\eta > 0$. Then there are $C_1, C_2 > 0$ such that for each u_0 fulfilling (2.1) and (1.3),*

$$\begin{aligned} \phi_t(s_0, t) &\geq 9 \int_0^{s_0} s^{\frac{4}{3}-\gamma} (s_0 - s) U_{ss} ds + C_1 \int_0^{s_0} s^{\lambda-\gamma} (s_0 - s) U U_s ds \\ &\quad - C_2 \int_0^{s_0} s^{-\gamma} (s_0 - s) \int_0^s \sigma^{\lambda-1} U(\sigma, t) d\sigma ds - 3 \int_0^{s_0} s^{-\gamma} (s_0 - s) V U_s ds \\ &\quad - \phi(s_0, t) - 3^{\alpha-1} K \int_0^{s_0} s^{-\gamma} (s_0 - s) \int_0^s U_s^\alpha(\sigma, t) w(\sigma^{\frac{1}{3}}, t) d\sigma ds \\ &=: I_1(s_0, t) + I_2(s_0, t) + I_3(s_0, t) + I_4(s_0, t) + I_5(s_0, t) + I_6(s_0, t) \end{aligned} \quad (4.4)$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$, where $\lambda = (1 - \alpha)(3\alpha - 1 + \frac{\eta}{3})$.

The choice $\gamma = \frac{1}{3}$ in the above lemma makes treating I_1 particularly simple.

Lemma 4.3. *Let $\gamma = \frac{1}{3}$. Then there is $C_3 > 0$ such that for each u_0 fulfilling (2.1) and with I_1 as in (4.4),*

$$I_1(s_0, t) \geq -C_3 s_0 \quad \text{for all } (s_0, t) \in (0, R^3) \times (0, T).$$

Proof. We integrate by parts and rely on (2.4) to conclude that

$$\begin{aligned} \frac{I_1(s_0, t)}{9} &= \int_0^{s_0} s(s_0 - s) U_{ss}(s, t) ds = - \int_0^{s_0} (s_0 - 2s) U_s(s, t) ds \\ &\geq -s_0 \int_0^{s_0} U_s(s, t) ds = -s_0 U(s_0, t) \geq -\frac{m}{\omega_3} s_0 \end{aligned}$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$, where $\omega_3 := 3|B_R|$. $\square$

In order to show (1.4) we will estimate I_3, I_4 and I_6 by I_2 . Moreover, we need the relation between I_2 and ϕ . The following lemma enables us to derive such estimates, which is established as in [21, Lemma 4.2].

Lemma 4.4. *Let $s_0 > 0$ and $\gamma, \lambda \in \mathbb{R}$ with $\gamma > \lambda$, and suppose that $\varphi \in C^1([0, s_0]; [0, \infty))$ fulfills $\varphi(0) = 0$ and $\varphi' \geq 0$ in $(0, s_0)$. Then*

$$\varphi(s) \leq \sqrt{2} s^{\frac{\gamma-\lambda}{2}} (s_0 - s)^{-\frac{1}{2}} \left(\int_0^{s_0} \sigma^{\lambda-\gamma} (s_0 - \sigma) \varphi(\sigma) \varphi'(\sigma) d\sigma \right)^{\frac{1}{2}} \quad \text{for all } s \in (0, s_0).$$

By making use of this lemma, we can provide some estimates for I_2, I_3 and I_6 .

Lemma 4.5. *Let $\gamma = \frac{1}{3}$. Then there exists $\eta > 0$ such that $\gamma > \lambda$, where λ is as given by Lemma 4.2. Moreover, there exist $C_4, C_5, C_6 > 0$ such that for each u_0 fulfilling (2.1) and with ϕ, I_2, I_3 and I_6 as in (4.2) and (4.4),*

$$I_2(s_0, t) \geq C_4 s_0^{\lambda+\gamma-3} \phi^2(s_0, t) \tag{4.5}$$

as well as

$$I_3(s_0, t) \geq -C_5 s_0^{\frac{3}{2} + \frac{\lambda-\gamma}{2}} I_2^{\frac{1}{2}}(s_0, t) \quad \text{and} \quad I_6(s_0, t) \geq -C_6 s_0^{3 - \frac{3\alpha}{2} - \gamma + \frac{(\gamma-\lambda)\alpha}{2}} I_2^{\frac{\alpha}{2}}(s_0, t) \tag{4.6}$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$.

Proof. It follows from (2.5) that $(1 - \alpha)(3\alpha - 1) < \frac{1}{3}$, and thus we can take $\eta > 0$ small enough such that $(1 - \alpha)(3\alpha - 1 + \frac{\eta}{3}) < \frac{1}{3}$, that is, $\lambda < \gamma$.

By Lemma 4.4 and as $-\frac{\lambda+\gamma}{2} > -\frac{1}{3} > -1$, we have

$$\begin{aligned} \phi(s_0, t) &\leq \sqrt{2} \left(\int_0^{s_0} s^{-\frac{\lambda+\gamma}{2}} (s_0 - s)^{\frac{1}{2}} ds \right) \left(\int_0^{s_0} s^{\lambda-\gamma} (s_0 - s) U U_s ds \right)^{\frac{1}{2}} \\ &= \sqrt{2} B \left(1 - \frac{\lambda + \gamma}{2}, \frac{3}{2} \right) s_0^{\frac{3-(\lambda+\gamma)}{2}} C_1^{-\frac{1}{2}} I_2^{\frac{1}{2}}(s_0, t) \end{aligned}$$

for all $(s_0, t) \in (0, R^n) \times (0, T)$, where B denotes Euler's Beta function and where C_1 is as in Lemma 4.2. Squaring yields (4.5). As to I_3 , by Fubini's theorem we may estimate

$$\begin{aligned} -\frac{I_3(s_0, t)}{C_2} &= \int_0^{s_0} \int_\sigma^{s_0} s^{-\gamma}(s_0 - s) ds \sigma^{\lambda-1} U(\sigma, t) d\sigma \\ &\leq \int_0^{s_0} \int_\sigma^{s_0} s^{-\gamma} ds \sigma^{\lambda-1} (s_0 - \sigma) U(\sigma, t) d\sigma \\ &\leq \frac{s_0^{1-\gamma}}{1-\gamma} \int_0^{s_0} \sigma^{\lambda-1} (s_0 - \sigma) U(\sigma, t) d\sigma \end{aligned}$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$, which together with Lemma 4.4 leads to the first estimate in (4.6). Finally we derive the estimate for I_6 . We see from Lemma 3.1 and Hölder's inequality that

$$-\frac{I_6(s_0, t)}{3^{\alpha-1}K} \leq K^* \int_0^{s_0} s^{-\gamma}(s_0 - s) \int_0^s U_s^\alpha(\sigma, t) d\sigma ds \leq K^* \int_0^{s_0} s^{1-\alpha-\gamma}(s_0 - s) U^\alpha ds$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$, and Lemma 4.4 implies the second estimate in (4.6). $\square$

We next give an estimate for I_4 . The idea of the proof is based on [21, Section 4.2], so that we only state the result here.

Lemma 4.6. *Let $\gamma = \frac{1}{3}$ and $\lambda > 0$ with*

$$\frac{1}{\alpha} \cdot \frac{4}{3} - 2 + \gamma < \lambda < \gamma. \quad (4.7)$$

Then there exists $C_7 > 0$ such that for each u_0 fulfilling (2.1) and with I_2, I_4 as in (4.4),

$$I_4(s_0, t) \geq -C_7 s_0^{2-\alpha-\frac{\lambda}{2}} I_2^{\frac{1}{2}}(s_0, t) - C_7 s_0^{\frac{11}{6}-\frac{3\alpha}{2}+\frac{(\gamma-\lambda)(\alpha+1)}{2}} I_2^{\frac{\alpha+1}{2}}(s_0, t)$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$.

As a consequence of these lemmas, we can derive the desired inequality.

Lemma 4.7. *Let $\gamma = \frac{1}{3}$. Then there exists $\eta > 0$ small enough such that λ satisfies (4.7), where λ is as given by Lemma 4.2. Moreover, there exist $C_8, C_9 > 0$ such that for each $s_0 > 0$ and u_0 fulfilling (2.1) and (1.3),*

$$\phi_t(s_0, t) \geq C_8 s_0^{\lambda+\gamma-3} \phi^2(s_0, t) - \phi(s_0, t) - C_9 s_0 \quad \text{for all } t \in (0, T).$$

Proof. Since $\alpha > \frac{2}{3}$, we have

$$\begin{aligned} (1-\alpha)(3\alpha-1) - \left(\frac{1}{\alpha} \cdot \frac{4}{3} - 2 + \gamma \right) &= \frac{1}{3\alpha} [3\alpha(1-\alpha)(3\alpha-1) - (4-5\alpha)] \\ &> \frac{1}{3\alpha} [3(1-\alpha) - (4-5\alpha)] = \frac{1}{3\alpha} (-1+2\alpha) > 0, \end{aligned}$$

and moreover we know that $(1 - \alpha)(3\alpha - 1) < \frac{1}{3}$. Therefore we can find $\eta > 0$ small enough such that (4.7) holds. From Lemmas 4.2, 4.3 and 4.6 as well as (4.6) we infer that

$$\begin{aligned}\phi_t(s_0, t) &\geq I_2(s_0, t) - \phi(s_0, t) \\ &\quad - C_3 s_0 - C_5 s_0^{\frac{3}{2} + \frac{\lambda - \gamma}{2}} I_2^{\frac{1}{2}}(s_0, t) - C_6 s_0^{3 - \frac{3\alpha}{2} - \gamma + \frac{(\gamma - \lambda)\alpha}{2}} I_2^{\frac{\alpha}{2}}(s_0, t) \\ &\quad - C_7 s_0^{2 - \alpha - \frac{\lambda}{2}} I_2^{\frac{1}{2}}(s_0, t) - C_7 s_0^{\frac{11}{6} - \frac{3\alpha}{2} + \frac{(\gamma - \lambda)(\alpha + 1)}{2}} I_2^{\frac{\alpha + 1}{2}}(s_0, t)\end{aligned}$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$. Because of the relation $\alpha < 1$, we can employ Young's inequality to obtain $c_1 > 0$ such that

$$\begin{aligned}\phi_t(s_0, t) &\geq \frac{1}{2} I_2(s_0, t) - \phi(s_0, t) \\ &\quad - c_1 \left(s_0 + s_0^{3 + \lambda - \gamma} + s_0^{\left(3 - \frac{3\alpha}{2} - \gamma + \frac{(\gamma - \lambda)\alpha}{2}\right) \frac{2}{2 - \alpha}} + s_0^{4 - 2\alpha - \lambda} + s_0^{\left(\frac{11}{6} - \frac{3\alpha}{2} + \frac{(\gamma - \lambda)(\alpha + 1)}{2}\right) \frac{2}{1 - \alpha}} \right)\end{aligned}$$

for all $(s_0, t) \in (0, R^3) \times (0, T)$. Here, $(3 + \lambda - \gamma) - 1 = 1 + \frac{2}{3} + \lambda > 0$. Also we see from the relations $\alpha < 1$ and $\lambda < \gamma = \frac{1}{3}$ that $(4 - 2\alpha - \lambda) - 1 \geq 2 - \frac{1}{3} - 1 > 0$, and moreover, $\left(3 - \frac{3\alpha}{2} - \gamma + \frac{(\gamma - \lambda)\alpha}{2}\right) \frac{2}{2 - \alpha} - 1 > 0$ as well as $\left(\frac{11}{6} - \frac{3\alpha}{2} + \frac{(\gamma - \lambda)(\alpha + 1)}{2}\right) \frac{2}{1 - \alpha} - 1 > 0$. Thus, for each $s_0 \in (0, R^n)$,

$$\phi_t(s_0, t) \geq \frac{1}{2} I_2(s_0, t) - \phi(s_0, t) - c_2 s_0 \quad \text{for all } t \in (0, T)$$

with some $c_2 > 0$, which together with Lemma 4.5 leads to the conclusion. $\square$

5. Outline of the proof

Let $m_1 > 0$, $\varepsilon \in (0, 1)$ and $\gamma = \frac{1}{3}$. Moreover, let $t_{**} > 0$ and $\lambda > 0$ be as given by Lemmas 3.2 and 4.7, respectively. From [21, estimate (5.5)] there exists $C_{10} > 0$ such that for each $s_0 \in (0, R^3)$ and u_0 fulfilling (2.1) as well as $\int_{B_{r_1}} u_0(x) dx \geq m_1$, where $r_1 := (\frac{s_0}{4})^{\frac{1}{3}}$, it follows that

$$\phi(s_0, 0) \geq C_{10} s_0^{2 - \gamma}.$$

Thus, by virtue of Lemmas 4.7 we see that

$$\begin{cases} \phi_t(s_0, t) \geq c_1(s_0) \phi^2(s_0, t) - \phi(s_0, t) - c_2(s_0), & t \in (0, T), \\ \phi(s_0, 0) \geq \phi_0(s_0), \end{cases}$$

where $c_1(s_0) := C_9 s_0^{\lambda + \gamma - 3}$, $c_2(s_0) := C_{10} s_0$ and $\phi_0(s_0) := C_{10} s_0^{2 - \gamma}$. Here, we choose $s_0 > 0$ small enough satisfying

$$\phi_0(s_0) \geq \frac{1 + \sqrt{c_1(s_0) c_2(s_0)}}{c_1(s_0)} + \frac{2}{c_1(s_0) t_{**}}.$$

Then we can apply [5, Lemma 3.2] to make sure that $T = T_{\max} \leq t_{**}$, which together with (2.3) and Lemma 3.1 yields the conclusion of Theorem 1.1. $\square$

References

- [1] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, *Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci., **25** (2015), 1663–1763.
- [2] N. Bellomo, K. J. Painter, Y. Tao, M. Winkler, *Occurrence vs. absence of taxis-driven instabilities in a May–Nowak model for virus infection*, SIAM J. Appl. Math., **79** (2019), 1990–2010.
- [3] H. Brézis, W. A. Strauss, *Semi-linear second-order elliptic equations in L^1* , J. Math. Soc. Japan, **25** (1973), 565–590.
- [4] M. Fuest, *Boundedness enforced by mildly saturated conversion in a chemotaxis–May–Nowak model for virus infection*, J. Math. Anal. Appl., **472** (2019), 1729–1740.
- [5] M. Fuest, *Finite-time blow-up in a two-dimensional Keller–Segel system with an environmental dependent logistic source*, Nonlinear Anal. Real World Appl., **52** (2020), Paper No. 103022, 14 pp.
- [6] M. Fuest, *On the optimality of upper estimates near blow-up in quasilinear Keller–Segel systems*, Appl. Anal., **101** (2022), 3515–3534.
- [7] M. Fuest, Y. Tanaka, *A critical conversion exponent in the three-dimensional chemotaxis–May–Nowak model*, in preparation.
- [8] T. H. Harris, E. J. Banigan, D. A. Christian, C. Konradt, E. D. Tait Wojno, K. Norose, E. H. Wilson, B. John, W. Weninger, A. D. Luster, A. J. Liu, C. A. Hunter, *Generalized Levy walks and the role of chemokines in migration of effector CD8 + T cells*, Nature, **486** (2012), 545–548.
- [9] B. Hu, J. Lankeit, *Boundedness of solutions to a virus infection model with saturated chemotaxis*, J. Math. Anal. Appl., **468** (2018), 344–358.
- [10] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.
- [11] A. Korobeinikov, *Global properties of basic virus dynamics models*, Bull. Math. Biol., **66** (2004), 879–883.
- [12] F. Lin, E. C. Butcher, *T cell chemotaxis in a simple microfluidic device*, Lab Chip, **6** (2006), 1462–1469.
- [13] M. J. Miller, S. H. Wei, M. D. Cahalan, I. Parker, *Autonomous T cell trafficking examined in vivo with intravital two-photon microscopy*, Proc. Nat. Acad. Sci. USA, **100** (2003), 2604–2609.
- [14] M. A. Nowak, R. May, *Virus dynamics: Mathematical principles of immunology and virology*, Oxford University Press (2000).

- [15] O. Stancevic, C. N. Angstmann, J. M. Murray, B. I. Henry, *Turing patterns from dynamics of early HIV infection*, Bull. Math. Biol., **75** (2013), 774–795.
- [16] Y. Tanaka, *Blow-up in a quasilinear parabolic–elliptic Keller–Segel system with logistic source*, Nonlinear Anal. Real World Appl., **63** (2022), Paper No. 103396, 29 pp.
- [17] Y. Tanaka, *Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production*, J. Math. Anal. Appl., **506** (2022), Paper No. 125654, 29 pp.
- [18] Y. Tanaka, T. Yokota, *Blow-up in a parabolic–elliptic Keller–Segel system with density-dependent sublinear sensitivity and logistic source*, Math. Methods Appl. Sci., **43** (2020), 7372–7396.
- [19] Y. Tao, M. Winkler, *Taxis-driven formation of singular hotspots in a May–Nowak type model for virus infection*, SIAM J. Math. Anal., **53** (2021), 1411–1433.
- [20] M. Winkler, *Boundedness in a chemotaxis–May–Nowak model for virus dynamics with mildly saturated chemotactic sensitivity*, Acta Appl. Math., **163** (2019), 1–17.
- [21] M. Winkler, *Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation*, Z. Angew. Math. Phys., **69** (2018), Paper No. 69, 40 pp.

Department of Mathematical Sciences
 Kwansei Gakuin University
 1 Gakuen Uegahara, Sanda, Hyogo 669-1330
 JAPAN
 E-mail address: yuya.tns.6308@gmail.com

関西学院大学・理学部数理科学科 田中 悠也